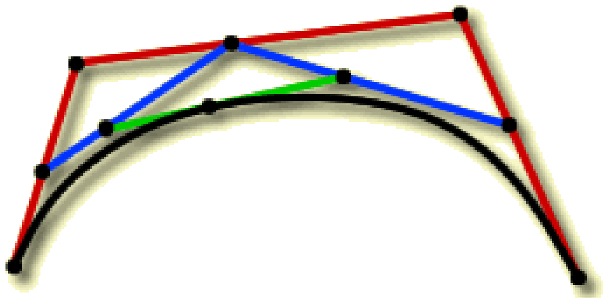
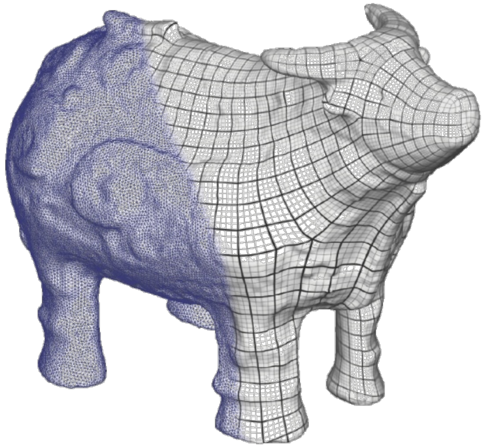
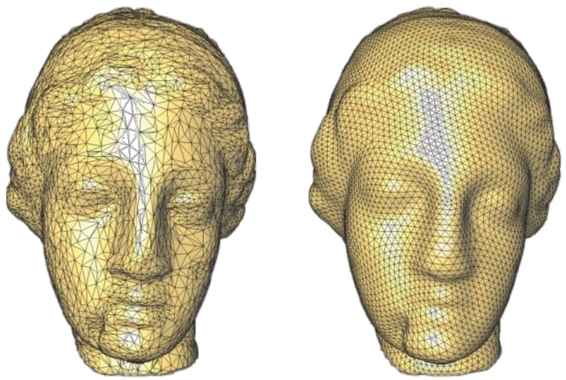
CS348a: Geometric Modeling and Processing







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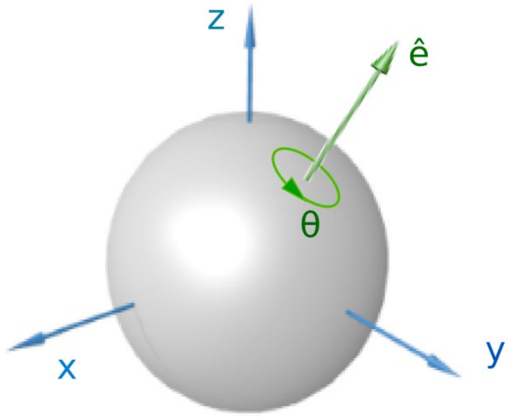
Geometric Computing

Last Time:
3D Rotations via Quaternions

Multiple 3D Rotation Representations

Rotations of 3-space around an axis through the origin

- Axis and angle
- Matrices
- Euler angles
- **Quaternions**



Quaternion Advantages

- Unit norm quaternions are a double cover of the group of rotations through the origin in 3D

$$q \mapsto C_q \mapsto M(q) \qquad\qquad M(-q) = M(q)$$

- Compact representation 1 x 4: $q = a + bI + cJ + dK$
  - efficient multiplication

- Homomorphism: $M(pq) = M(p)M(q)$

- Relatively easy conversion to other representations

- Continuity – important for gradient descent when regressing rotations

Scaled Quaternion Conjugation

$$C_q(r) := qr\bar{q}$$

$$M(q) = M(a + bI + cJ + dK) =$$

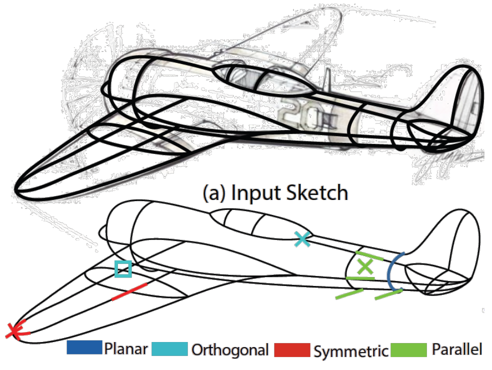
$$\begin{pmatrix} a^2+b^2+c^2+d^2 & 0 & 0 & 0 \\ 0 & a^2+b^2-c^2-d^2 & 2bc-2ad & 2bd+2ac \\ 0 & 2bc+2ad & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 0 & 2bd-2ac & 2cd+2ab & a^2-b^2-c^2+d^2 \end{pmatrix}$$

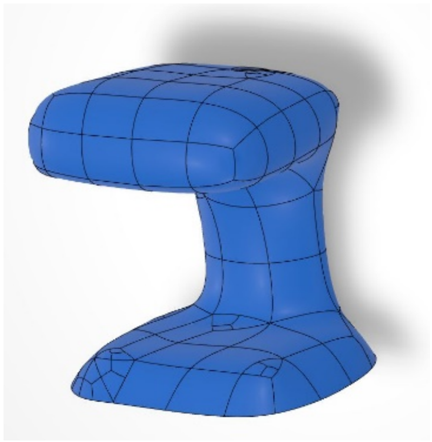
A rotation matrix if $||q|| = \sqrt{a^2+b^2+c^2+d^2} = 1$

Today:
Designed Shapes, Polynomial Curves

Shapes Representations for Human Design

- Boundary-based or volume-based?
- One piece or many? – Splines
- What class of mathematical functions?
- Of what degree?
- Parametric or implicit?





