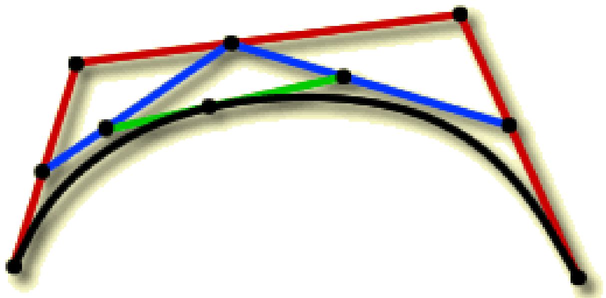
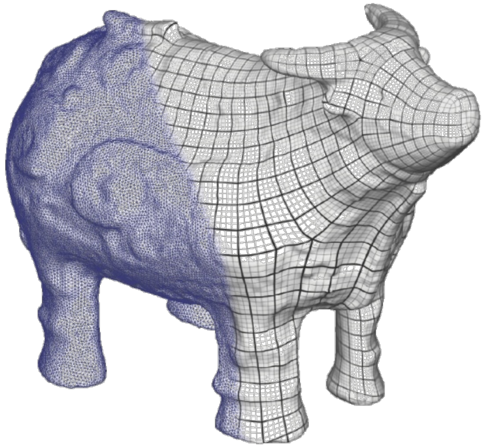
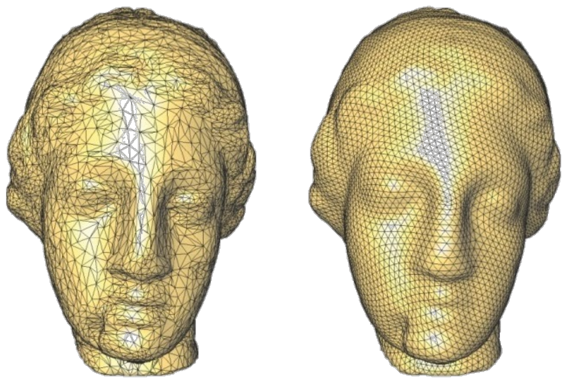
CS348a: Geometric Modeling and Processing







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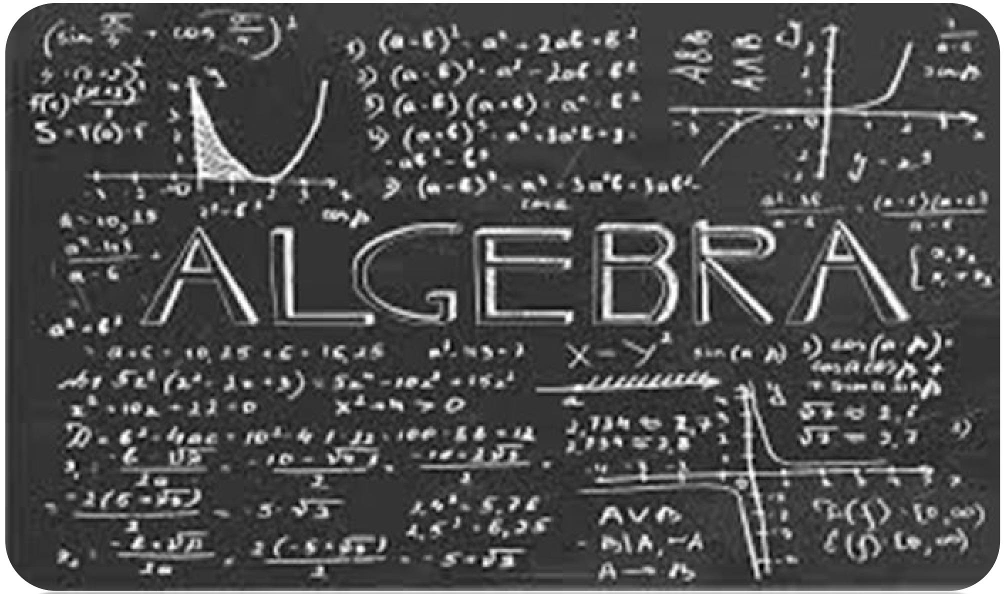
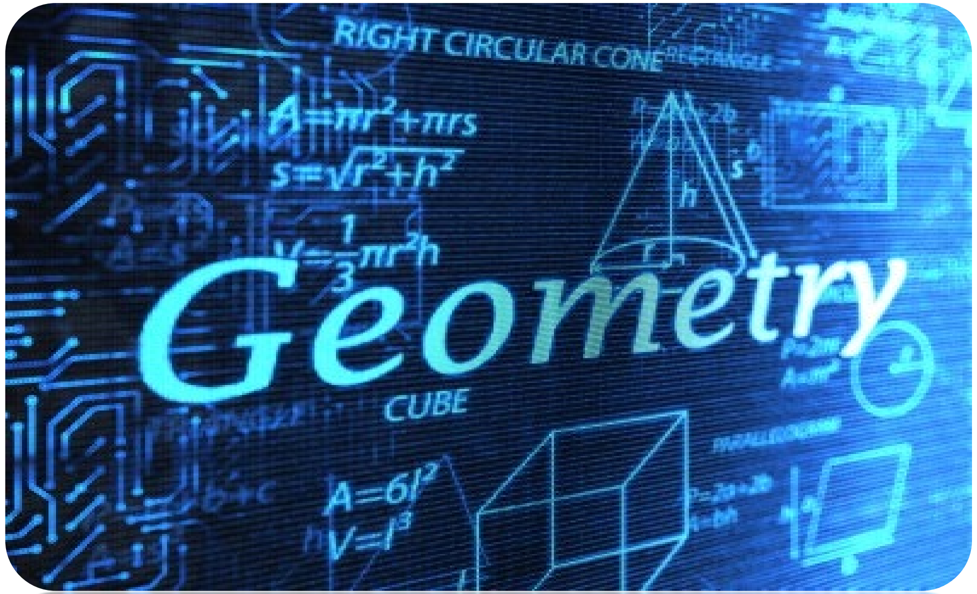
Geometric Computing

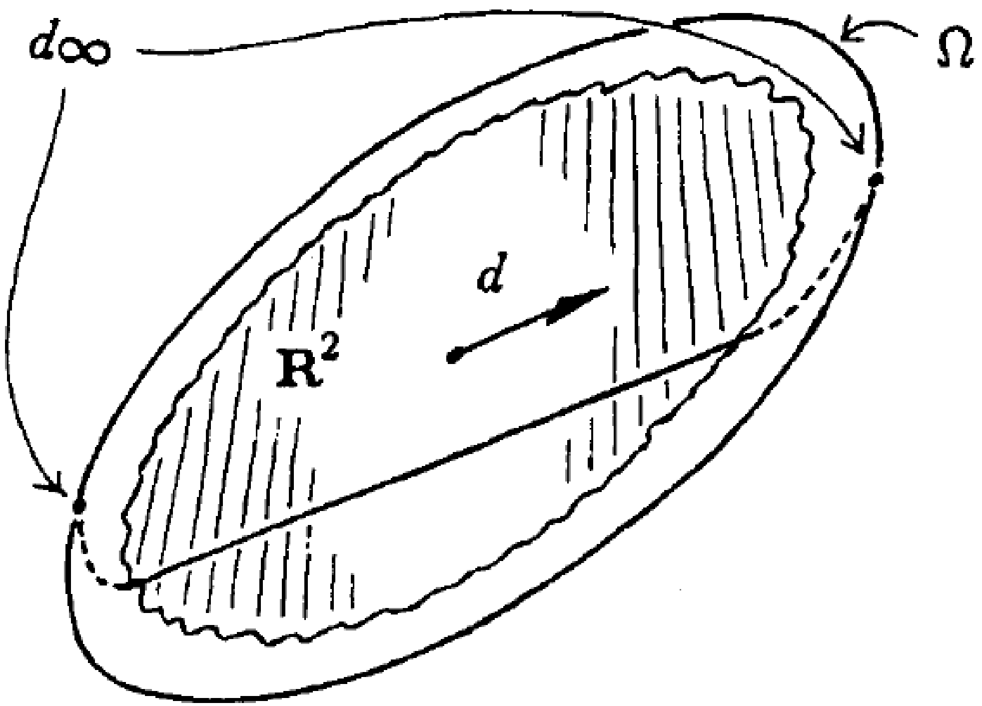
Last Time:
Projective Spaces,
Homogeneous Coordinates



