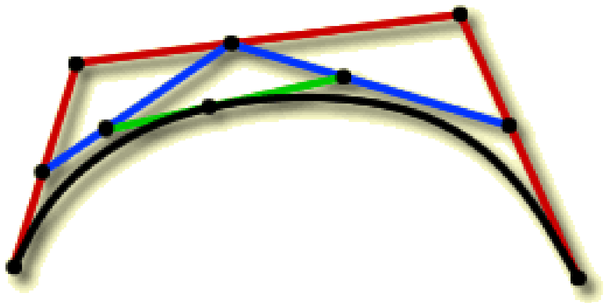
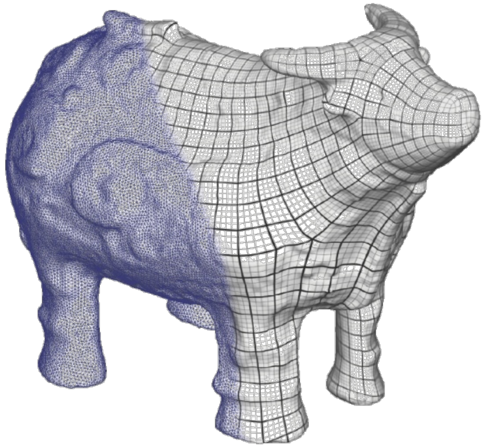
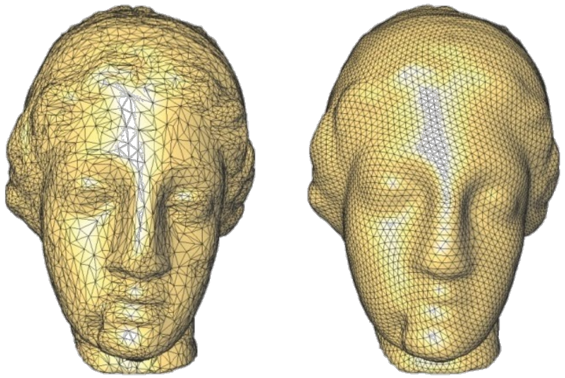
CS348a: Geometric Modeling and Processing