Whiteboard

Class of Mathematical Functions

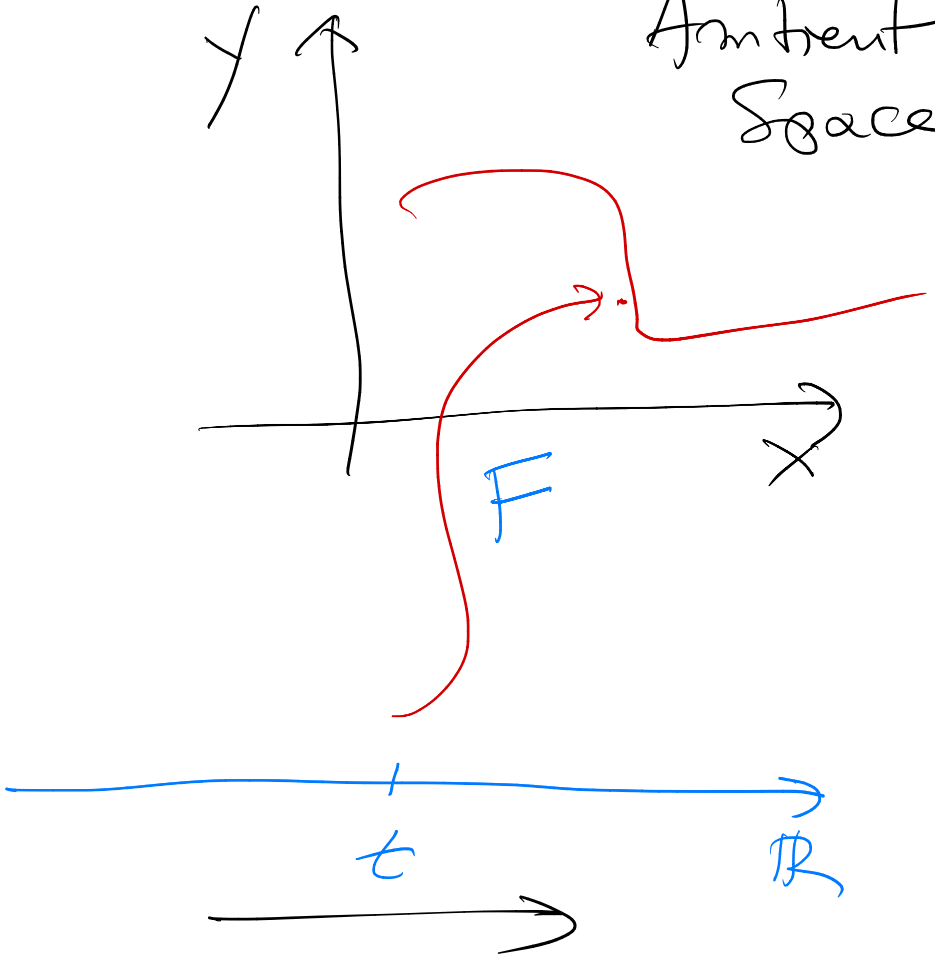
- Polynomial
- Rational

} of degree ≤ 3

Parametric

Implicit

Ambient Space

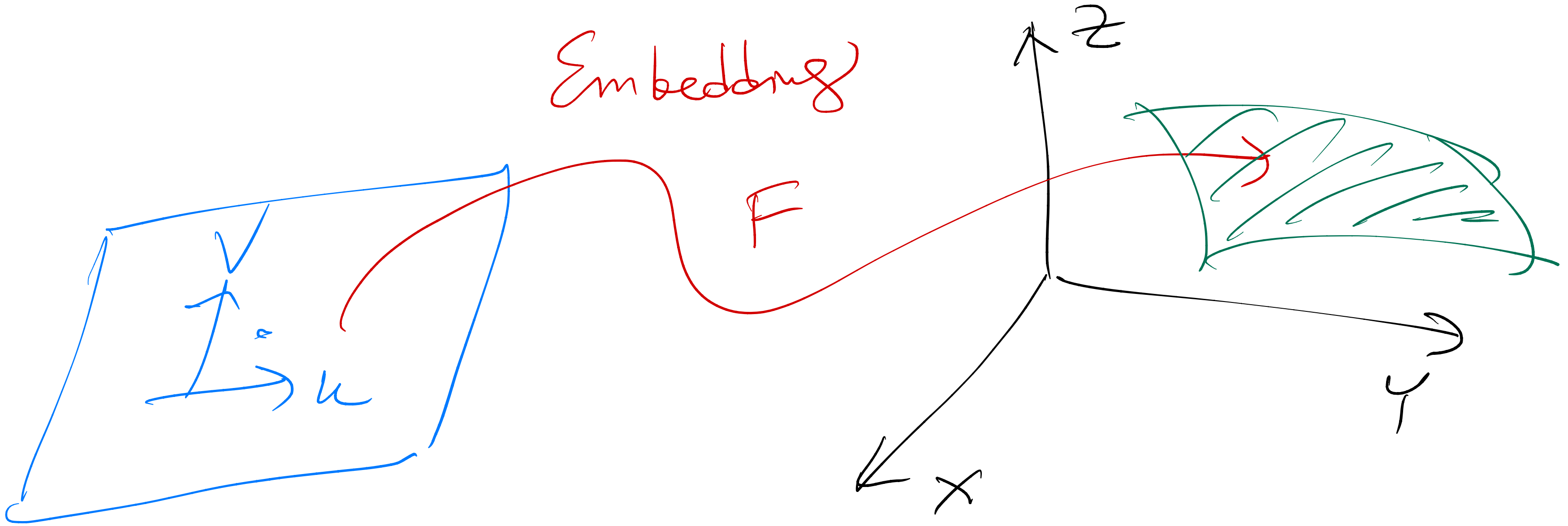


$F(t) = (x(t), y(t))$

$(x(t), y(t), z(t))$

two polynomials

Parameter Space



$$F(u,v) = (X(u,v), Y(u,v), Z(u,v))$$

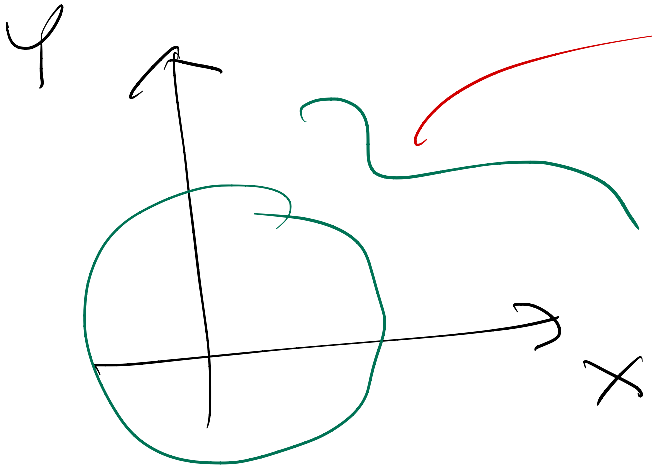
$$F : R^2 \longrightarrow R^3 \leftarrow \text{ambient space}$$