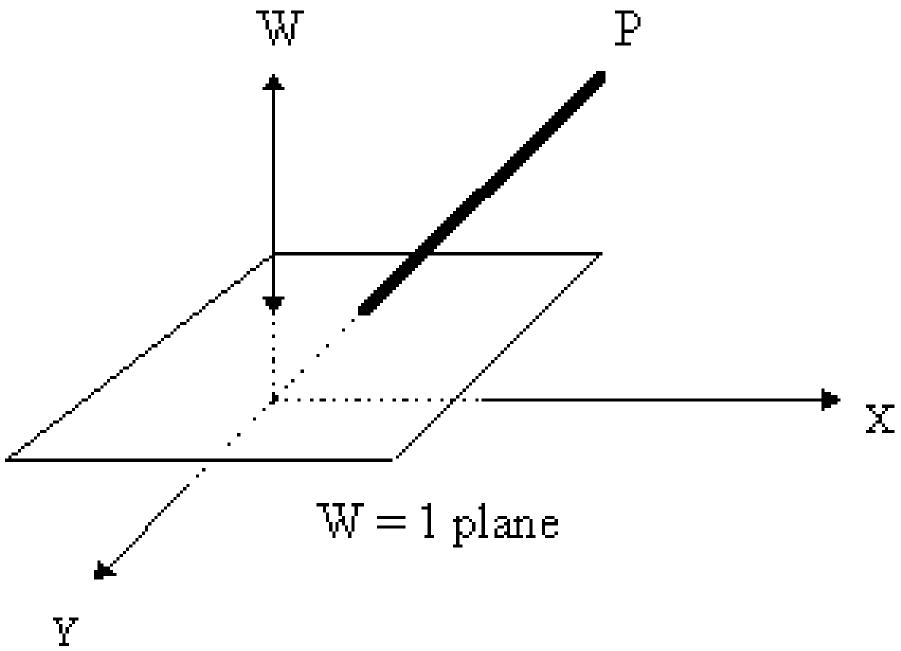


René Descartes

Straight Model of the Projective Plane



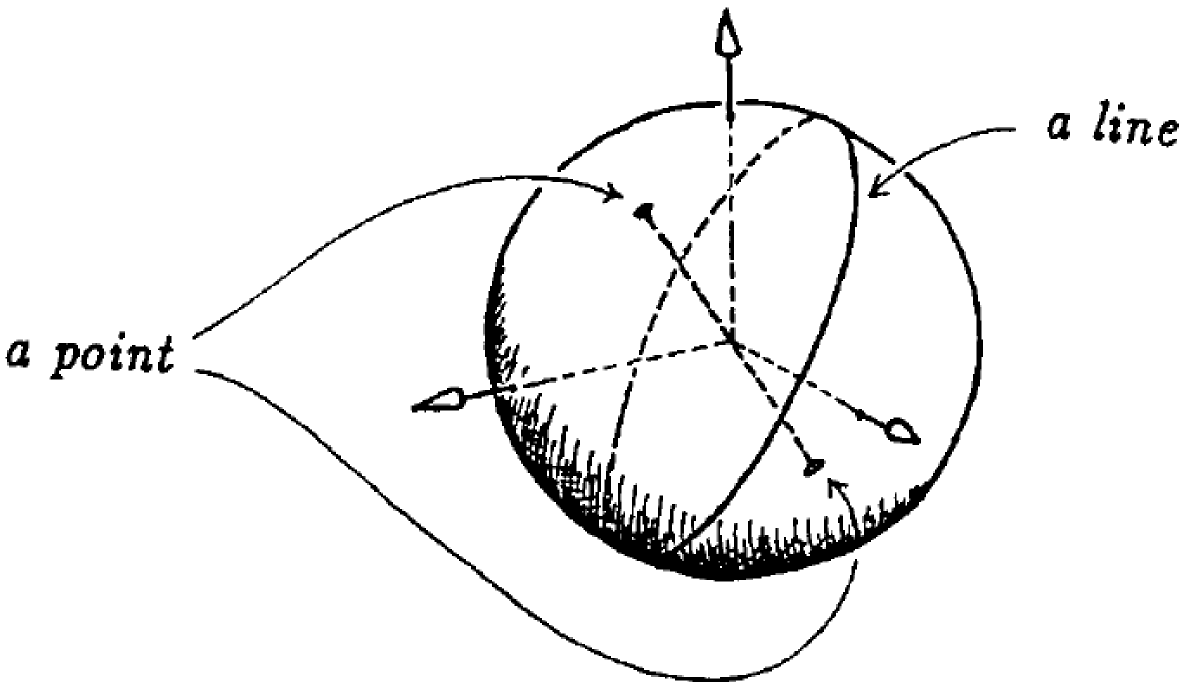
3D Line Model of the Projective Plane



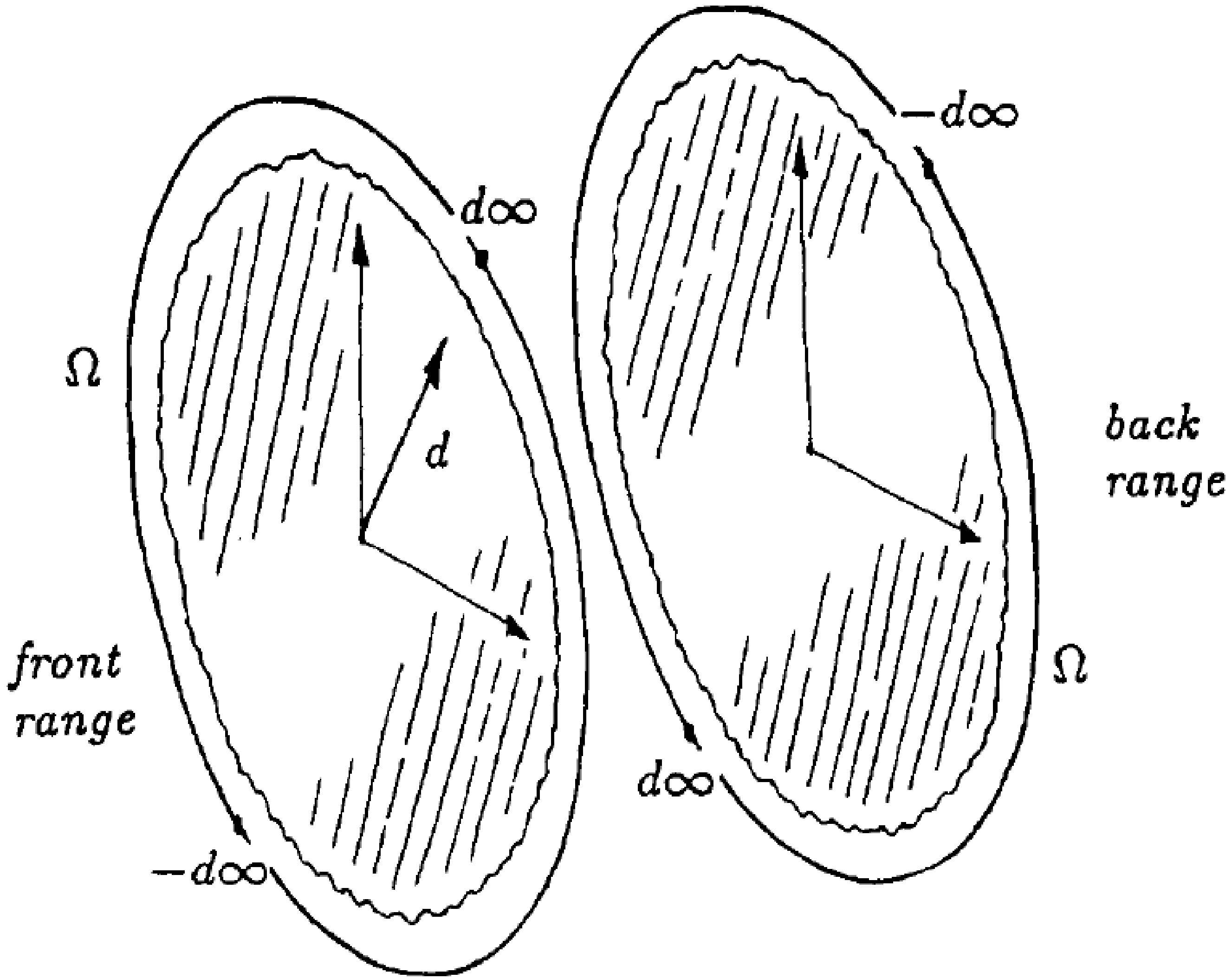
Pencil of lines through the origin in $[w, x, y]$ 3D space