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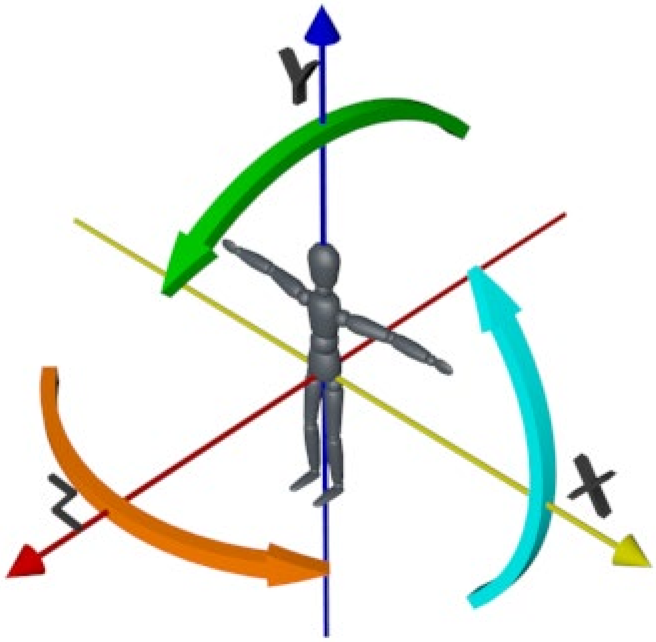
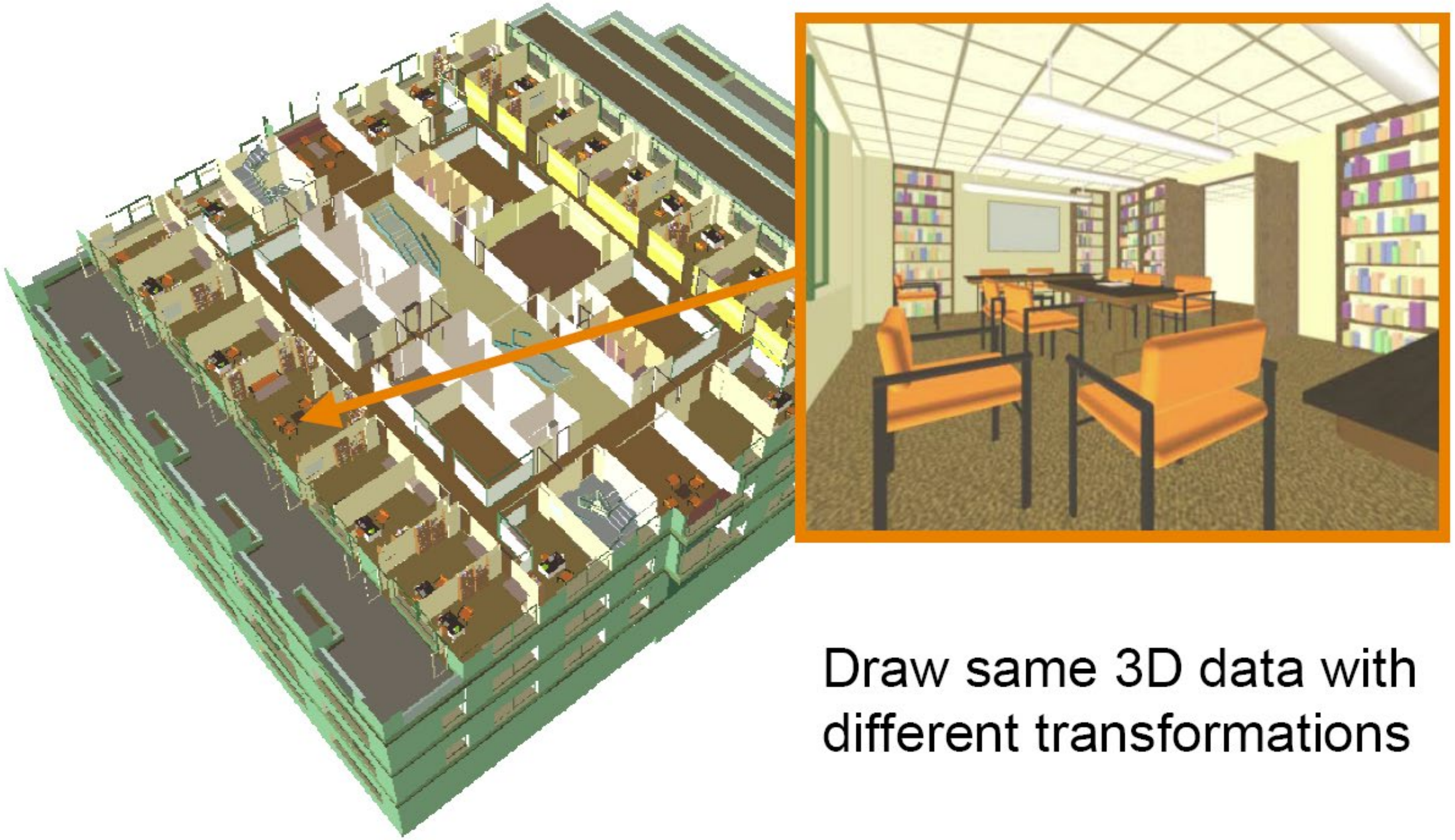
Last Time:
2D/3D Affine and Projective Transforms

Transforms in Graphics, Vision, Robotics



- An object may appear in a scene multiple times



Draw same 3D data with different transformations

2D Transforms in Homogeneous Coordinates

The Affine Group

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline a & b & c \\ d & e & f \end{array}\right) \begin{pmatrix} 1 \\ X \\ Y \end{pmatrix} = \begin{pmatrix} 1 \\ a + bX + cY \\ d + eX + fY \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & S_X & 0 \\ 0 & 0 & S_Y \end{pmatrix}$$

Rotation, Scaling

$$\begin{pmatrix} 1 & 0 & 0 \\ t_X & 1 & 0 \\ t_Y & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ X \\ Y \end{pmatrix} = \begin{pmatrix} 1 \\ X + t_X \\ Y + t_Y \end{pmatrix}$$

Translation

2D Transforms in Homogeneous Coordinates

The Projective Group

$$\left(\begin{array}{c|cc} 1 & a & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)\begin{pmatrix} w \\ x \\ y \end{pmatrix} \begin{pmatrix} w + ax \\ x \\ y \end{pmatrix}$$

$$X' = \frac{X}{1 + aX}$$

$$Y' = \frac{Y}{1 + aX}$$

3D Homegeneous Coordinates

$$[w, x, y, z] = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$$

$$X = \frac{x}{w}$$

$$Y = \frac{y}{w}$$

$$Z = \frac{z}{w}$$

4×4 Matrices

$$\langle \alpha, \beta, \gamma, \delta \rangle [w, x, y, z] = \alpha w + \beta x + \gamma y + \delta z = 0$$