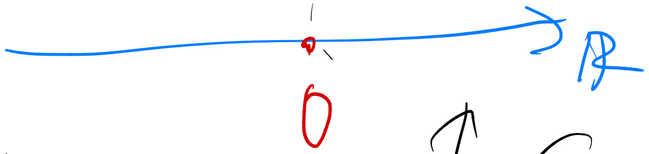
$\uparrow$ parameter space

Implicit Forms

F $\quad$ F $\quad$ x^2+y^2-1

Neural Implicits



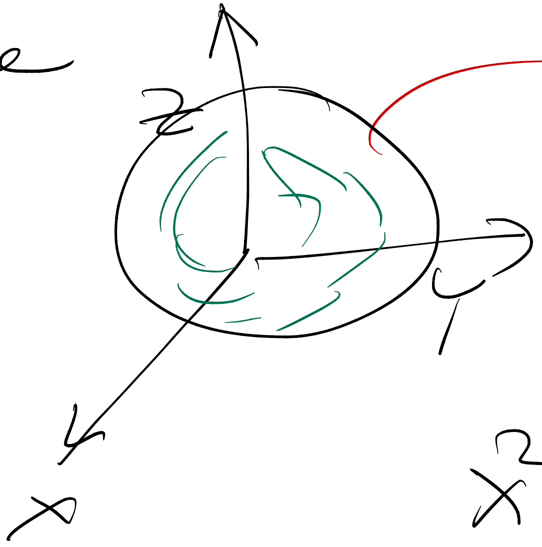


Ambient Space

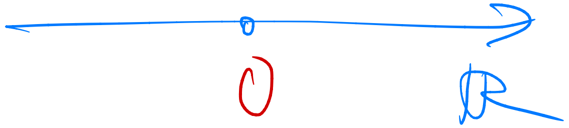
$\uparrow$ Gauge space

$F^{-1}(0)$

$$\boxed{x^2+y^2-1=0}$$



Gauge Space

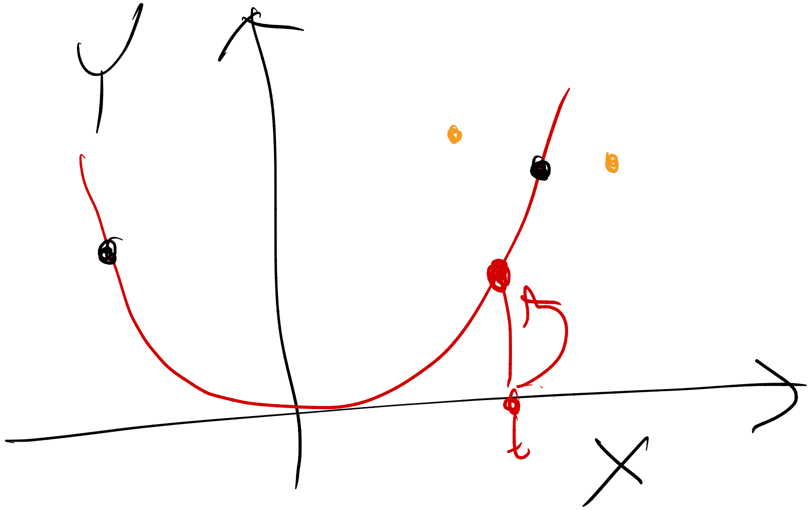


$F: N_\theta$

$$x^2+y^2+z^2-1=0$$

Ambient

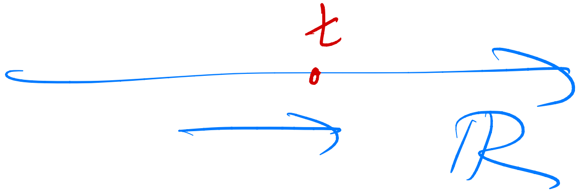
$F^{-1}(0)$



$$\left.\begin{aligned} Y &= X^2 \\ Y - X^2 &= 0 \end{aligned}\right\} \text{implicit}$$

$$\boxed{F(t) = (t, t^2)} \quad \text{Parametric}$$

$X(t)$ $Y(t)$



Parametric Curves

Polynomial

$d=1$
$d=2$
$d=3$

$d=1$ $\quad \left.\begin{aligned} x(t) &= x_1 t + x_0 \\ y(t) &= y_1 t + y_0 \end{aligned}\right\}$ straight line

$d=2$ $\quad \left.\begin{aligned} x(t) &= x_2 t^2 + x_1 t + x_0 \\ y(t) &= y_2 t^2 + y_1 t + y_0 \end{aligned}\right\}$ parabolas

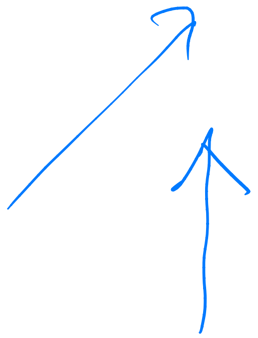
$\alpha x + \beta y + \gamma = 0$

$t \to \infty$ $\quad \frac{y_2}{x_2}$

$t \to -\infty$

$$x(t) = x_2 t^2 + x_1 t + x_0$$
$$y(t) = y_2 t^2 + y_1 t + y_0$$

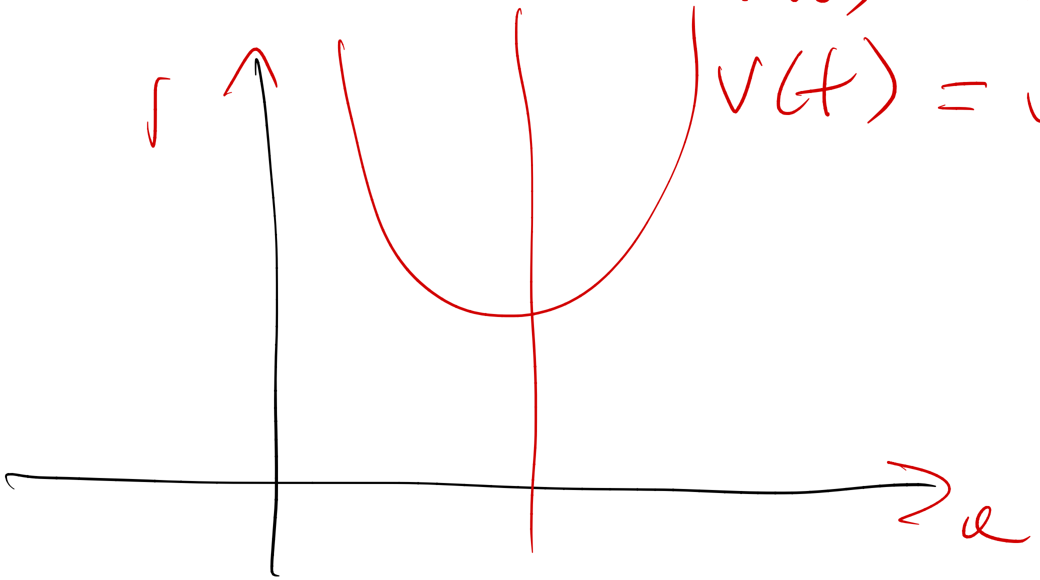
$t \to \infty \qquad \frac{y_2}{x_2}$



$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} y_2 & -x_2 \\ x_2 & y_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$u_2 = 0$

$$u(t) = u_1 t + u_0$$
$$v(t) = v_2 t^2 + v_1 t + v_0$$



$d=3$

$F(t) = (x(t), y(t))$

$$\left.\begin{aligned} x(t) &= x_3 t^3 + x_2 t^2 + x_1 t + x_0 \\ y(t) &= y_3 t^3 + y_2 t^2 + y_1 t + y_0 \end{aligned}\right\}$$

$t \to \infty \qquad \dfrac{y_3}{x_3}$

$K_{12} = x_1 y_2 - x_2 y_1$

$K_{23} = x_2 y_3 - x_3 y_1$

$K_{31} = x_3 y_1 - x_1 y_3$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} y_3 & -x_3 \\ x_3 & y_3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$F(t) = (u(t), v(t))$$

$$u(t) = \underline{\underline{u_2}}\, t^2 + \underline{\underline{u_1}}\, t + u_0$$

$$v(t) = v_3 t^3 + v_2 t^2 + v_1 t + v_0$$

$u_2 = k_{23}$

$u_1 = -k_{31}$

α

$k_{23} = u_2 = 0, \quad k_{31} = -u_1 \neq 0$

$$u(t) = u_1 t + u_0$$

$$v(t) = v_3 t^3 + v_2 t^2 + v_1 t + v_0$$

"S-shaped"