Intersect with the $w = 1$ plane

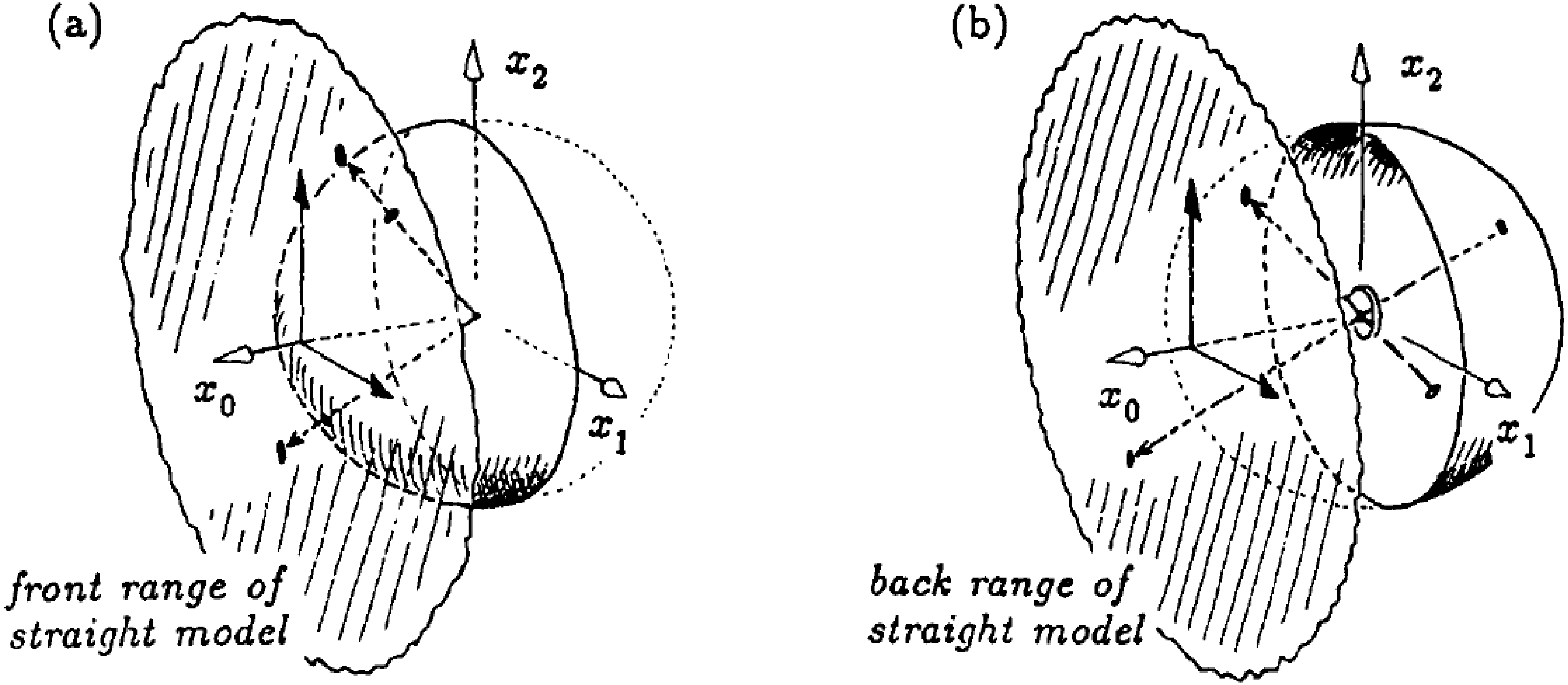
Spherical Model of the Projective Plane



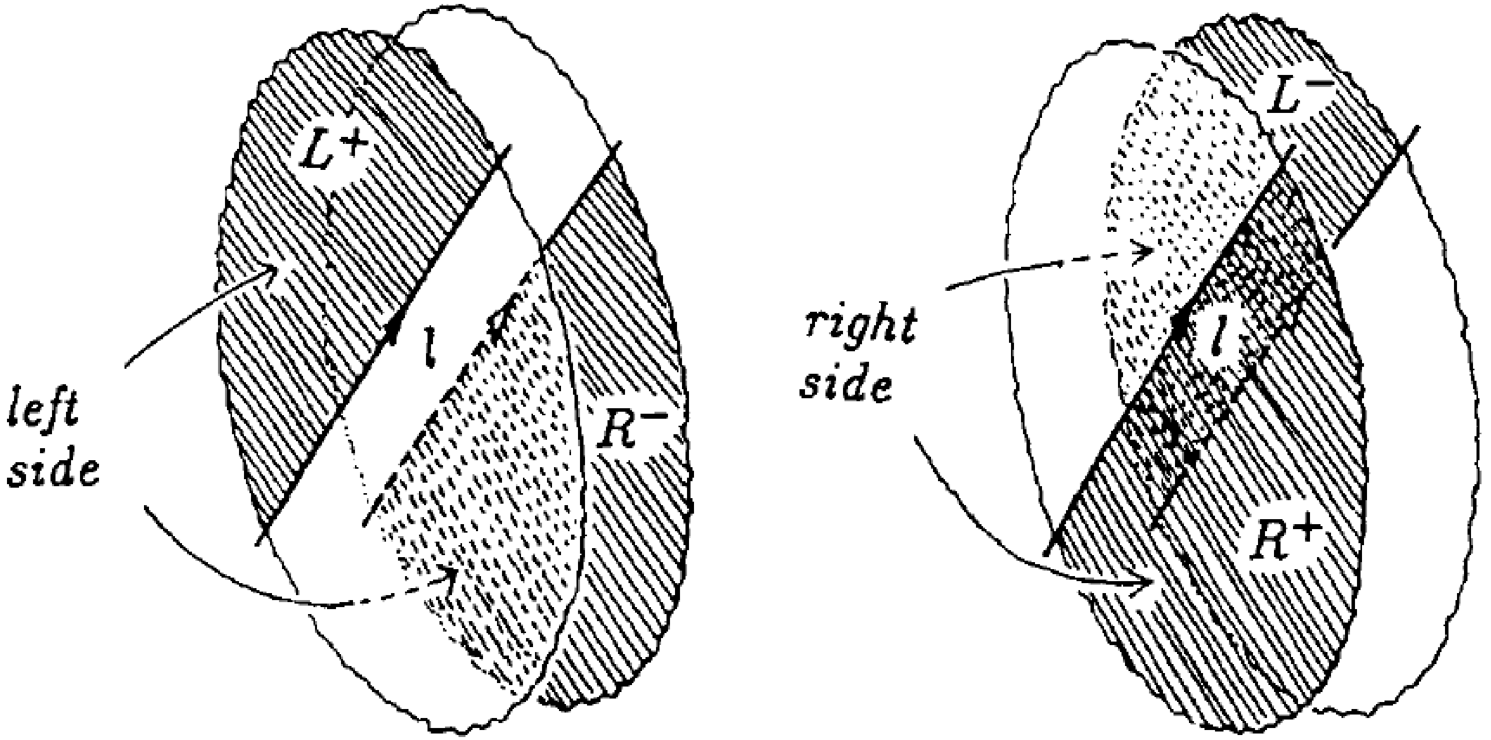
Straight Model for the Oriented Projective Plane



Stereographic Projection in the Oriented Case

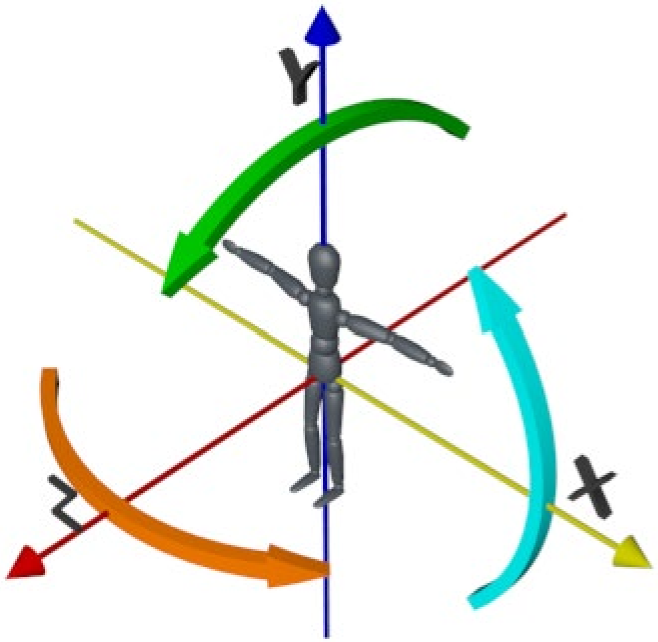


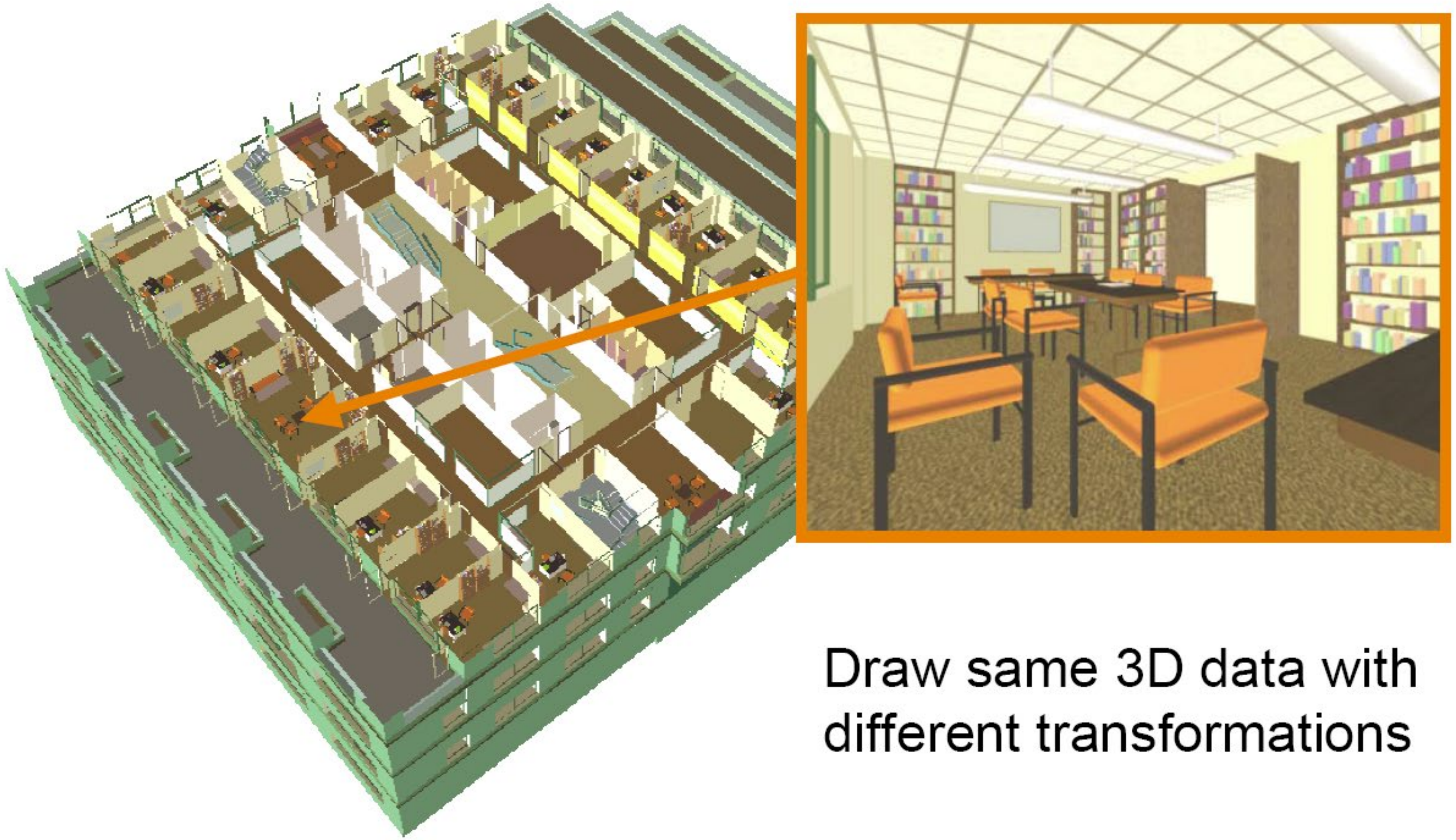
Left and Right Sides in the Oriented Case



