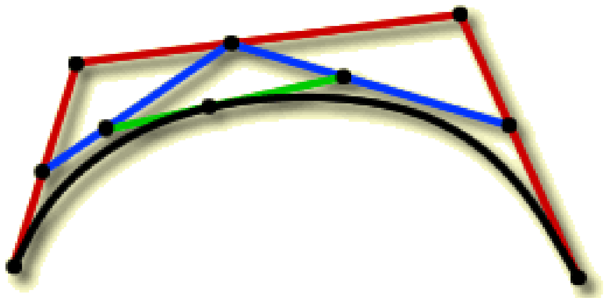
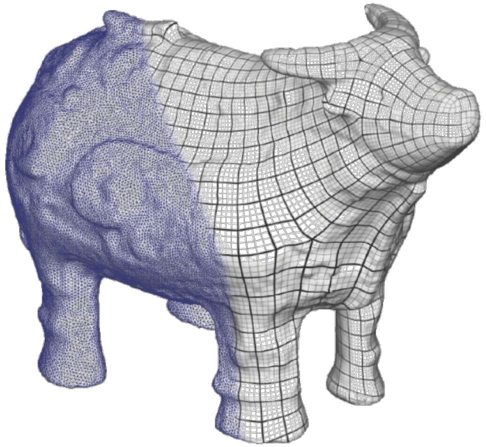
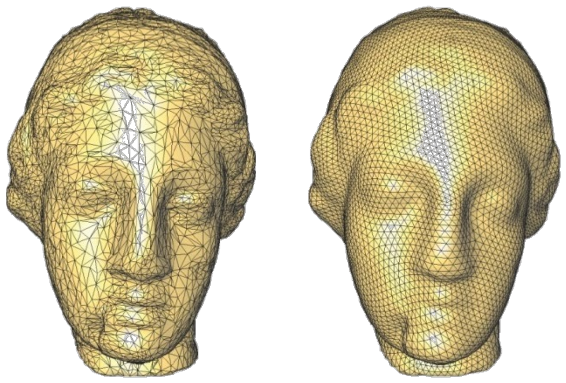
CS348a: Geometric Modeling and Processing







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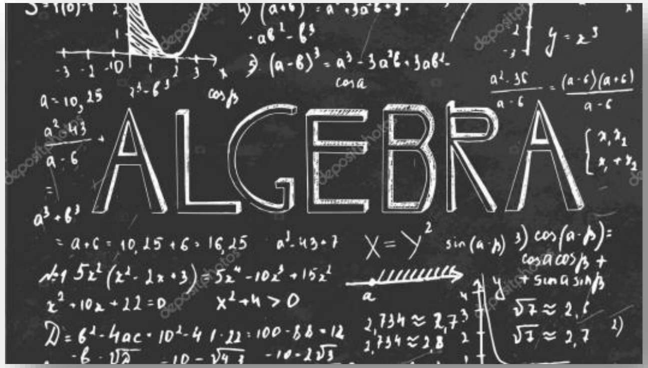
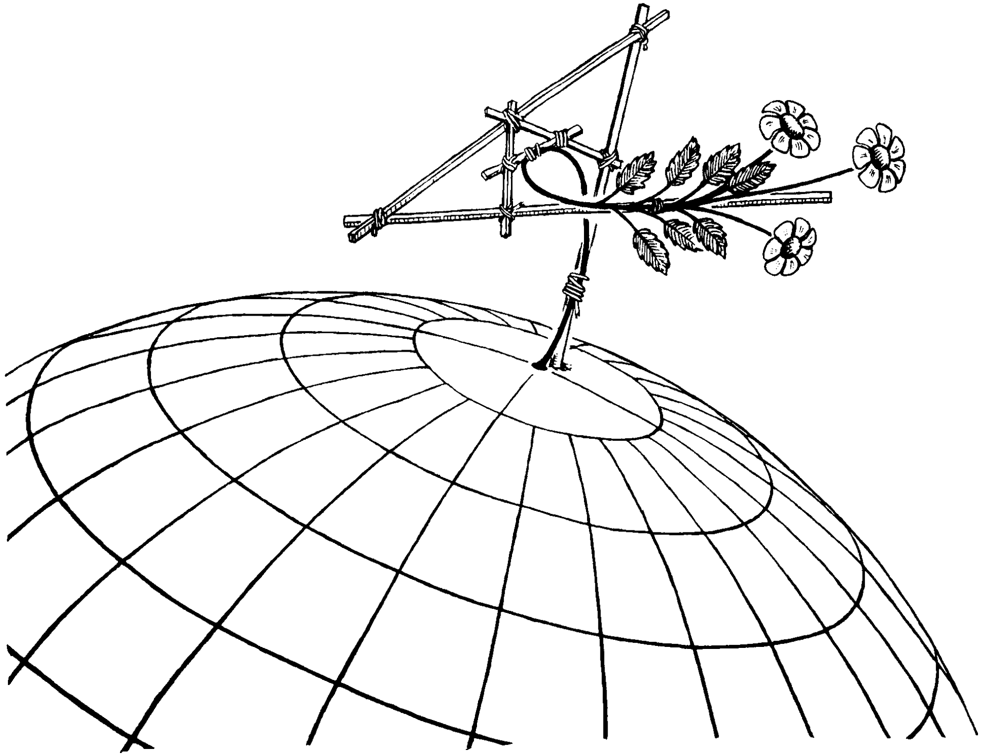
Geometric Computing