β

$k_{23} = k_{31} = 0 \qquad k_{12} \neq 0$

$$0 = \begin{vmatrix} \overset{Y_1}{\cancel{X_1}} & \overset{Y_2}{\cancel{X_2}} & \overset{Y_3}{\cancel{X_3}} \\ X_1 & X_2 & X_3 \\ Y_1 & Y_2 & X_3 \end{vmatrix}$$

$$\cancel{X_1 k_{23}} + \cancel{X_2 k_{31}} + X_3 k_{12} = 0$$

$$\cancel{Y_1 k_{23}} + \cancel{Y_2 k_{31}} + Y_3 k_{12} = 0$$

$$\Rightarrow X_3 = 0$$

$$Y_3 = 0$$

Parabola

8

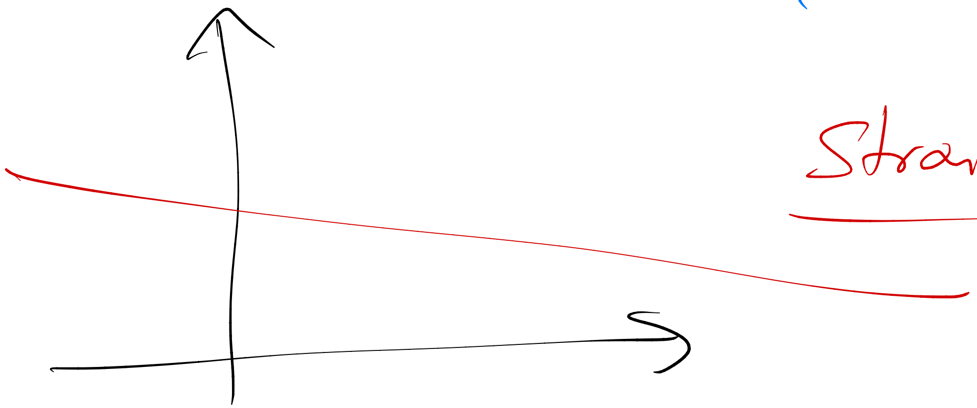
$$k_{12} = k_{23} = k_{31} = 0$$

$$\frac{y_3}{x_3} = \frac{y_2}{x_2} = \frac{y_1}{x_1} = \lambda$$