2D and 3D Transformations

- An object may appear in a scene multiple times

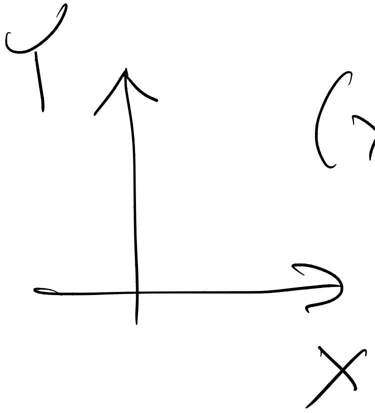




Draw same 3D data with different transformations

Whiteboard

Algebraic



(x, y)

$[w, x, y]$

$X = \frac{x}{w}, \quad Y = \frac{y}{w}$

$[w, x, y] = [\lambda w, \lambda x, \lambda y]$ ← Unoriented $\mathbb{P}^2$

$\lambda \neq 0$

~~$[0, 0, 0]$~~

$[w, x, y]$

$w \neq 0$ Euclidean

$[0, x, y]$ Ideal Pts

$w = 0$ or Pts at ∞

Geometric

$\mathbb{E}^2 \to \mathbb{P}^2$

$\mathbb{E}^3 \to \mathbb{P}^3$

$[w, x, y] =$

$[\lambda w, \lambda x, \lambda y]$

$\lambda > 0$

↑ Oriented

Perfect Duality

Transformations in E^2, P^2

- translations
- rotations
- scalings
- ...

} Affine, Projective

$$[w, x, y] = \begin{pmatrix} w \\ x \\ y \end{pmatrix}$$

3x3 matrices

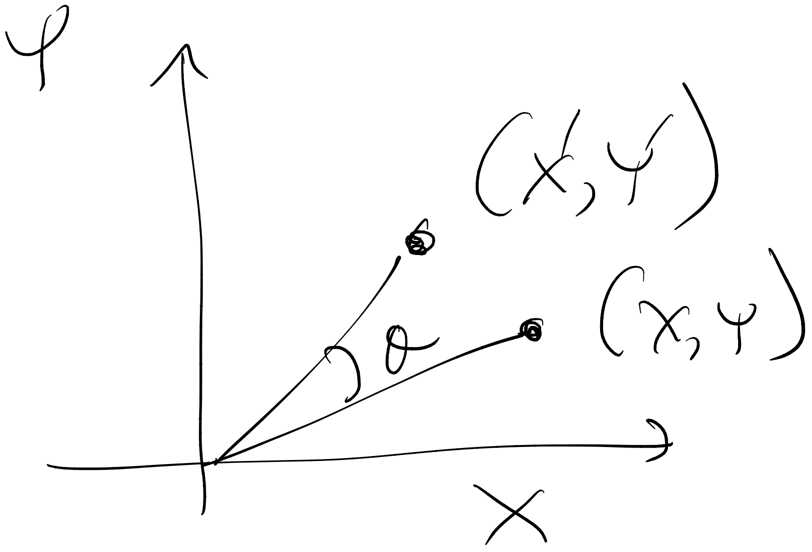
$$M \underset{3\times3}{\begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}} \underset{3\times1}{\begin{pmatrix} w \\ x \\ y \end{pmatrix}} = \underset{3\times1}{\begin{pmatrix} w' \\ x' \\ y' \end{pmatrix}}$$

$$Mp = p'$$

CS348a conventions:

1. homo coordinate is 1st
2. Transforms premultiply

2-d rotation



$$X' = X\cos\theta - Y\sin\theta$$
$$Y' = X\sin\theta + Y\cos\theta$$

$$R \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} w \\ x \\ y \end{bmatrix} = \begin{bmatrix} w \\ x' \\ y' \end{bmatrix}$$

Transformation Composition
≡ Matrix Multiplication

2-d scaling

$$S \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & S_x & 0 \\ 0 & 0 & S_y \end{bmatrix} \begin{bmatrix} w \\ x \\ y \end{bmatrix} = \begin{bmatrix} w \\ x' \\ y' \end{bmatrix}$$

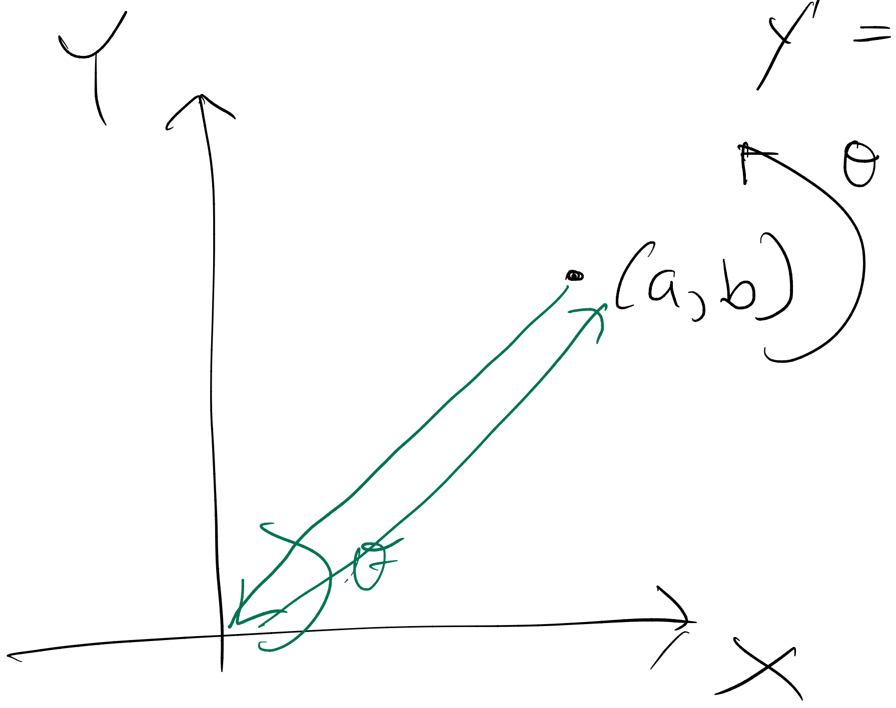
Translation

$$\begin{bmatrix} 1 & 0 & 0 \\ t_x & 1 & 0 \\ t_y & 0 & 1 \end{bmatrix} \begin{pmatrix} w \\ x \\ y \end{pmatrix} = \begin{pmatrix} w' \\ x' \\ y' \end{pmatrix}$$

$$x' = w t_x + x$$
$$y' = w t_y + y$$

Translation

$$x' = x + t_x$$
$$y' = y + t_y$$



$$T_{(a,b)} \, R_\theta \, T_{-(a,b)}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b & 0 & 1 \end{bmatrix}$$

$$[w, x, y] \quad \langle \alpha, \beta, \gamma \rangle \qquad (\alpha, \beta, \gamma)\begin{pmatrix} w \\ x \\ y \end{pmatrix} = 0$$

$$\alpha w + \beta x + \gamma y = 0$$

$$\underset{3\times 3}{M} \; \underset{3\times 1}{[w, x, y]} \rightarrow 3\times 1$$

$$\langle \alpha, \beta, \gamma \rangle \cdot [w, x, y] = 0$$

$$I$$

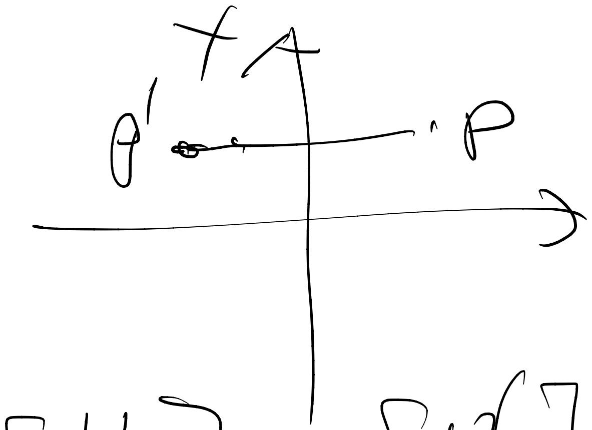
$$\underset{1\times 3}{\langle \alpha, \beta, \gamma \rangle} \underset{3\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{3\times 1}{[w, x, y]} \longrightarrow \underset{1\times 1}{0}$$

$$M^{-1} M$$

$$\left(\langle \alpha, \beta, \gamma \rangle M^{-1}\right)\left(M \, [w, x, y]\right) = 0$$