Some Issues

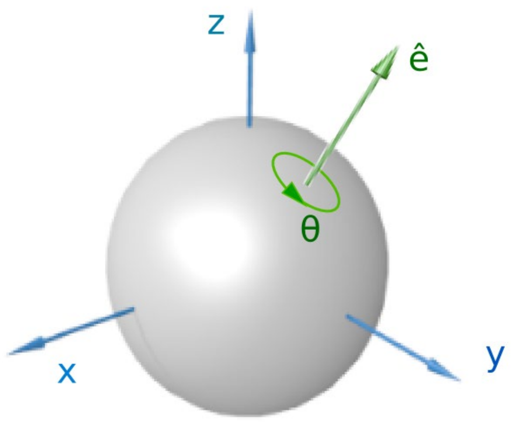
- Orientability vs Separability
- Real vs Complex Projective Spaces

3D Rotations via Quaternions

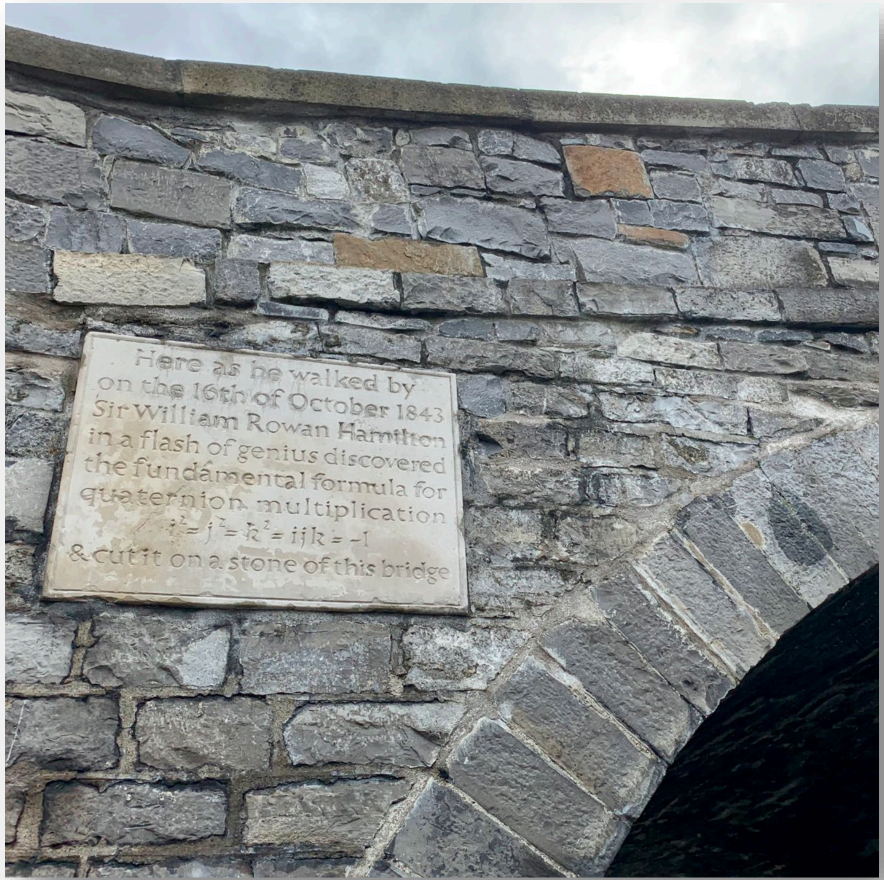
Multiple 3D Rotation Representations

Rotations of 3-space around an axis through the origin

- Axis and angle
- Matrices
- Euler angles
- Quaternions

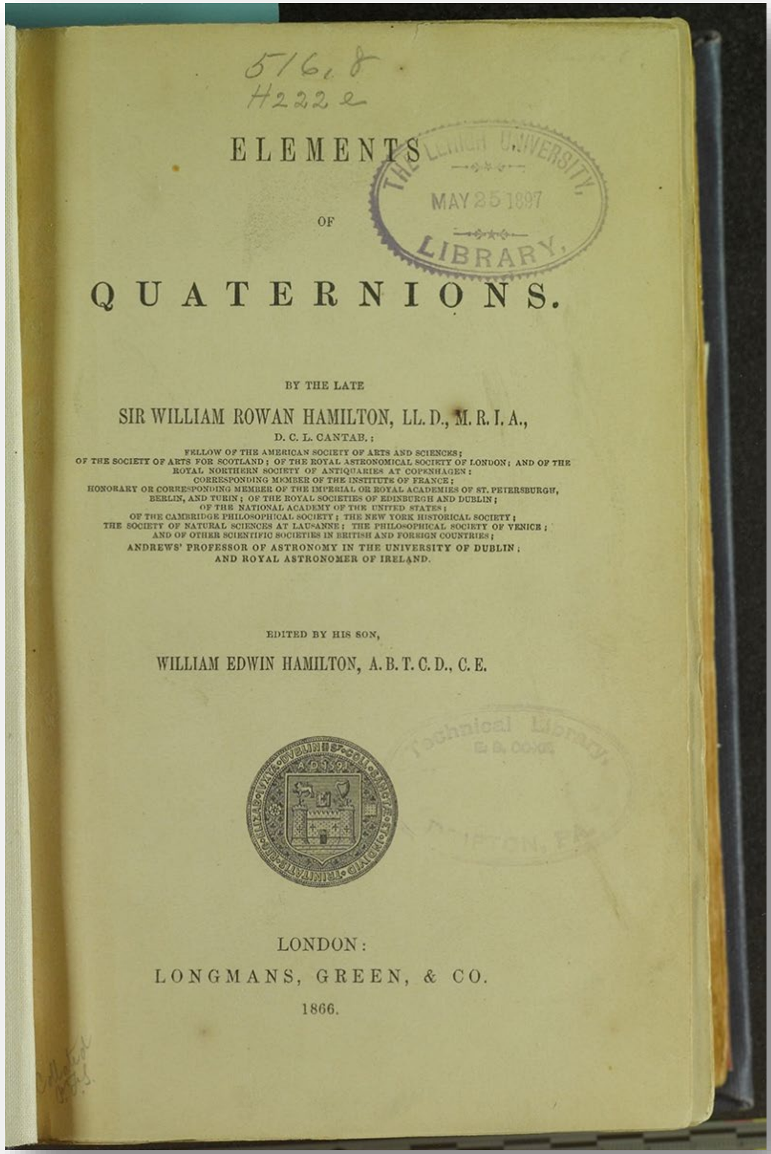


Quaternions



William Rowan Hamilton (1805-1865)

Quaternions



ELEMENTS

OF

QUATERNIONS.

BY THE LATE

SIR WILLIAM ROWAN HAMILTON, LL.D., M.R.I.A.,

D.C.L. CANTAB.;

FELLOW OF THE AMERICAN SOCIETY OF ARTS AND SCIENCES;
OF THE SOCIETY OF ARTS FOR SCOTLAND; OF THE ROYAL ASTRONOMICAL SOCIETY OF LONDON; AND OF THE ROYAL NORTHERN SOCIETY OF ANTIQUARIES AT COPENHAGEN;
CORRESPONDING MEMBER OF THE INSTITUTE OF FRANCE;
HONORARY OR CORRESPONDING MEMBER OF THE IMPERIAL OR ROYAL ACADEMIES OF ST. PETERSBURGH, BERLIN, AND TURIN; OF THE ROYAL SOCIETIES OF EDINBURGH AND DUBLIN;
OF THE NATIONAL ACADEMY OF THE UNITED STATES;
OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY; THE NEW YORK HISTORICAL SOCIETY;
THE SOCIETY OF NATURAL SCIENCES AT LAUSANNE; THE PHILOSOPHICAL SOCIETY OF VENICE;
AND OF OTHER SCIENTIFIC SOCIETIES IN BRITISH AND FOREIGN COUNTRIES;
ANDREWS' PROFESSOR OF ASTRONOMY IN THE UNIVERSITY OF DUBLIN;
AND ROYAL ASTRONOMER OF IRELAND.