$$x(t) = x_3 t^3 + x_2 t^2 + x_1 t + x_0$$

$$y(t) = \lambda(x_3 t^3 + x_2 t^2 + x_1 t) + y_0$$

$$y(t) = \lambda(x(t) - x_0) + y_0$$

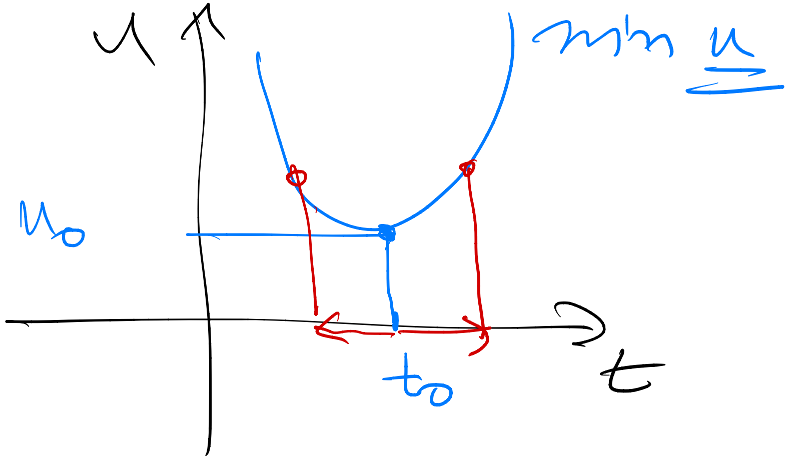


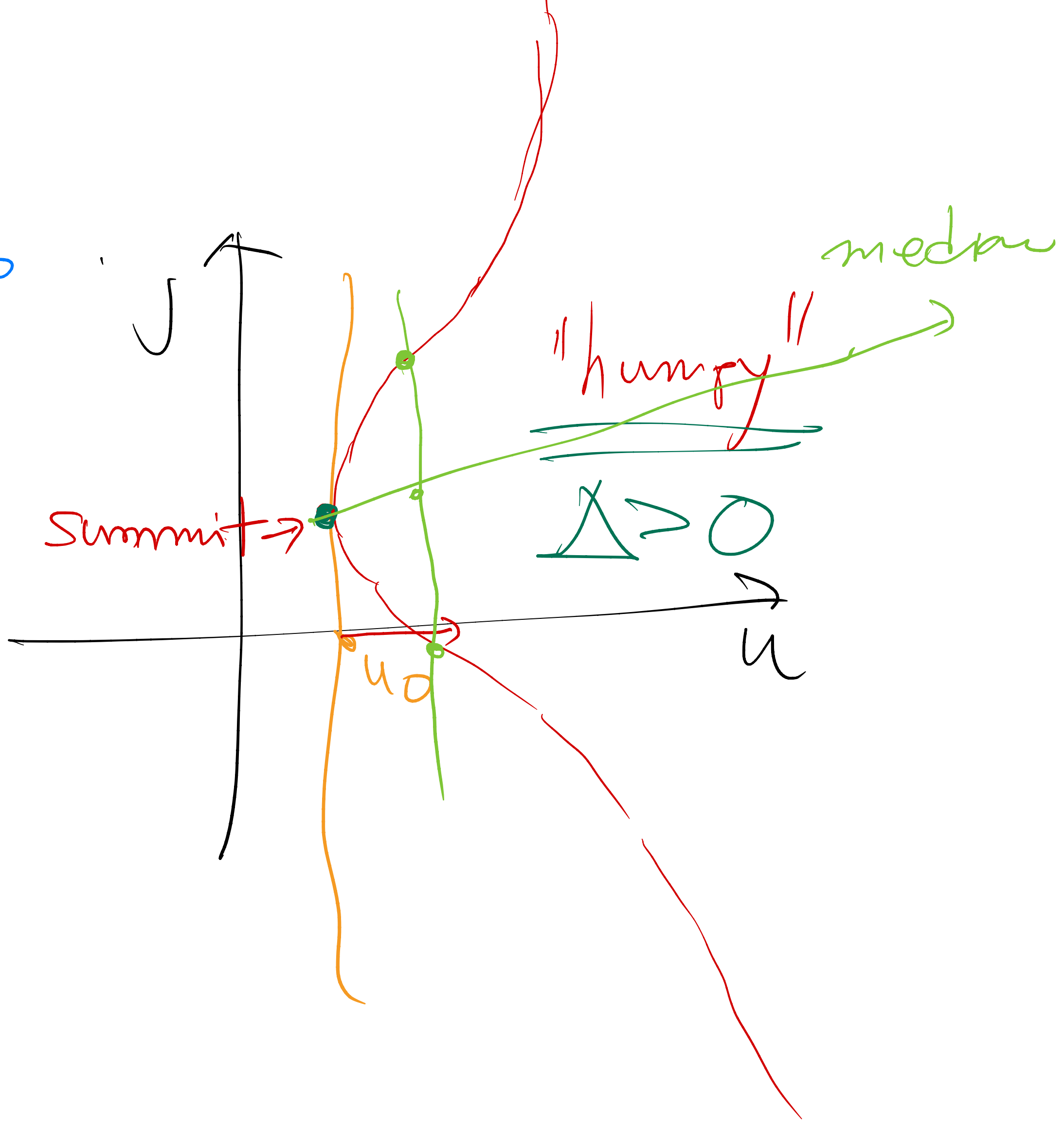
Straight line

F

$k_{23} = u_2 \neq 0$ $\quad$ $\underline{u_2 > 0}$

$$u(t) = u_2 t^2 + u_1 t + u_0$$





$$\Delta = 3k_{31}^2 - 4k_{12}k_{23}$$

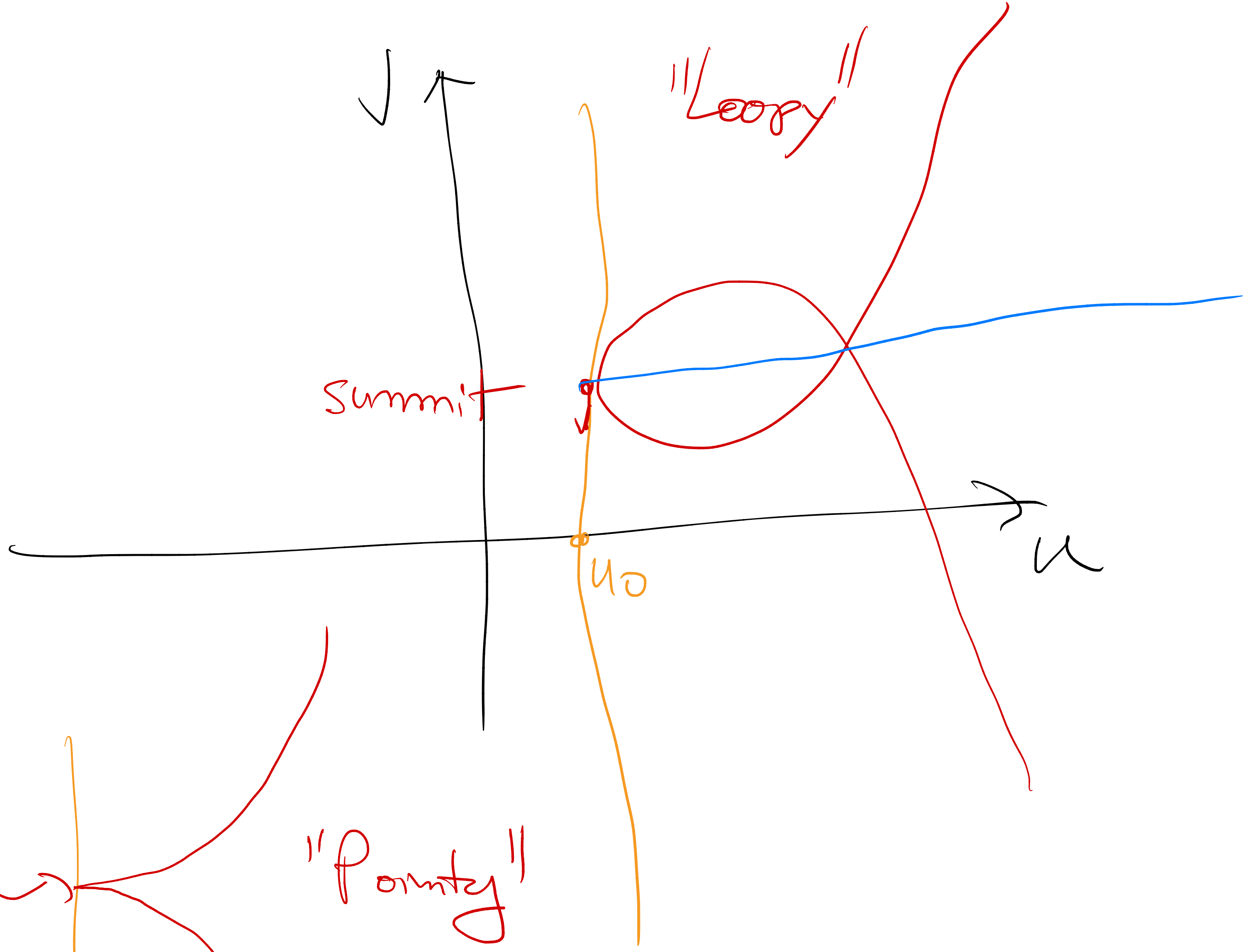
discriminant

$\Delta > 0$

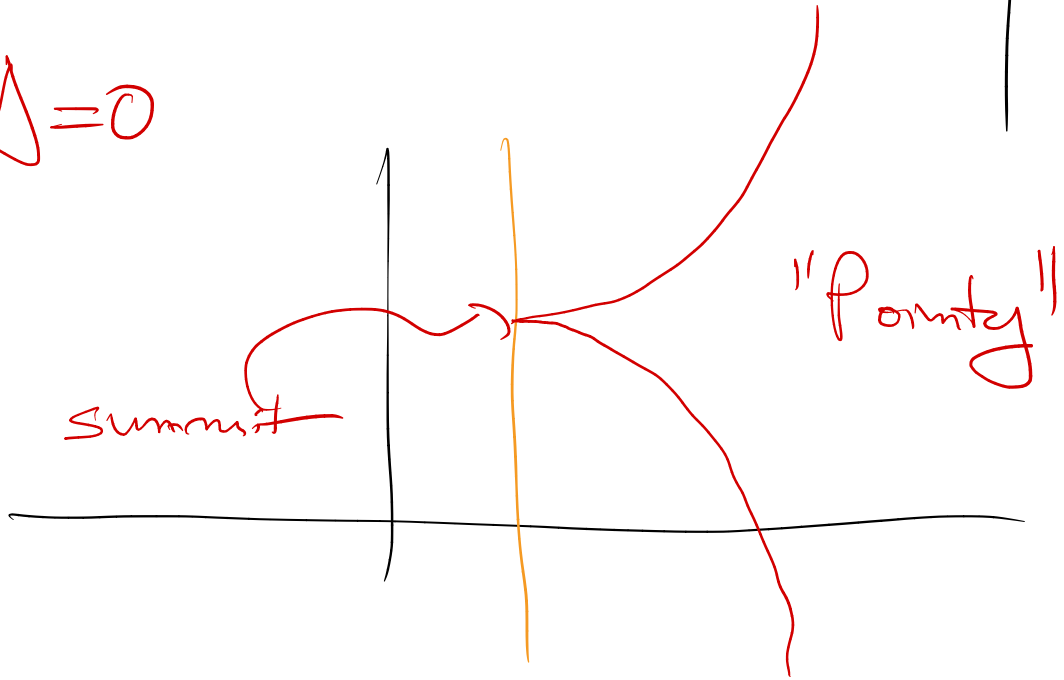
$\Delta = 0$

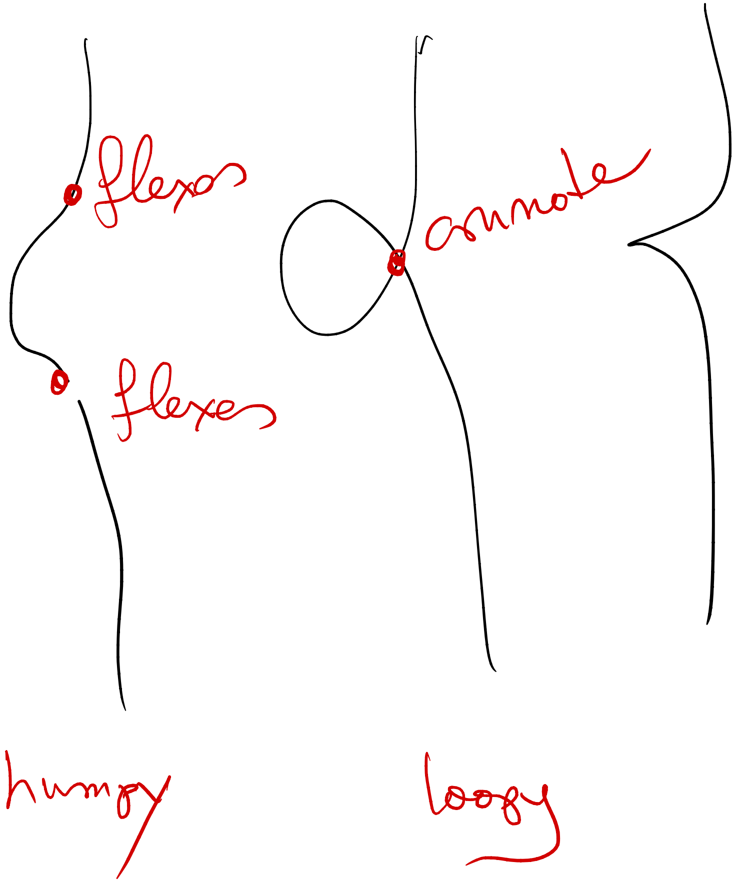
$\Delta < 0$

(3) $\Delta < 0$