Reflection

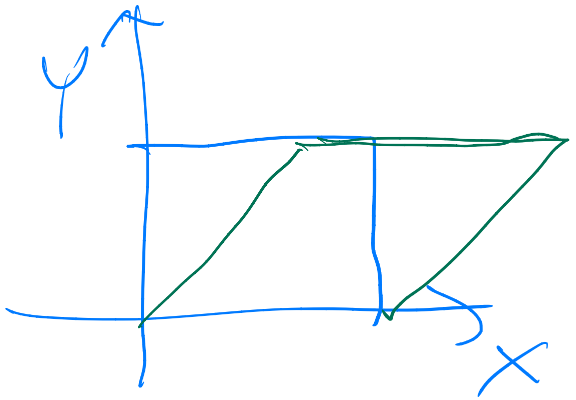
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Shearing

$$\left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & S \\ 0 & 0 & 1 \end{array}\right] \begin{bmatrix} w \\ x \\ y \end{bmatrix} = \begin{bmatrix} w' \\ x' \\ y' \end{bmatrix}$$

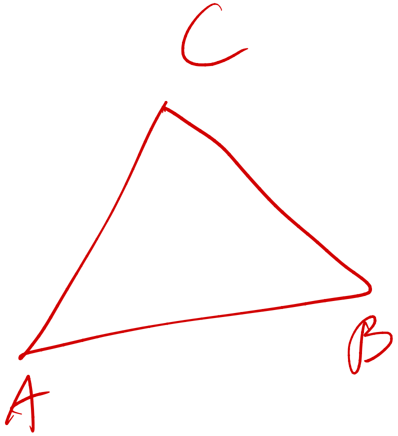
$$w' = w$$
$$x' = x + Sy$$
$$y' = y$$

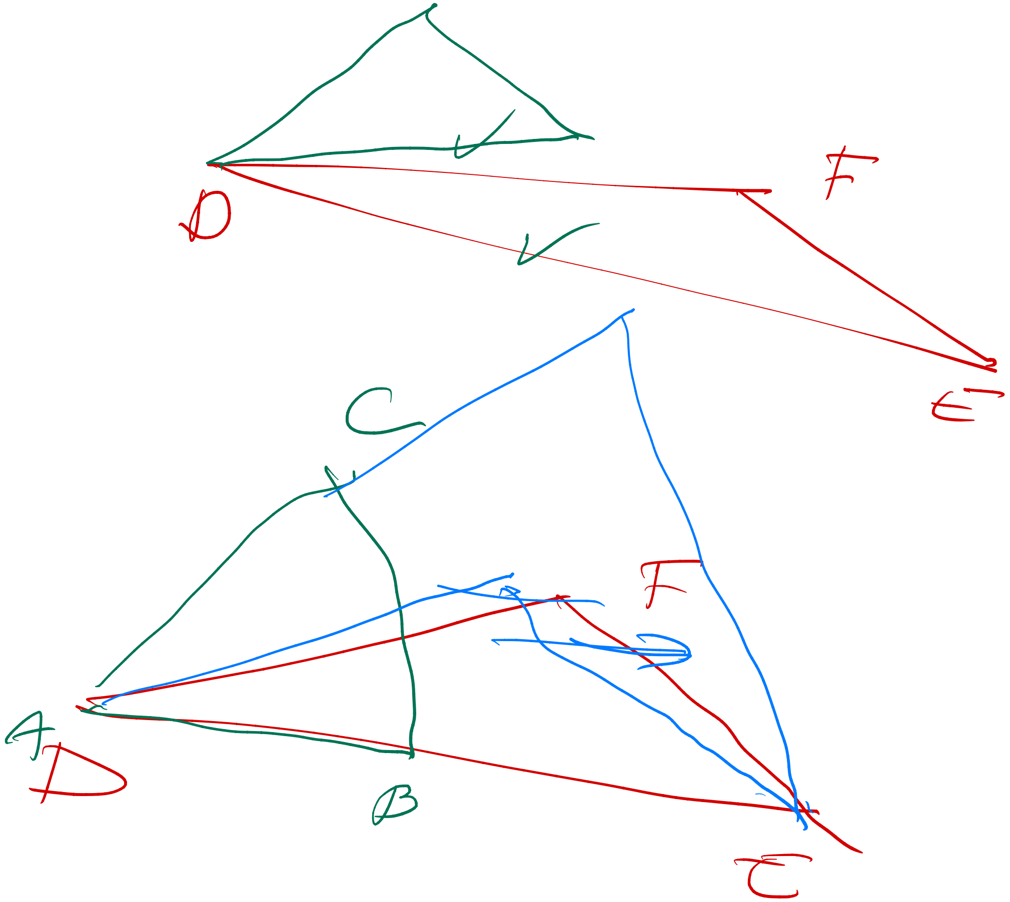


Affine Transformation Group

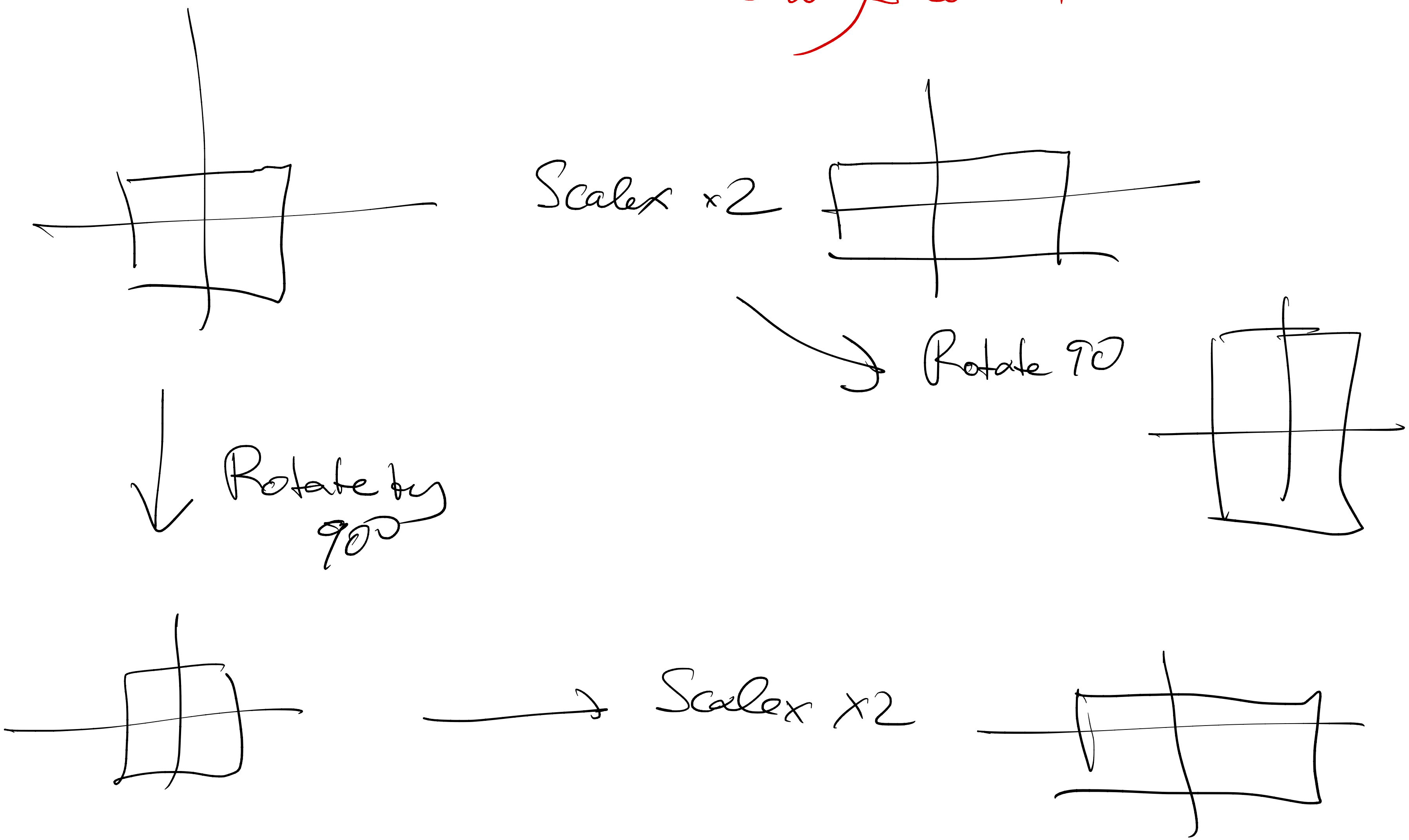
$$\begin{bmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