Last Time: Polar Forms, Bézier Control Points, de Casteljau Subdivision

Polar Forms and Blossoms

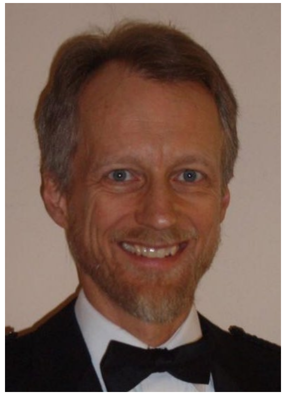
- Homogenization
- Polarization





Pierre
Bézier

Paul
de Casteljau

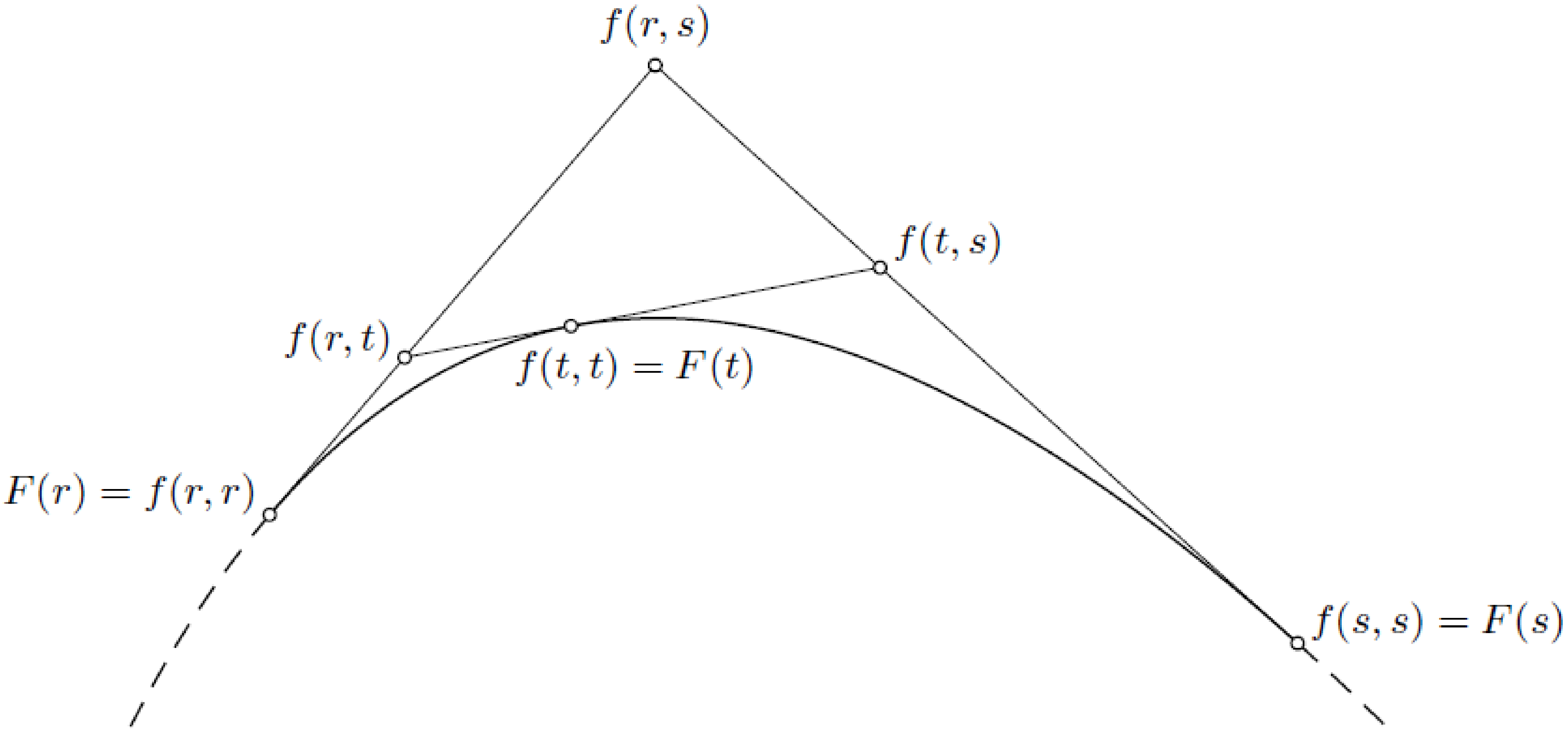


Lyle
Ramshaw

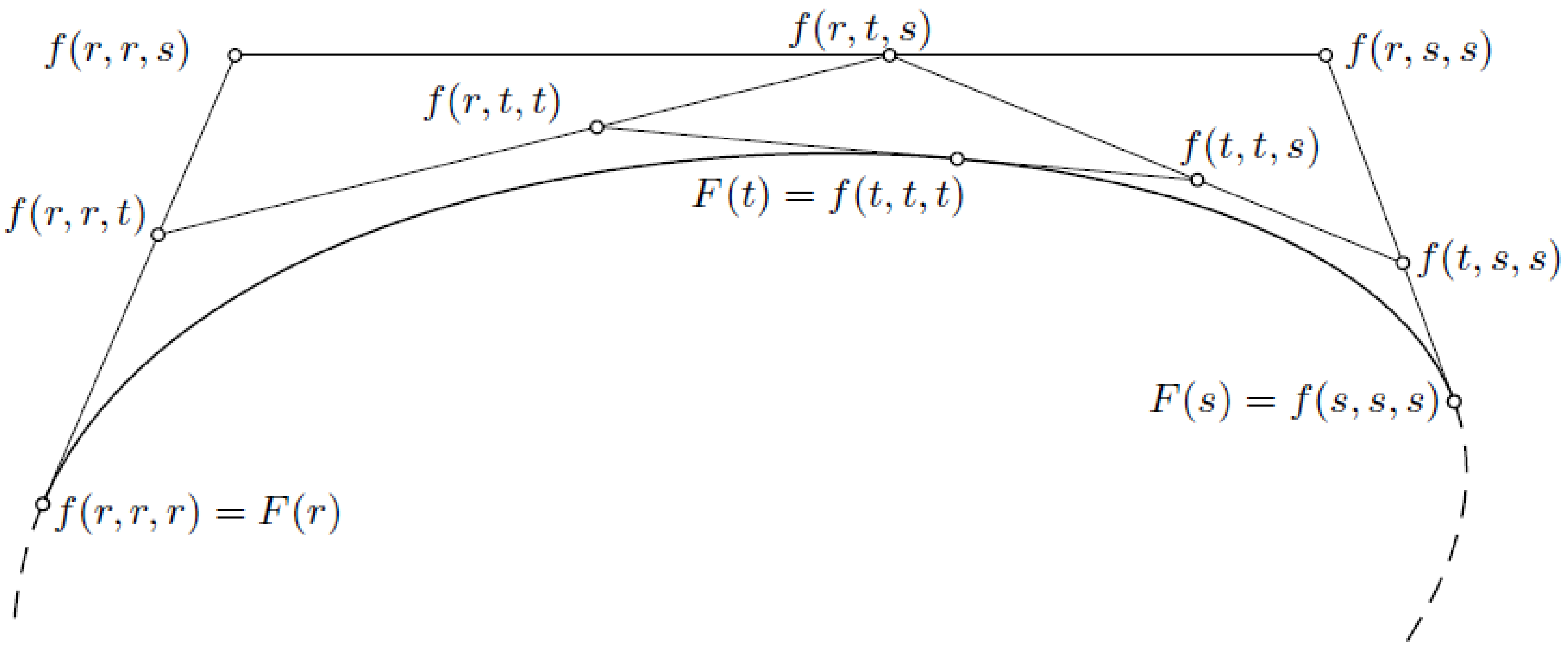
The Algebra of Polar Forms and Blossoms

- $F(t)$ [degreed d poly] vs $f(t_1, t_2, \dots t_d)$ – f is the polar form of blossom of F
    - $F(t) = f(t, t, \dots, t)$ – diagonal property
    - $f(t_1, t_2, \dots t_d) = f(t_{\sigma(1)}, t_{\sigma(2)}, \dots t_{\sigma(d)})$ – σ a permutation; symmetry property
    - $f(., \dots, t_i, \dots, .) = \alpha_i \, t_i + \beta_i$, for all i – multi-affine property