(S) $\Delta = 0$

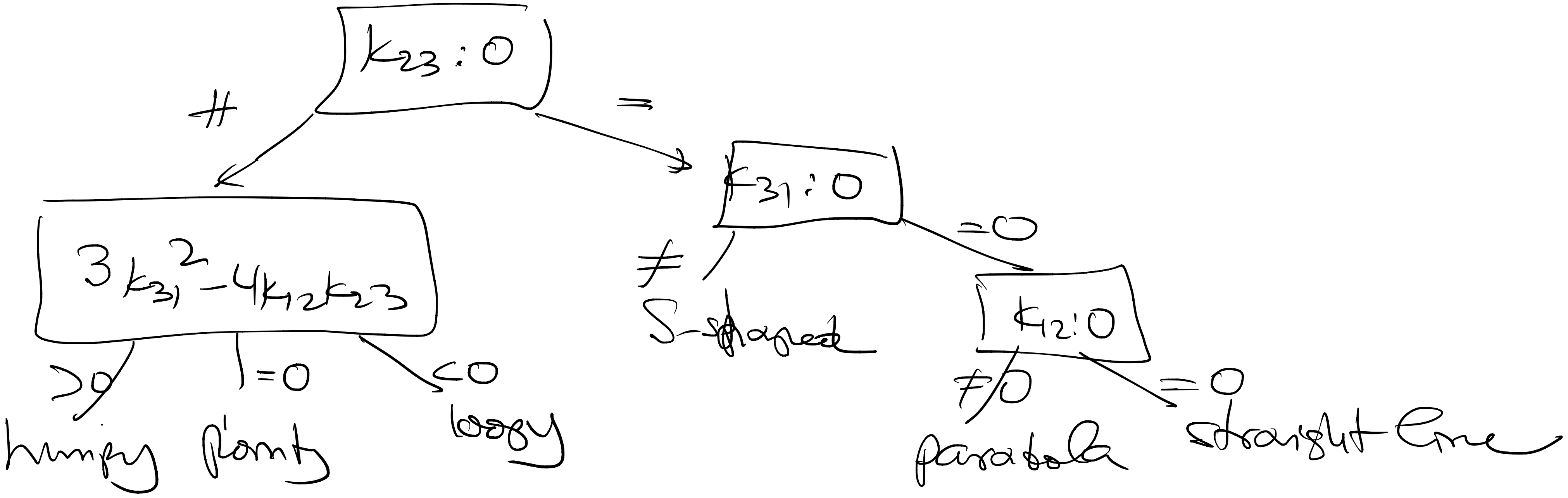


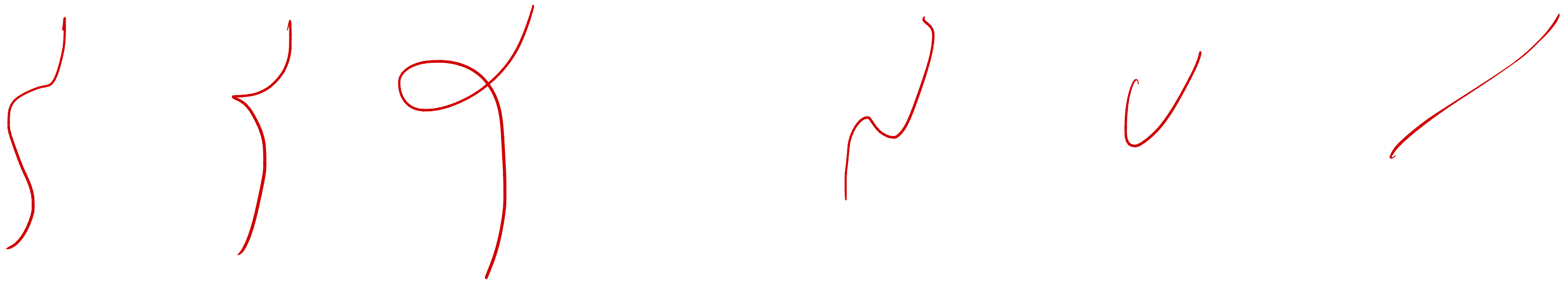


$$u(t) = u_2 t^2 + u_0$$

$$v(t) = v_3 t^3 + v_2 t^2 + v_1 t + v_0$$

Classification Tree



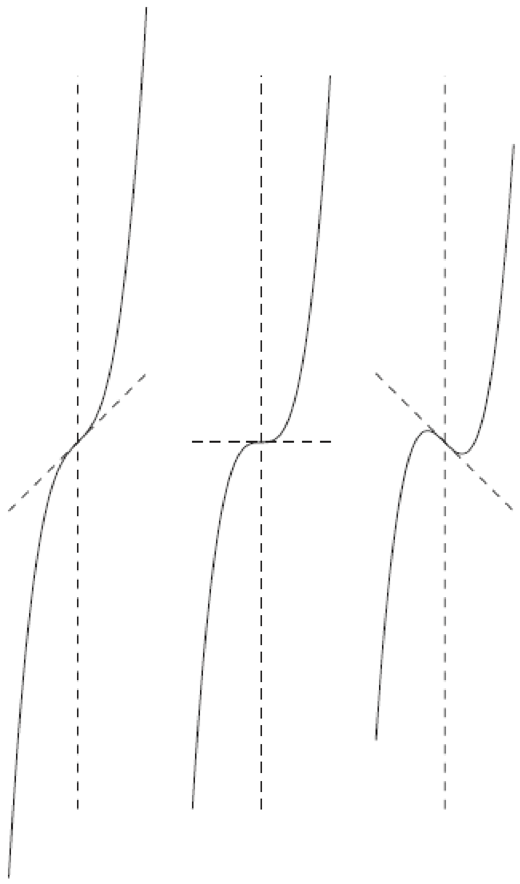


Classification of Cubics

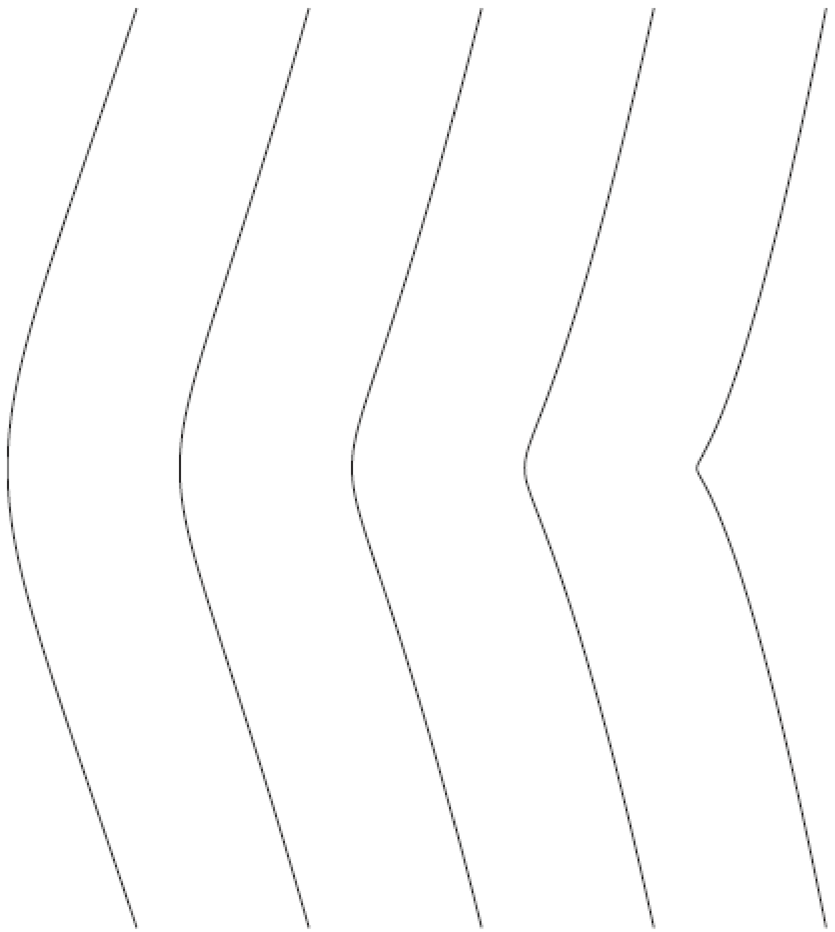
- Standard humpy: $H(r) := < r^2, r^3 + r >$
- Standard loopy: $L(r) := < r^2, r^3 - r >$
- Standard pointy: $P(r) := < r^2, r^3 >$
- Standard S-shaped: $S(r) := < r, r^3 >$
- Standard parabola: $Q(r) := < r, r^2 >$
- Standard line: $A(r) := < r, r >$