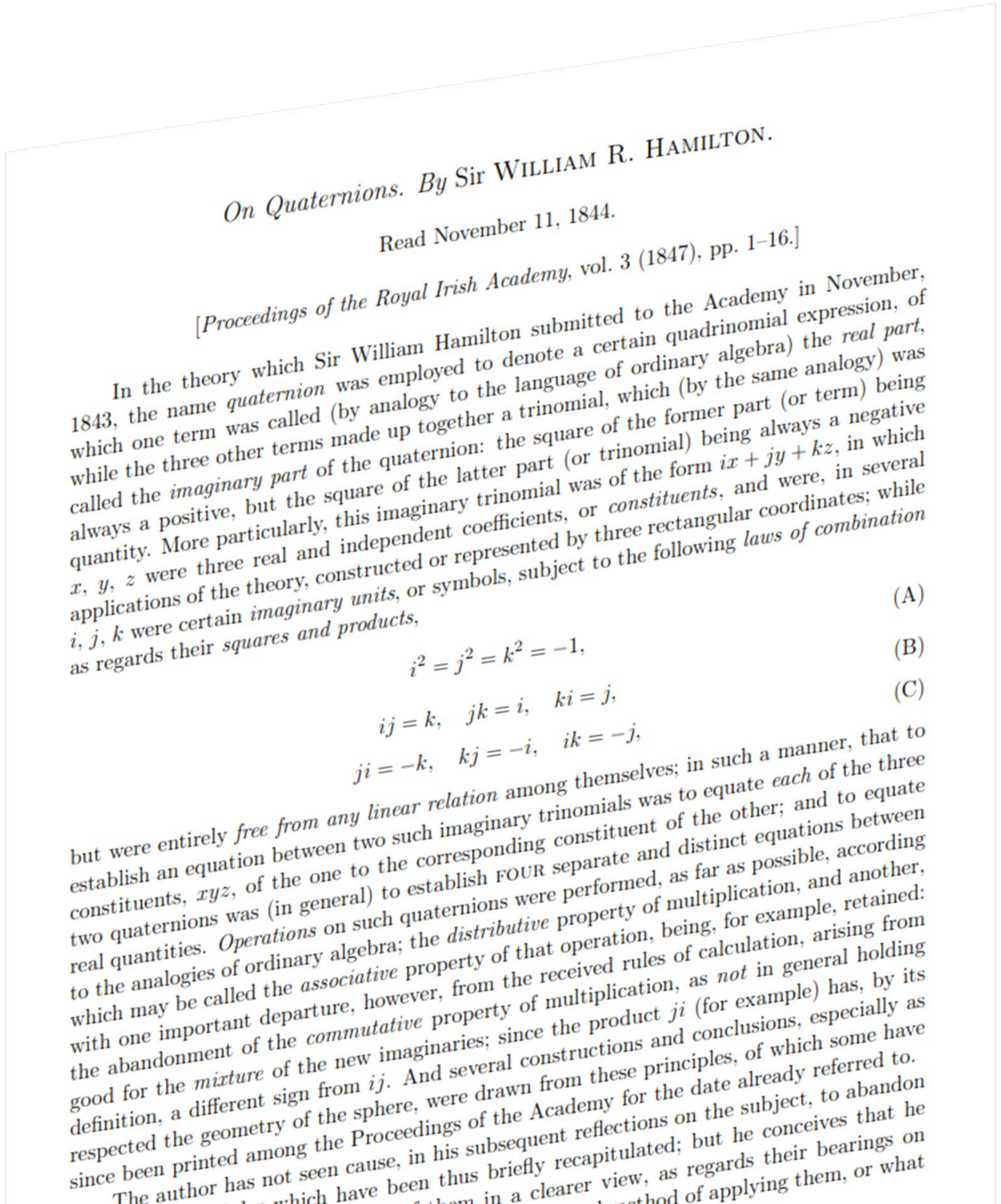
EDITED BY HIS SON,

WILLIAM EDWIN HAMILTON, A.B.T.C.D., C.E.

LONDON:

LONGMANS, GREEN, & CO.

1866.



On Quaternions. By Sir WILLIAM R. HAMILTON.

Read November 11, 1844.

[*Proceedings of the Royal Irish Academy*, vol. 3 (1847), pp. 1–16.]

In the theory which Sir William Hamilton submitted to the Academy in November, 1843, the name *quaternion* was employed to denote a certain quadrinomial expression, of which one term was called (by analogy to the language of ordinary algebra) the *real part*, while the three other terms made up together a trinomial, which (by the same analogy) was called the *imaginary part* of the quaternion: the square of the former part (or term) being always a positive, but the square of the latter part (or trinomial) being always a negative quantity. More particularly, this imaginary trinomial was of the form $ix + jy + kz$, in which x, y, z were three real and independent coefficients, or *constituents*, and were, in several applications of the theory, constructed or represented by three rectangular coordinates; while i, j, k were certain *imaginary units*, or symbols, subject to the following *laws of combination* as regards their *squares and products*,

$$i^2 = j^2 = k^2 = -1, \tag{A}$$

$$ij = k, \quad jk = i, \quad ki = j, \tag{B}$$

$$ji = -k, \quad kj = -i, \quad ik = -j, \tag{C}$$

but were entirely *free from any linear relation* among themselves; in such a manner, that to establish an equation between two such imaginary trinomials was to equate *each* of the three constituents, xyz, of the one to the corresponding constituent of the other; and to equate two quaternions was (in general) to establish FOUR separate and distinct equations between real quantities. *Operations* on such quaternions were performed, as far as possible, according to the analogies of ordinary algebra; the *distributive* property of multiplication, and another, which may be called the *associative* property of that operation, being, for example, retained: with one important departure, however, from the received rules of calculation, arising from the abandonment of the *commutative* property of multiplication, as *not* in general holding good for the *mixture* of the new imaginaries; since the product ji (for example) has, by its definition, a different sign from ij. And several constructions and conclusions, especially as respected the geometry of the sphere, were drawn from these principles, of which some have since been printed among the Proceedings of the Academy for the date already referred to.