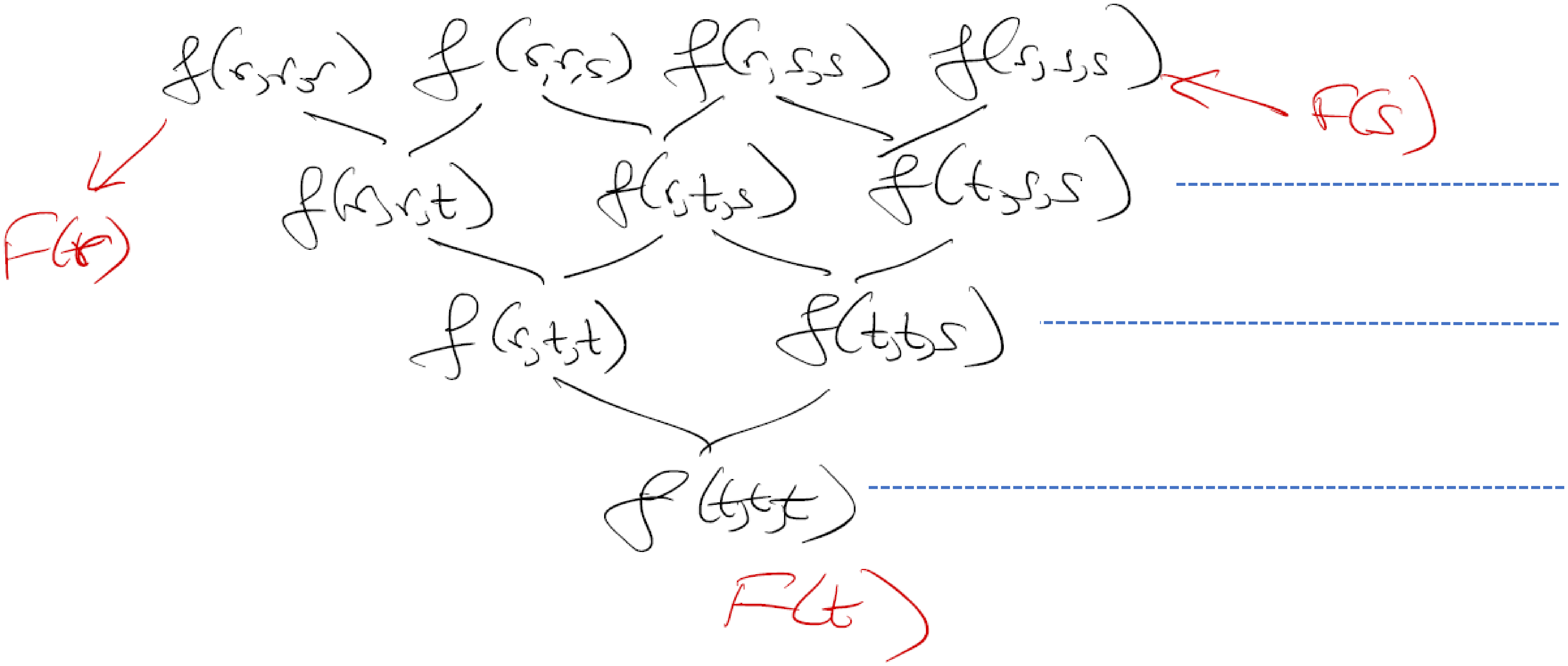
Bézier and de Casteljau, $d = 2$



Bézier and de Casteljau, $d = 3$



de Casteljau Subdivision



one *t*

two *t*’s

three *t*’s

The Bernstein Basis

$$F(t) = \left(\frac{s-t}{s-r}\right)^3 f(r,r,r) + 3\left(\frac{s-t}{s-r}\right)^2\left(\frac{t-r}{s-r}\right) f(r,r,s) + 3\left(\frac{s-t}{s-r}\right)\left(\frac{t-r}{s-r}\right)^2 f(r,s,s) + \left(\frac{t-r}{s-r}\right)^3 f(s,s,s)$$