6D space

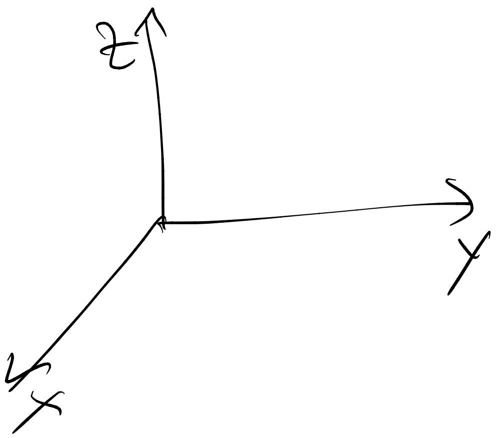




Transformation don't always commute



$$E^3 \to \mathbb{P}^3$$



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \qquad [w, x, y, z]$$

$$\langle \alpha, \beta, \gamma, \delta \rangle \quad [w, x, y, z]$$

$$\alpha w + \beta x + \gamma y + \delta z = 0$$

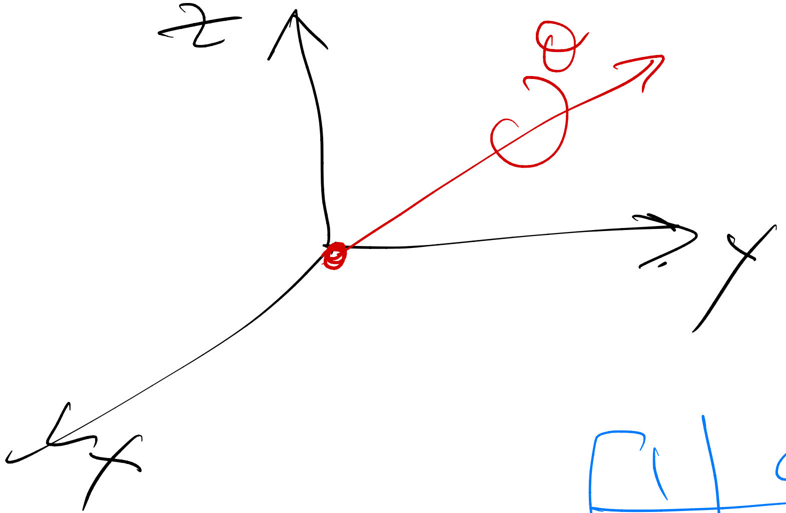
4x4

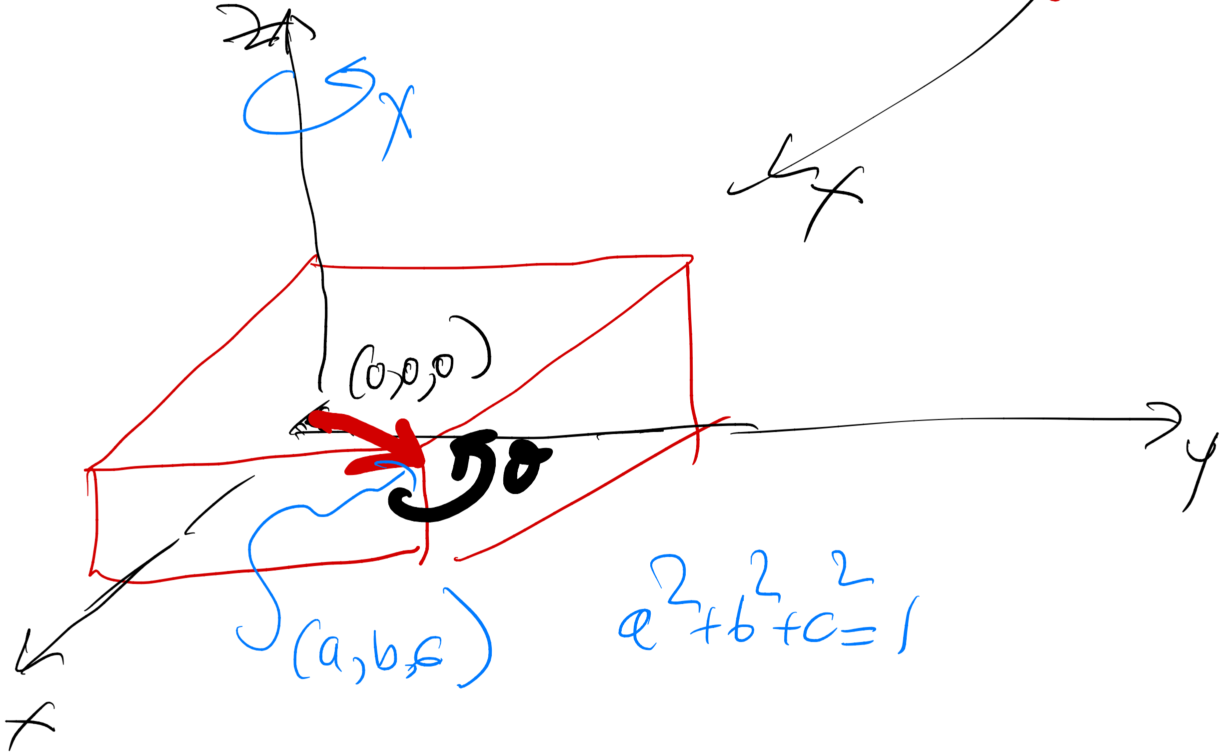
$$\left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}\right]$$

$$\left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline t_x & 1 & 0 & 0 \\ t_y & 0 & 1 & 0 \\ t_z & 0 & 0 & 1 \end{array}\right]$$ Translation

$$\left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & S_x & 0 & 0 \\ 0 & 0 & S_y & 0 \\ 0 & 0 & 0 & S_z \end{array}\right]$$ Scaling

3D Rotations



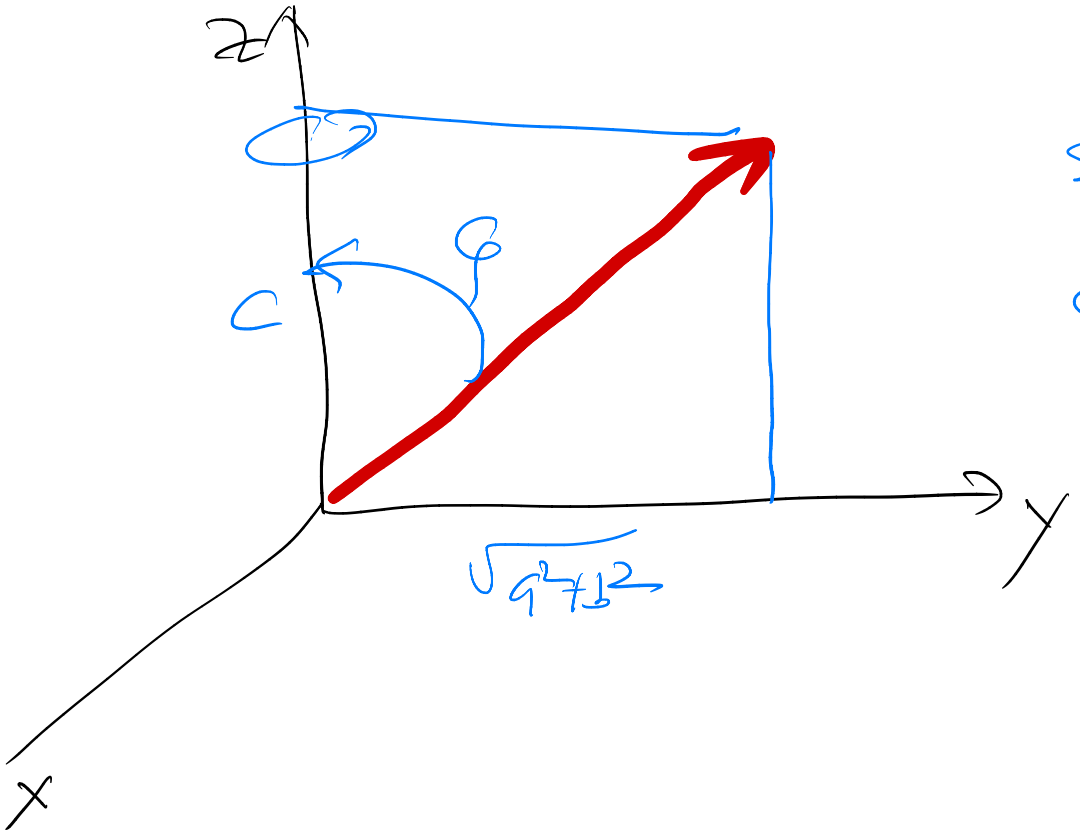


$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \end{bmatrix} ??$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

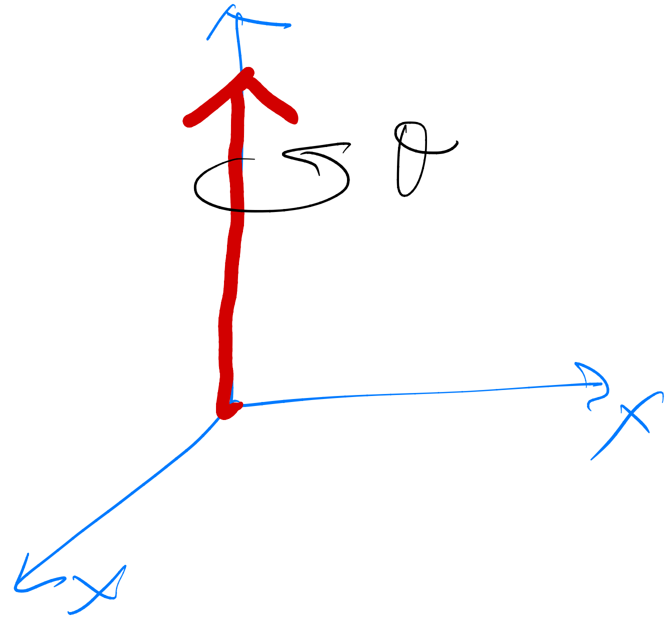
$R^x \quad R^y \quad R^z_\theta$

$$\cos\gamma = \frac{b}{\sqrt{a^2+b^2}} \qquad \sin\gamma = \frac{a}{\sqrt{a^2+b^2}} \qquad R_x^z$$



$$\sin\varphi = \sqrt{a^2+b^2} \Big/ \sqrt{a^2+b^2+c^2}$$

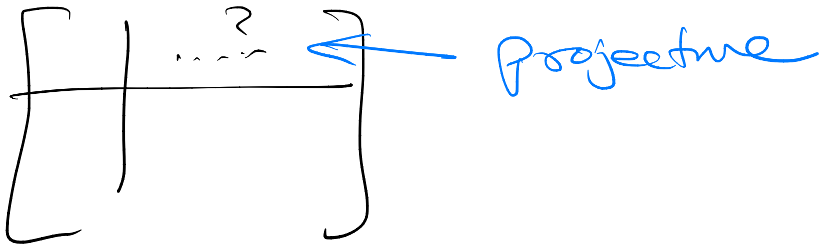
$$\cos\varphi = c \Big/ \sqrt{a^2+b^2+c^2}$$



$$\underbrace{R_z(-\gamma)\,R_x(-\varphi)\,R_z(\theta)\,R_x(\varphi)\,R_z(\gamma)}$$

$$\left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot \end{array}\right]$$