Every planar cubic is affinely equivalent to one of the above

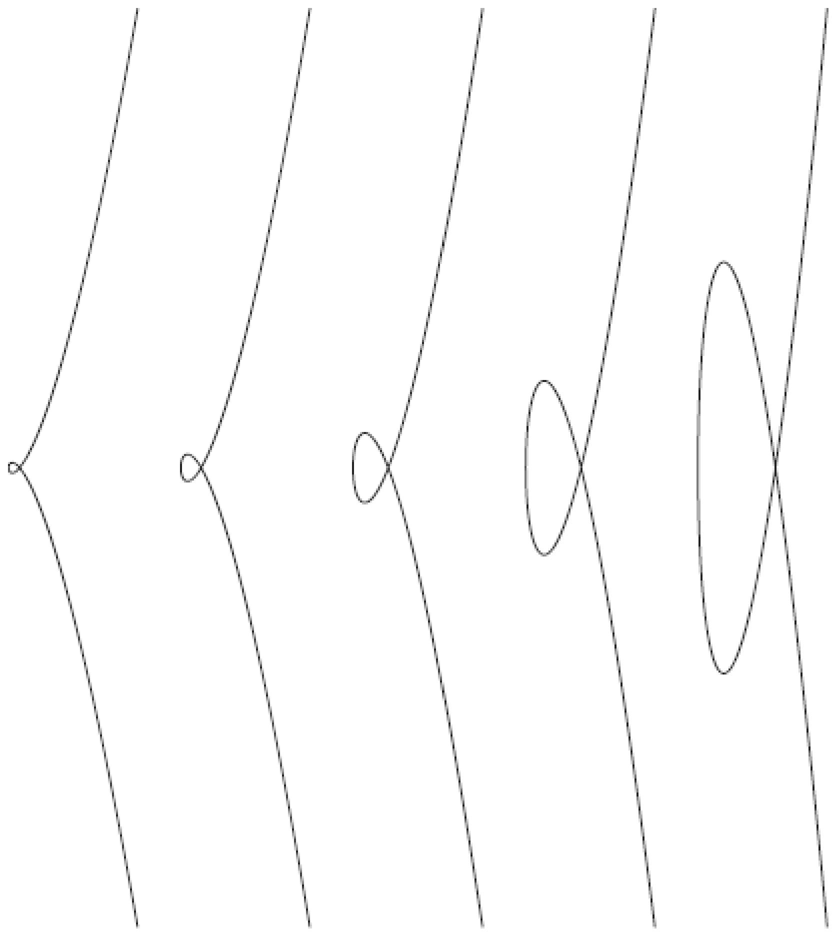
S-shaped Cubics



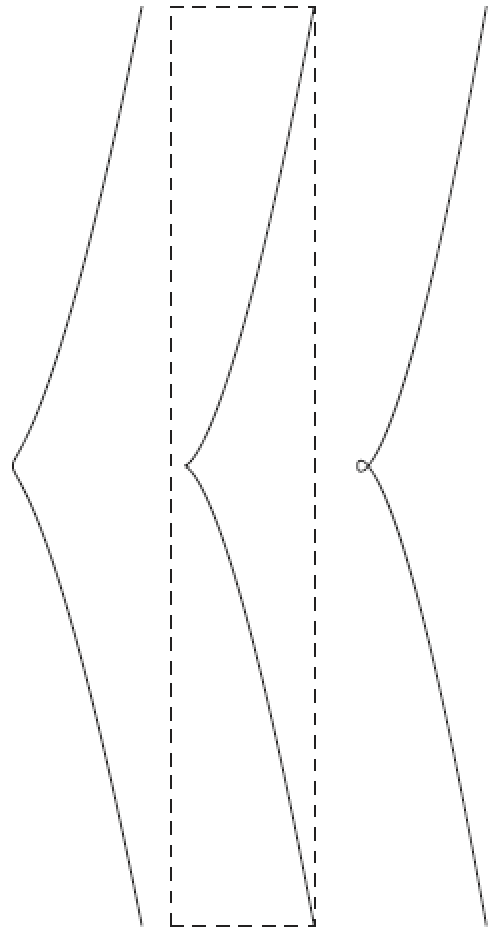
Humpy Cubics



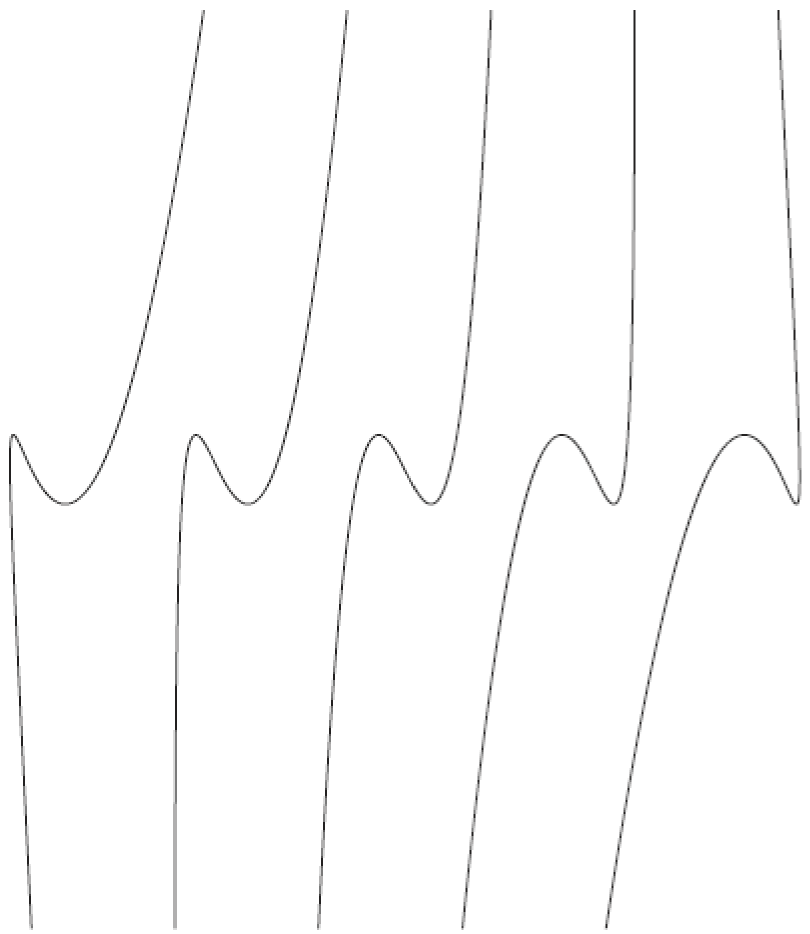
Loopy Cubics



Humpy to Loopy though Pointy



Humpy to Humpy through S-shaped



Interconversions Between Parametric and Implicit

Whiteboard

Standard humpy (r^2, r^3+r)