$r=0$
$s=1$

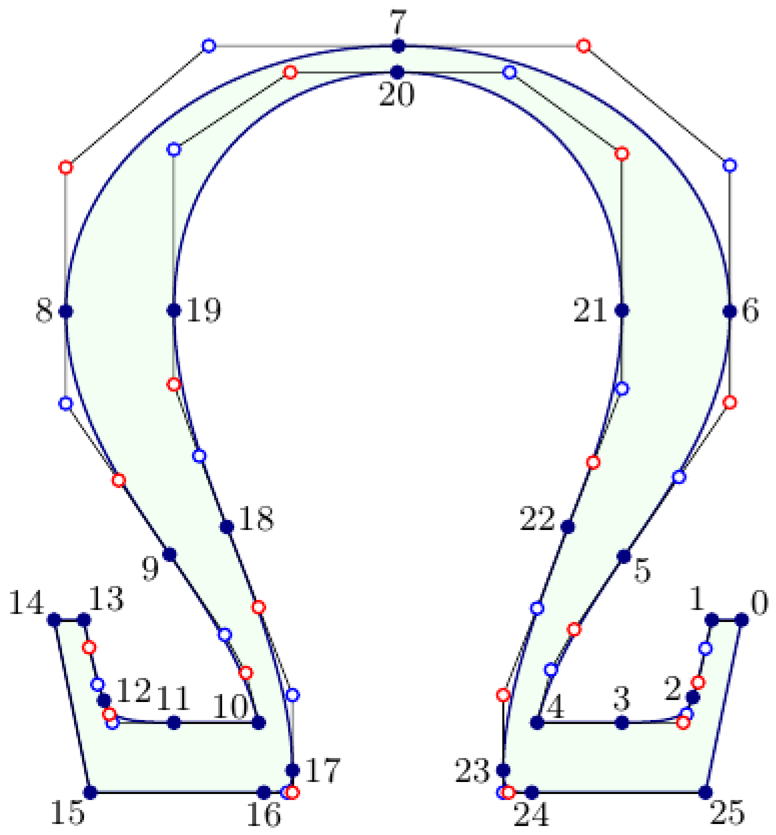
$$(1-t)^3 P + 3(1-t)^2 t\, Q + 3(1-t)t^2 R + t^3 S$$