The author has not seen cause, in his subsequent reflections on the subject, to abandon

Scaled Quaternion Conjugation

$$C_q(r) := qr\bar{q}$$

$$M(q) = M(a + bI + cJ + dK) =$$

$$\begin{pmatrix} a^2+b^2+c^2+d^2 & 0 & 0 & 0 \\ 0 & a^2+b^2-c^2-d^2 & 2bc-2ad & 2bd+2ac \\ 0 & 2bc+2ad & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 0 & 2bd-2ac & 2cd+2ab & a^2-b^2-c^2+d^2 \end{pmatrix}$$

Whiteboard

Projective Spaces

Real Projective

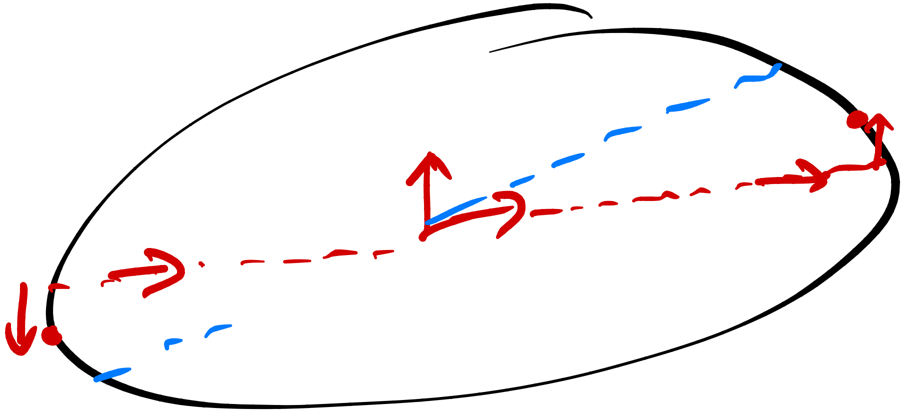
Complex Projective

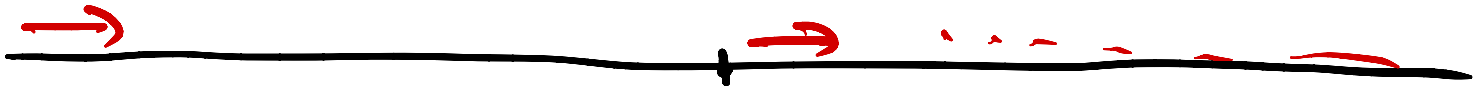
P^2 $\quad \mathbb{R}P^2$ $\quad \mathbb{C}P^2$

P^n is orientable iff n is odd

P^2 **not** orientable

$(-1)^{n-1}$





$$z \longrightarrow \frac{\alpha z + \beta}{\gamma z + \delta}$$

$$z \rightarrow \frac{1}{z}$$

$$z \rightarrow \frac{1}{\bar{z}}$$

Rotations in 3D

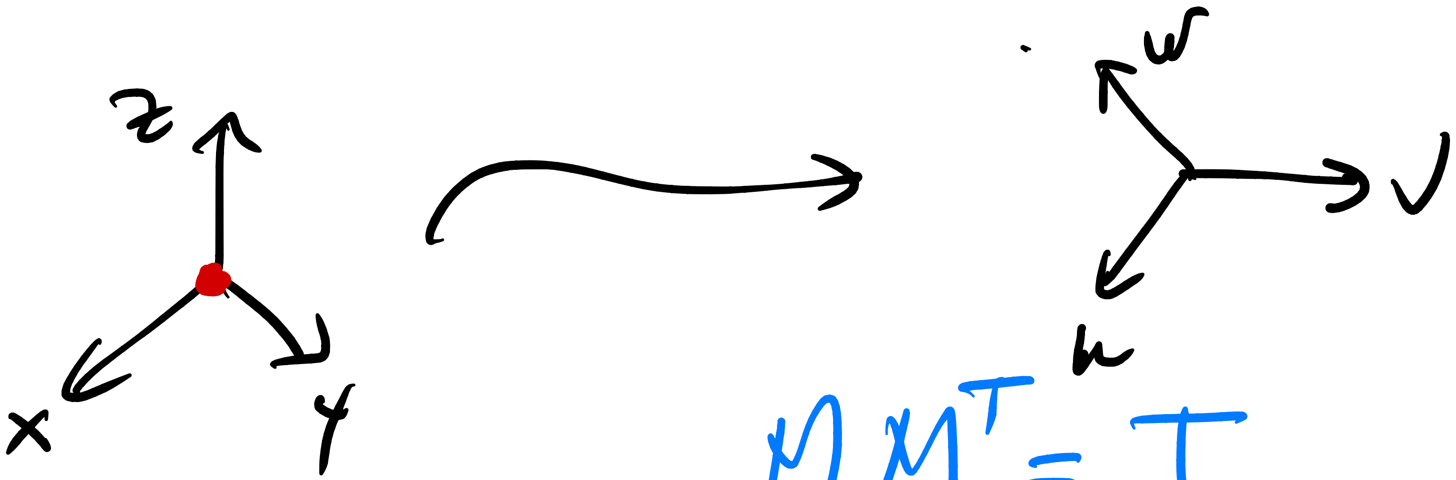
1. Axis & angle
2. Euler angles
3. Matrix representation
4. Quaternions