$$x(t) = r^2$$

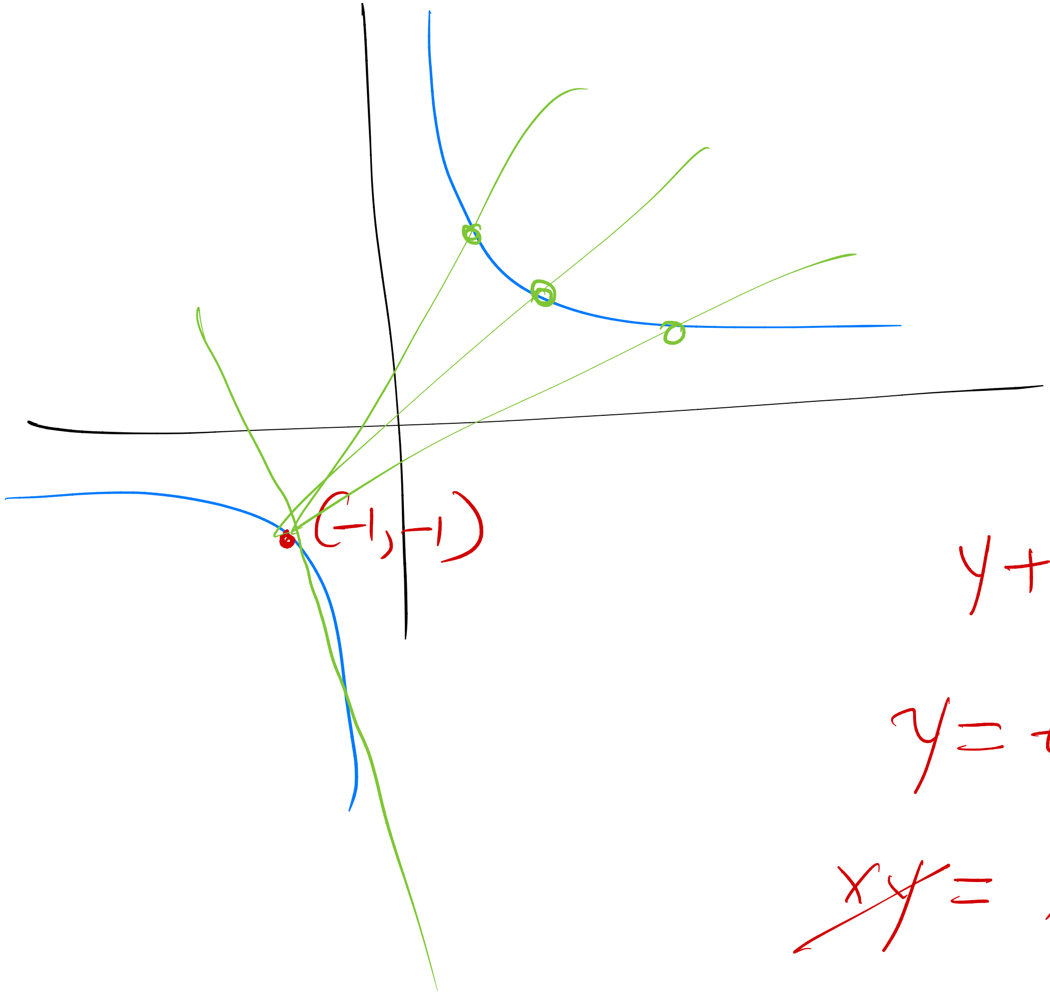
$$y(r) = r^3 + r = r(r^2+1)$$

$$= r(x+1)$$

$$y = r(x+1)$$

$$y^2 = r^2(x+1)^2$$

$$\boxed{y^2 = x(x+1)^2}$$ Implicit



$$xy - 1 = 0$$

$$x(t) = \frac{1}{t}$$

$$y(t) = t$$

$$y + 1 = t(x+1)$$

$$y = t(x+1) - 1$$

$$\cancel{xy} = xt(x+1) - x$$

$$= 1$$

$$\hookrightarrow (x+1)(tx-1) = 0$$

$$x = 1/t$$

$$y = t$$

That’s All