Bernstein poly

$t \in [r,s]$

Today:
Continuity Between Arcs, Derivatives and Polar Forms

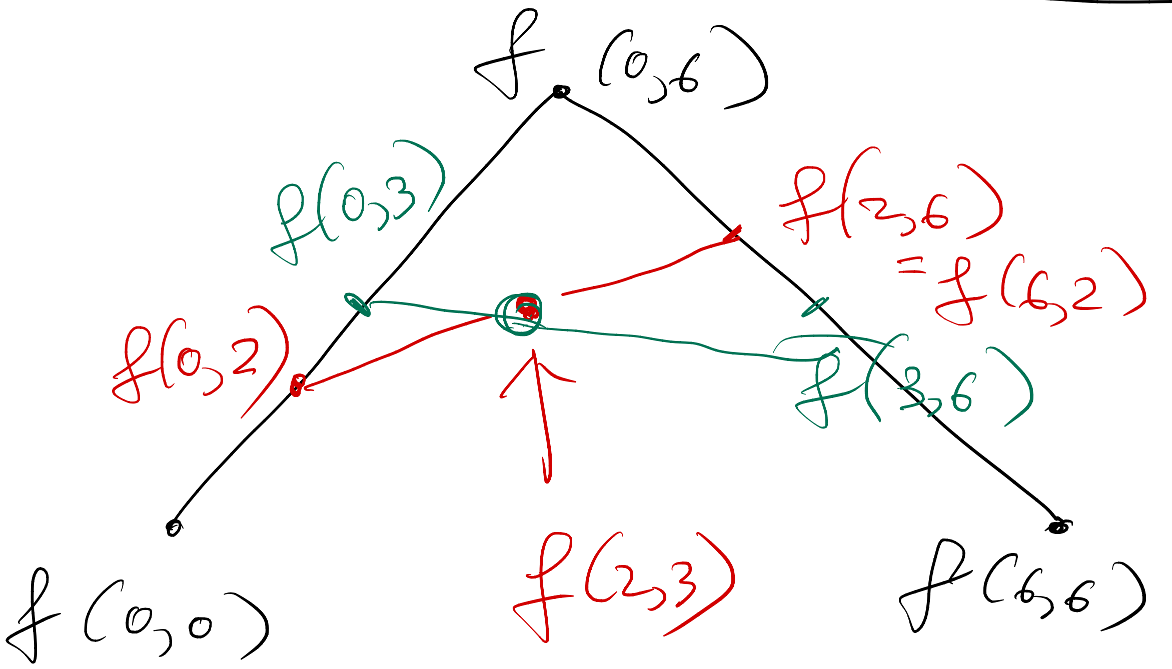
Modeling 2D Shapes with Spline Curves



The fonts we use …

Whiteboard

$f(t,t,t)$ $\quad$ $f(t_1,t_2,t_3)$



$f(2,3)$

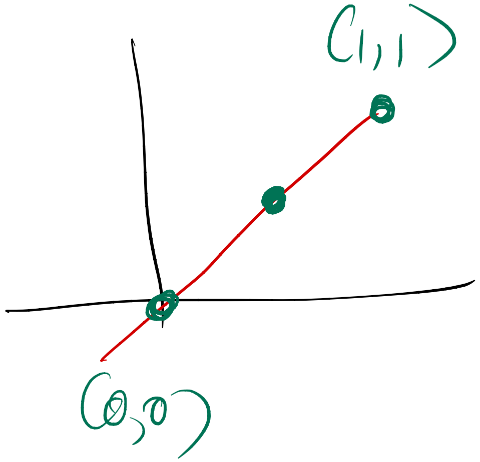
$f(3,2)$

$f(3,2) = f(2,3)$

Polar forms are degree dependent

$t^2 \to t_1 t_2$ as $d=2$ $\quad < t_1, t_2$

$t^2 \to \dfrac{t_1t_2+t_2t_3+t_3t_1}{3}$ as $d=3$ $\quad < t_1, t_2, t_3$



$F(t) = (t,t)$
↑ $d=1$

$f(t) = F(t)$ $\quad d=1$

$\left(\dfrac{t_1+t_2}{2}, \dfrac{t_1+t_2}{2}\right)$

$G(t) = (t,t)$

$d=2$

$g(t_1,t_2) = \dfrac{f(t_1)+f(t_2)}{2}$

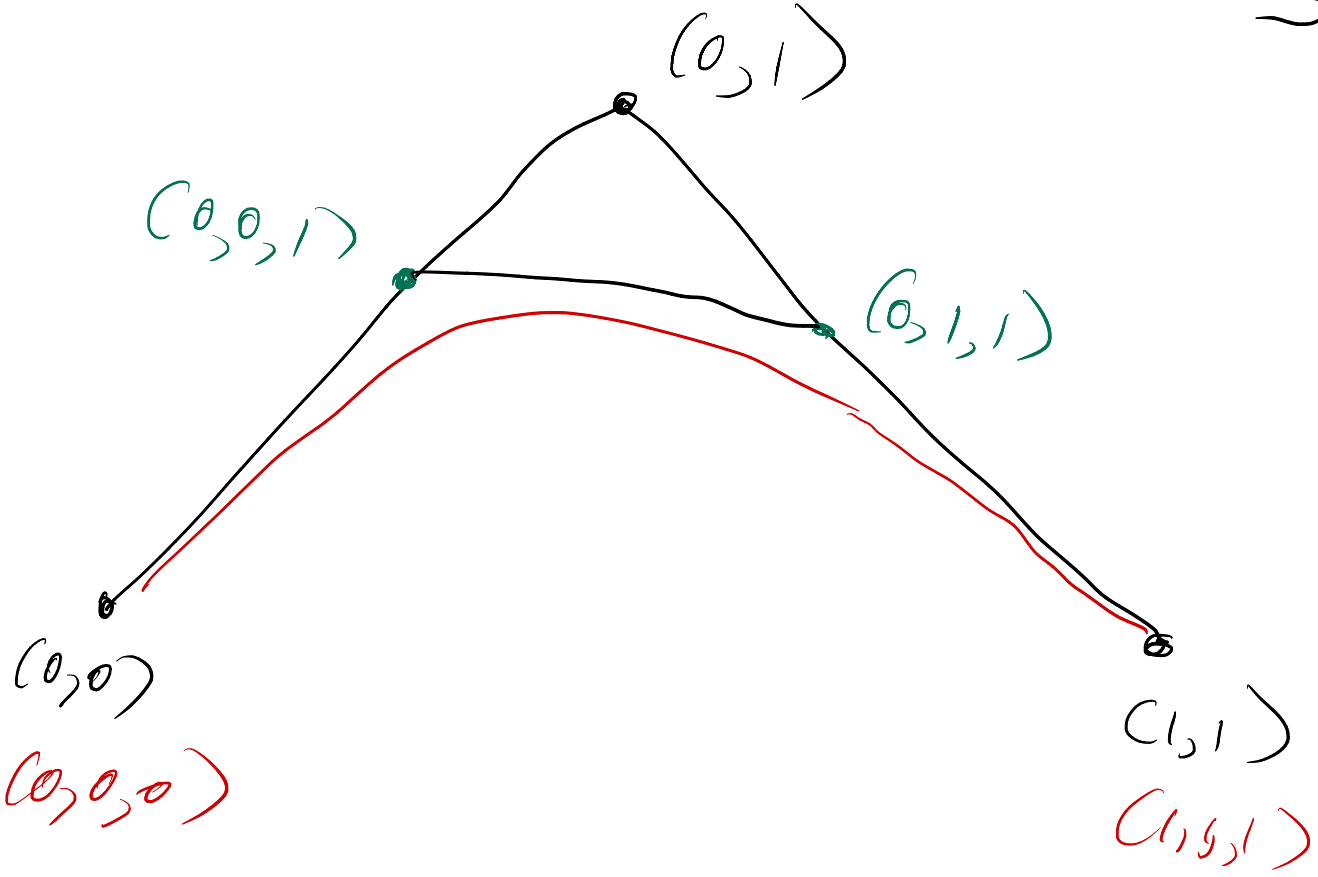
$(0,1) = \left(\frac{1}{2}, \frac{1}{2}\right)$