Orthogonal Matrices

M

$$M = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & u_x & u_y & u_z \\ 0 & v_x & v_y & v_z \\ 0 & w_x & w_y & w_z \end{array}\right) \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ u_x \\ u_y \\ u_z \end{pmatrix}$$

All rows have unit length
All distinct rows are orthogonal



$M^T = M^{-1}$

$MM^T = I$

$\det^2 M = 1 \Rightarrow \det M = \pm 1$

$\det M = 1$ rotations

$\det M = -1$ reflections

3D Rotation

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u_x & u_y & u_z \\ 0 & v_x & v_y & v_z \\ 0 & w_x & w_y & w_z \end{pmatrix}$$

9 #s

$u_x^2+u_y^2+u_z^2=1$

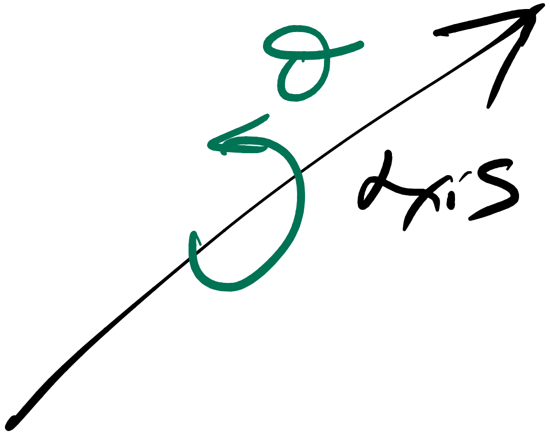
v

w

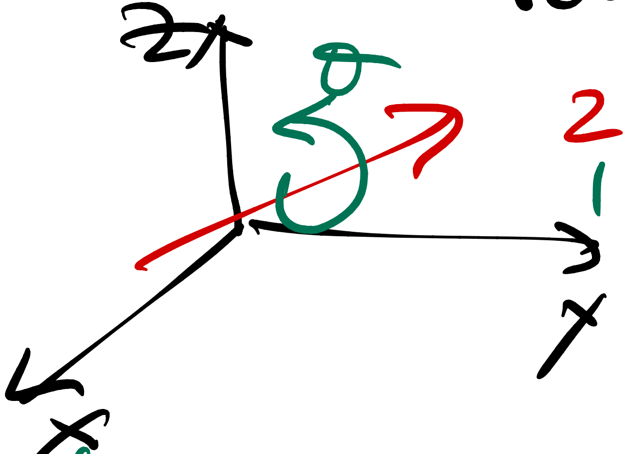
$u_xv_x+u_yv_y+u_zv_z=0$

6 conditions

$M^{-1}=M^T$



Rotation around the origin



2 d.

1 d

3D rotations around the origin have 3 degrees of freedom

Quaternions

Complex #s
$\mathbb{R}^2 \hookrightarrow$ algebra

$\hookrightarrow \mathbb{R}^4 \hookrightarrow$ algebra

$I, J, K \qquad I^2 = J^2 = K^2 = IJK = -1$

$IJ = K, \quad JK = I, \quad KI = J$

$JI = -K, \quad KJ = -I, \quad IK = -J$

$$\boxed{a + bI + cJ + dK}$$

non-commutative division algebra, skew field

$$|a+bI+cJ+dK| = \sqrt{a^2+b^2+c^2+d^2}$$

$$q = a+\underbrace{bI+cJ+dK}_{\text{Imaginary}}$$

$$|q| = 0$$

$$\hookrightarrow a=b=c=d=0$$

$$z = a+bi$$

$$\bar{z} = a-bi$$

$$\bar{q} = a-bI-cJ-dK$$

$$q\bar{q} = \bar{q}q = a^2+b^2+c^2+d^2 = |q|^2 = |\bar{q}|^2$$

$$\frac{q\bar{q}}{|q|^2} = 1 \Rightarrow q^{-1} = \frac{\bar{q}}{|q|^2}$$

$$\overline{pq} = \bar{q}\,\bar{p}$$

---

p is a quaternion

$C_p : \quad r \longmapsto p\,r\,\bar{p}$ "scaled conjugation"

$\qquad r \longrightarrow p\,r\,p^{-1}$ "conjugation"