Projective



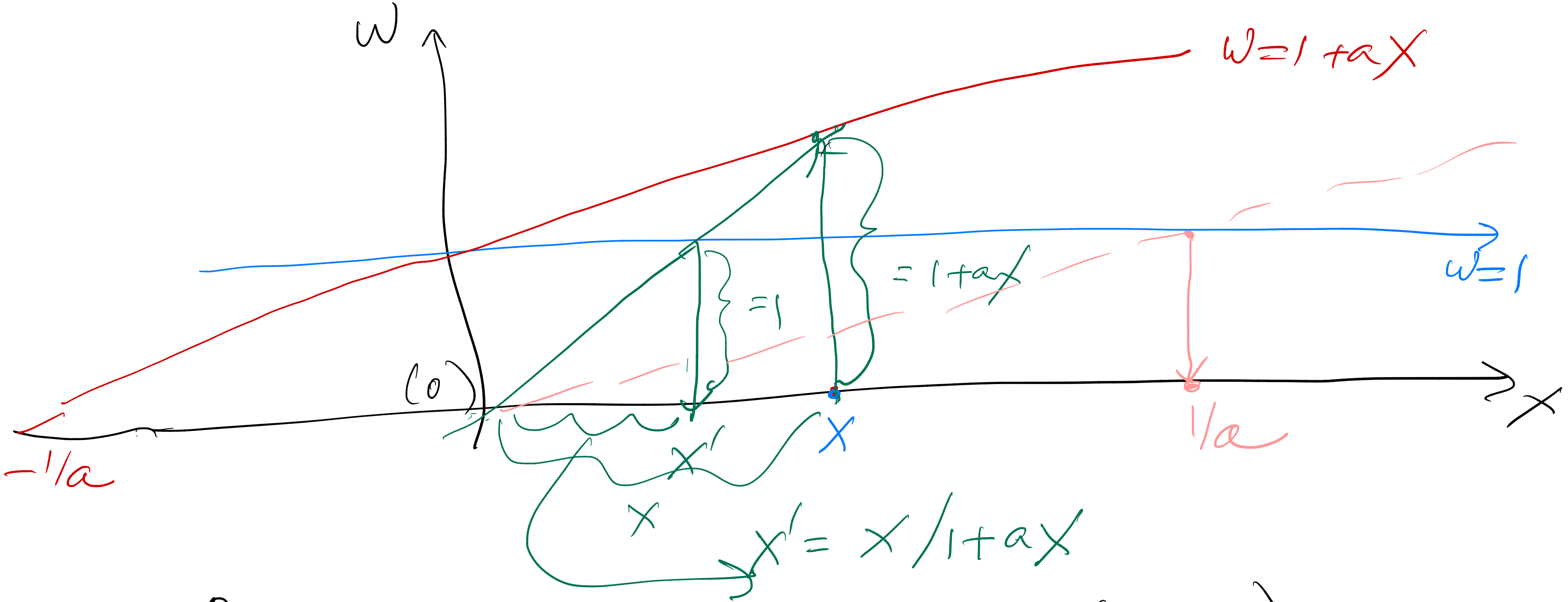
1-D projective transform

$$\left[\begin{array}{c|c} 1 & a \\ \hline 0 & 1 \end{array}\right] \begin{bmatrix} w \\ x \end{bmatrix} = \begin{bmatrix} w' \\ x' \end{bmatrix}$$

$$[w, x] \to [w + ax, x]$$

$$\frac{x}{w} \to \frac{x/w}{1 + a\,x/w}$$

$$x \to x / (1 + ax)$$



$0 \to 0$

$(0, +\infty) \longmapsto (0, \frac{1}{a})$

$(-\frac{1}{a}, 0) \longmapsto (-\infty, 0)$

$(-\infty, -\frac{1}{a}) \hookrightarrow (\frac{1}{a}, +\infty)$

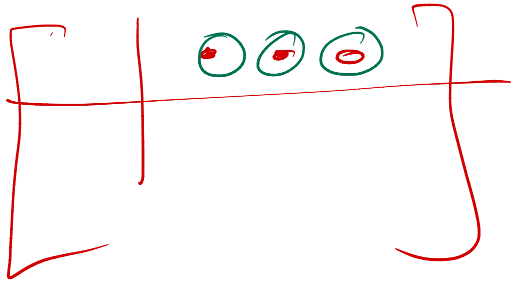
$$0 \to 0$$
$$\infty \to 1/a$$
$$-1/a \to \infty$$

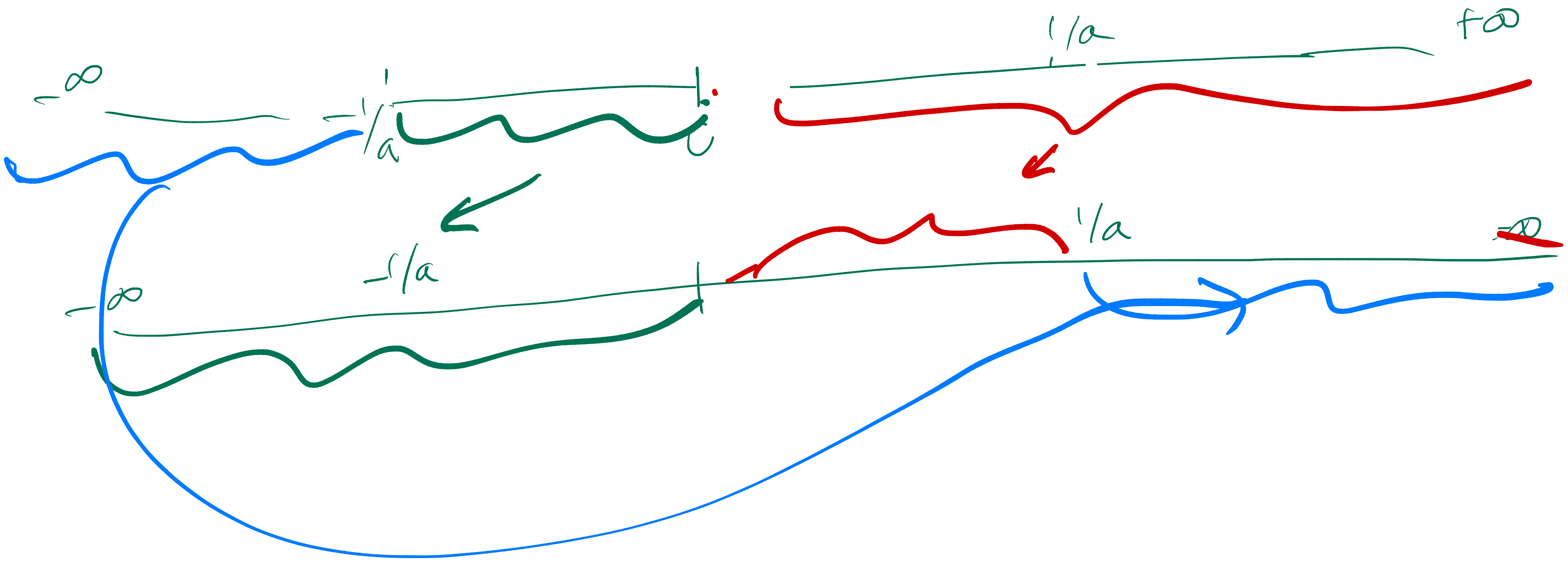
} mixing Euclidean & ideal pts

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \end{pmatrix} = \begin{pmatrix} w+ax \\ x \\ y \end{pmatrix}$$