$f(t_1, t_2)$

$t^2 \to t_1 t_2$

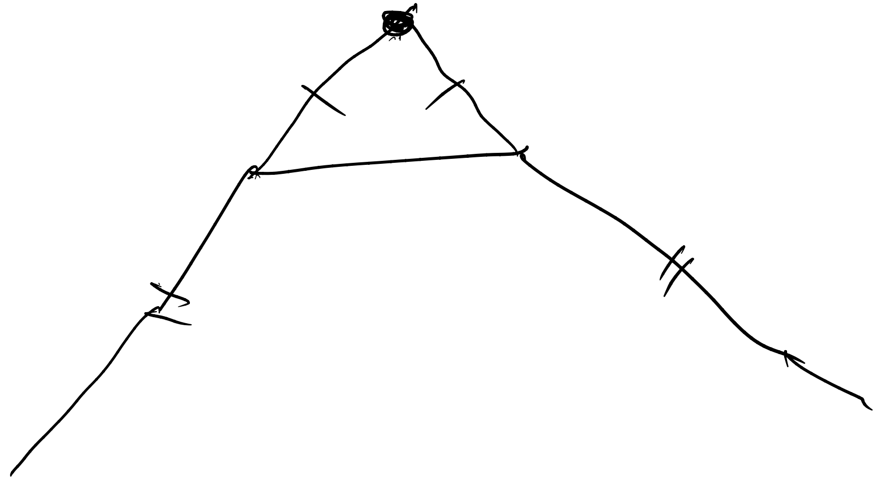
$g(t_1, t_2, t_3)$

$$g(t_1, t_2, t_3) = \frac{f(t_1, t_2) + f(t_2, t_3) + f(t_3, t_1)}{3}$$

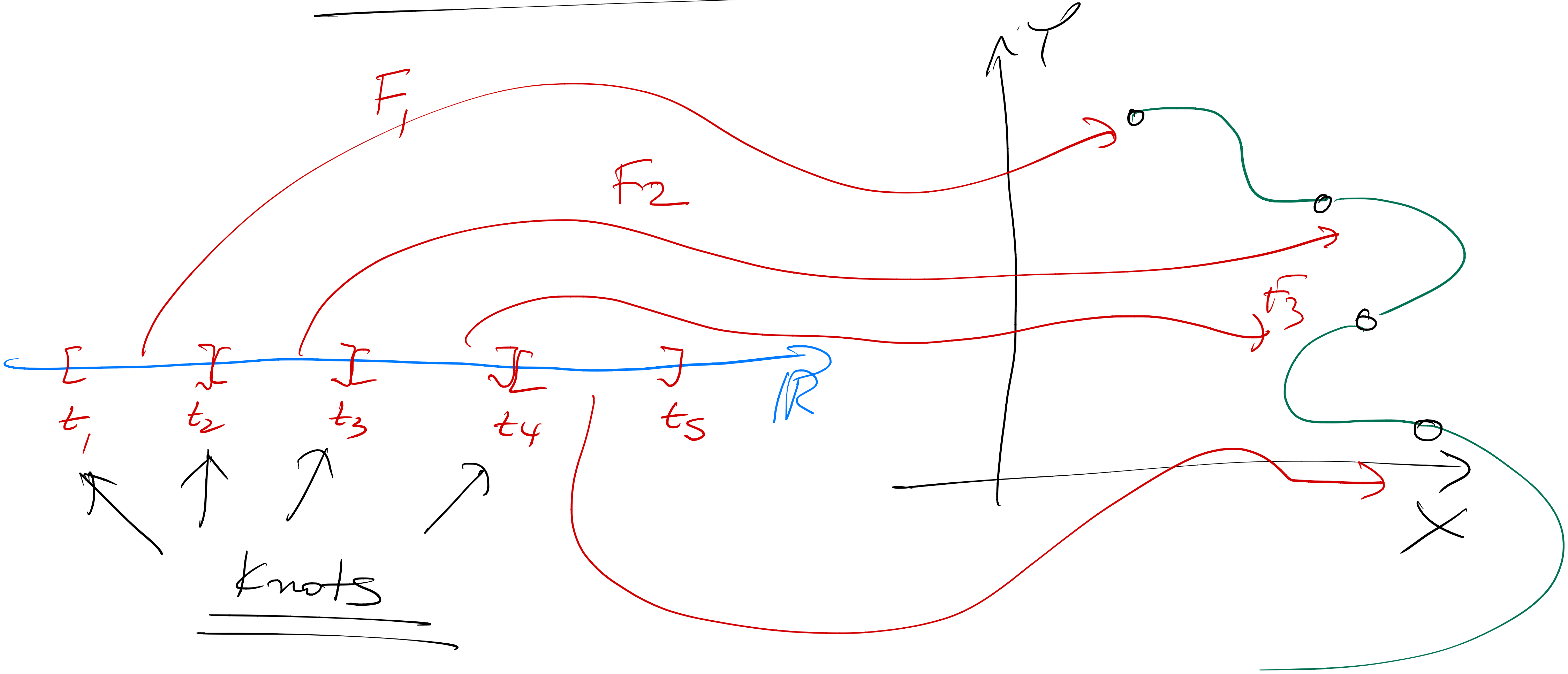