$\bar{p} = |p|^2 p^{-1}$

$$C_p(C_q(r)) = p\,(q\,r\,\bar{q})\,\bar{p} = (pq)\,r\,(\bar{q}\,\bar{p}) =$$

$$(pq)\,r\,\overline{(pq)} = C_{pq}(r)$$

$\underline{\underline{C.}}$

$$r \to p r \bar{p}$$

$$s \to p s \bar{p}$$

$$(r+s) \to p(r+s)\bar{p}$$

$$q = a + bI + cJ + dK \qquad C_q$$

$$M_q = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

$$q r \bar{q}$$

$M(q) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$

M is orthogonal & $|q|=1$ $a^2+b^2+c^2+d^2=1$

- inner prod. of any row or col with itself $= |q|^4 = (a^2+b^2+c^2+d^2)^2$
- inner prod. of any two distinct rows or cols $= 0$
- $\det(M(q)) = |q|^8 = (a^2+b^2+c^2+d^2)^4$

$$M(pq) = M(p)M(q)$$

unit norm quaternions are an encoding for 3D rotations around the origin

$$q = a + bI + cJ + dK$$

$$p = x + yI + zJ + wK \qquad p \to qp\bar{q}$$

$$\cancel{p} = qp\bar{q}$$

$$\begin{pmatrix} \equiv \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix}$$

$$q = a + bI + cJ + dK$$

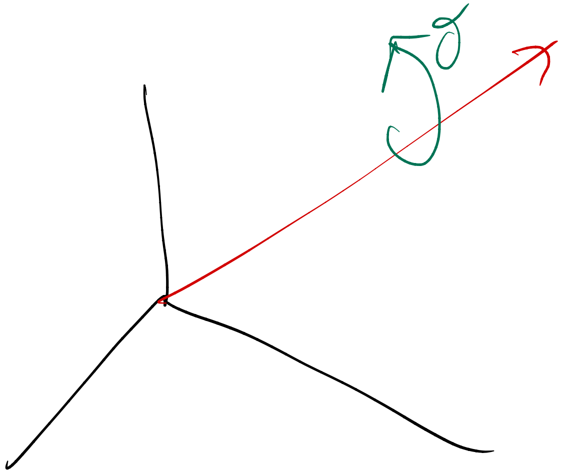
$$-q = -a - bI - cJ - dK$$

axis is parallel to $[b, c, d]$

$$[1, 0, 0, 0] \rightarrow [a, b, c, d]$$

$$a = \cos\frac{\theta}{2} \Longleftrightarrow \sqrt{b^2 + c^2 + d^2} = \sin\theta/2$$

$$a^2 + b^2 + c^2 + d^2 = 1$$



$q = 1$ $\quad 1 + 0I + 0J + 0K$

$\cos\theta/2 = 1$ $\quad \theta/2 = k \cdot 360^\circ$

$$\theta = k \cdot 720^\circ$$

$$p = \frac{1+I}{\sqrt{2}}$$

$$M_p = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

90° around the x-axis

$$q = \frac{1+J}{\sqrt{2}}$$

$$M_q = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

90° around the y-axis

$q\,p$

$$\cancel{\left(\frac{1+J}{\sqrt{2}}\right)\left(\frac{1+I}{\sqrt{2}}\right)} = \frac{1+I+J+K}{2}$$

$$\left(\frac{1+I}{\sqrt{2}}\right)\left(\frac{1+J}{\sqrt{2}}\right)$$

$$\longleftrightarrow M$$

$$\left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

axis $(b,c,d) = (1,1,1)$

$$\cos\frac{\theta}{2} = \frac{1}{2} \qquad \frac{\theta}{2} = 60 \rightarrow \theta = 120^\circ$$

$$\left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & r_{00} & r_{01} & r_{02} \\ 0 & r_{10} & r_{11} & r_{12} \\ 0 & r_{20} & r_{21} & r_{22} \end{array}\right)$$

$$1 + r_{00} + r_{11} + r_{22} = 4a^2$$

$$1 + r_{00} - r_{11} - r_{22} = 4b^2$$

$$1 - r_{00} + r_{11} - r_{22} = 4c^2$$

$$1 - r_{00} - r_{11} + r_{22} = 4d^2$$

$$r_{21} - r_{12} = 4ab$$

$$r_{02} - r_{20} = 4ac$$

$$r_{10} - r_{01} = 4ad$$

$$r_{10} + r_{10} = 4bc$$

$$r_{21} + r_{12} = 4cd$$

$$r_{02} + r_{20} = 4bd$$

Matrix

Axis Angle

Quaternion

$1 \quad I \quad J$

$I^2 = J^2 = -1$

$IJ = a + bI + cJ$

$IIJ = aI - b + cIJ$

$-J = -b + aI + \cancel{cIJ}\, c(a + bI + cJ)$

$0 = (ac - b) + (a + bc)I + \underline{\underline{(c^2 + 1)}}J$

$c^2 + 1 = 0$

That’s All