$$x' = \frac{x}{1+ax} \qquad y' = \frac{y}{1+ax}$$

Perspective foreshortening



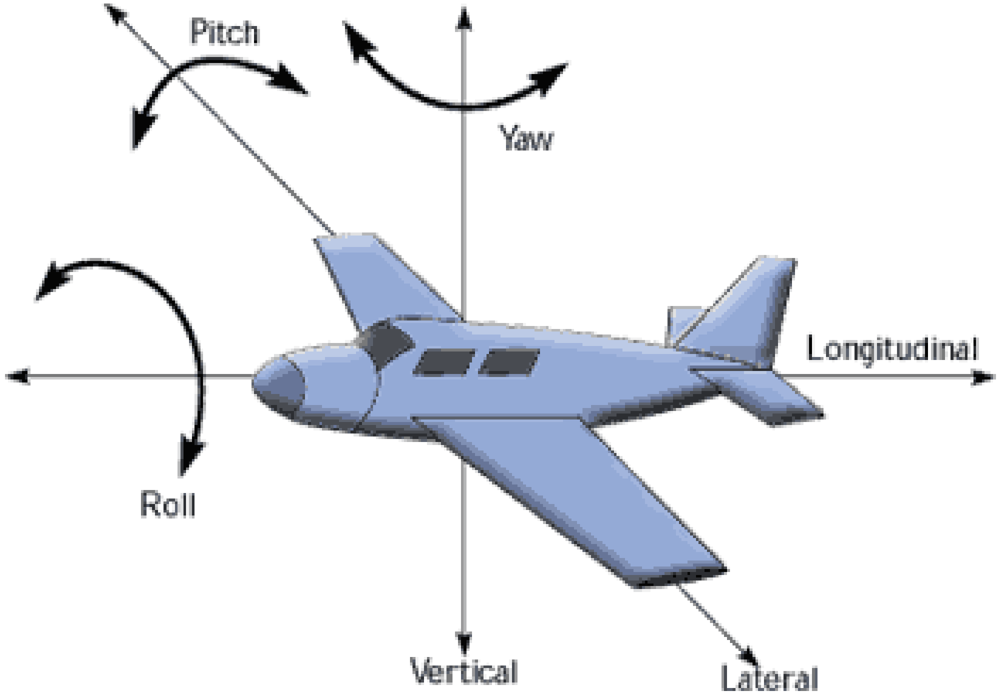

-∞
-1/a
1/a
+∞
-∞
-1/a
1/a
∞

Euler Angles

Leonhard Euler

1707 -- 1783

Pitch, Yaw, and Roll

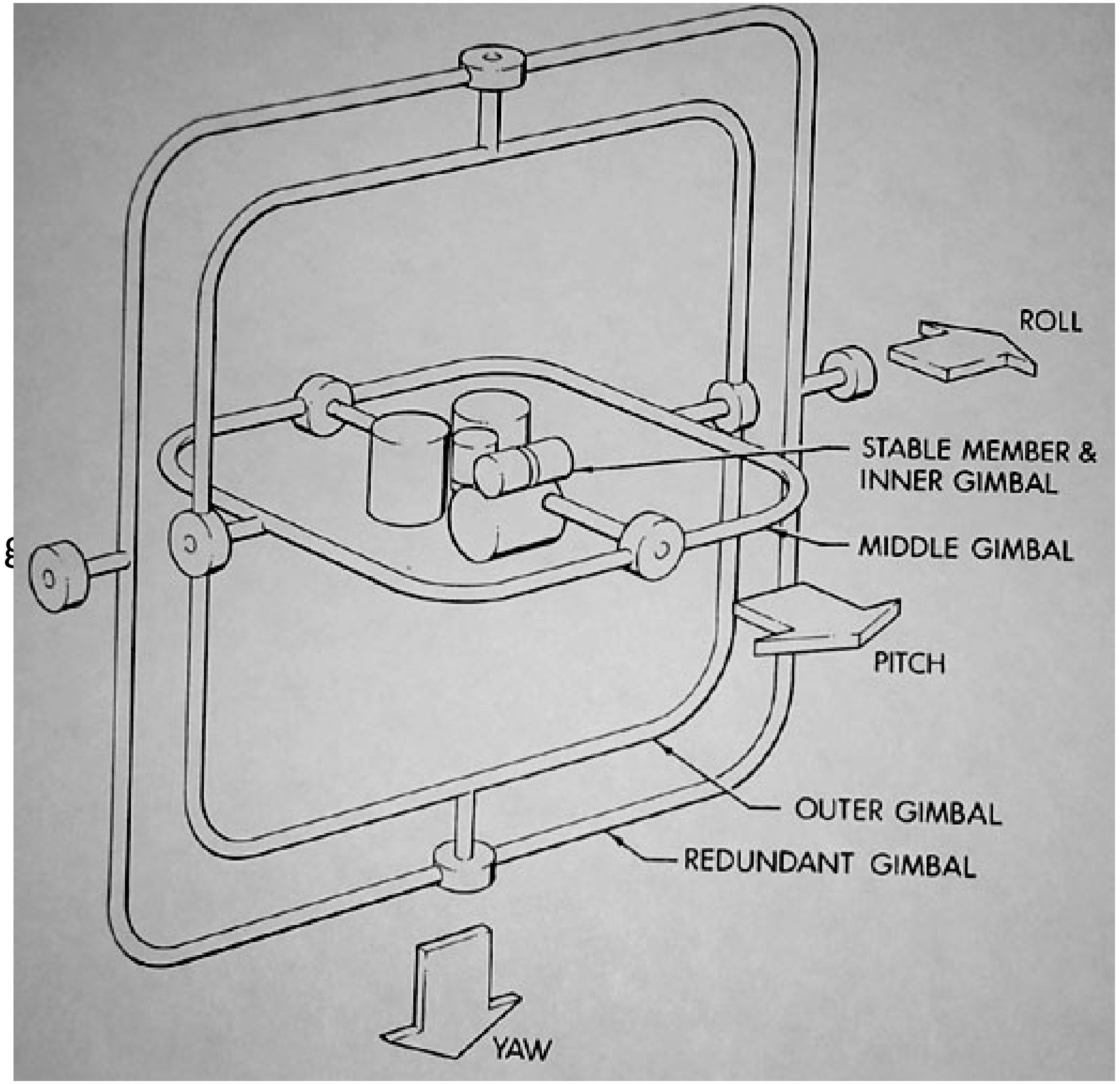


Euler Angles

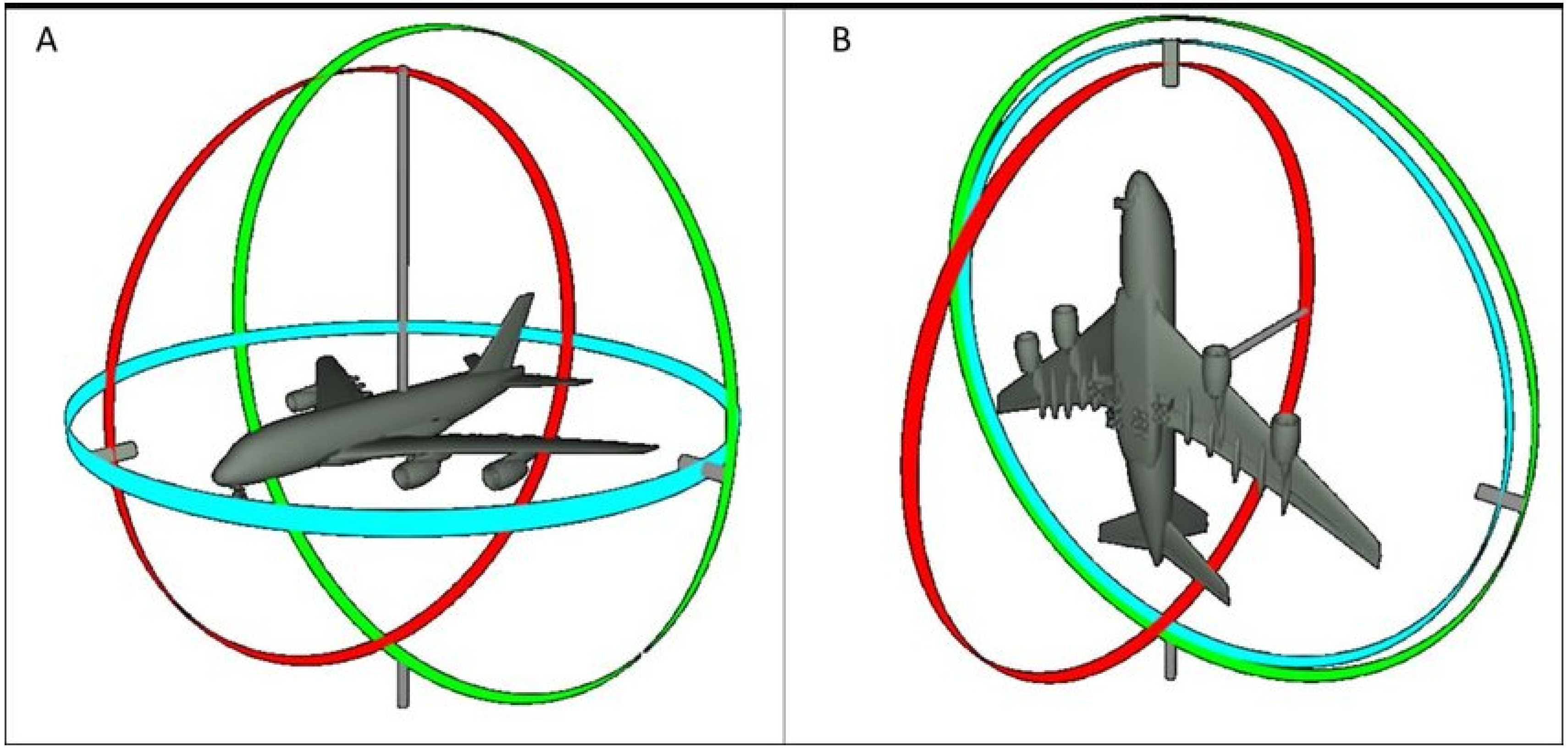
- ***Gimble (Gimbal)***
  - Hardware implementation of Euler angles
  - Aircraft, Camera

Euler Angles

- Rotation about three orthogonal axes
  - 12 combinations
    - XYZ, XYX, XZY, XZX
    - YZX, YZY, YXZ, YXY
    - ZXY, ZXZ, ZYX, ZYZ
- ***Gimble (or Gimbal) lock***
  - Coincidence of inner most and outmost g
  - Loss of degree of freedom



Gimbal Lock



That’s All