$$(0,0,1) = \frac{f(0,0) + f(0,1) + f(0,1)}{3}$$

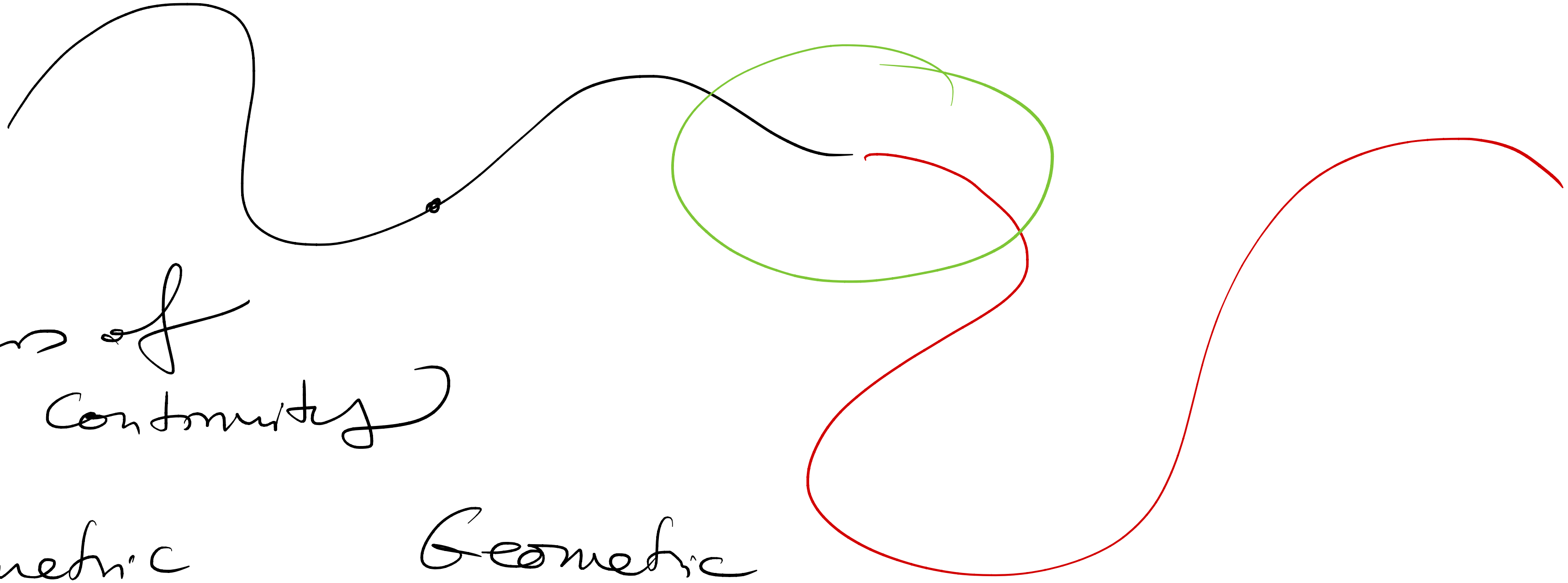
$$(0,1,1) \to \frac{f(0,1) + f(0,1) + f(1,1)}{3}$$



Junctions between Polynomial Arcs



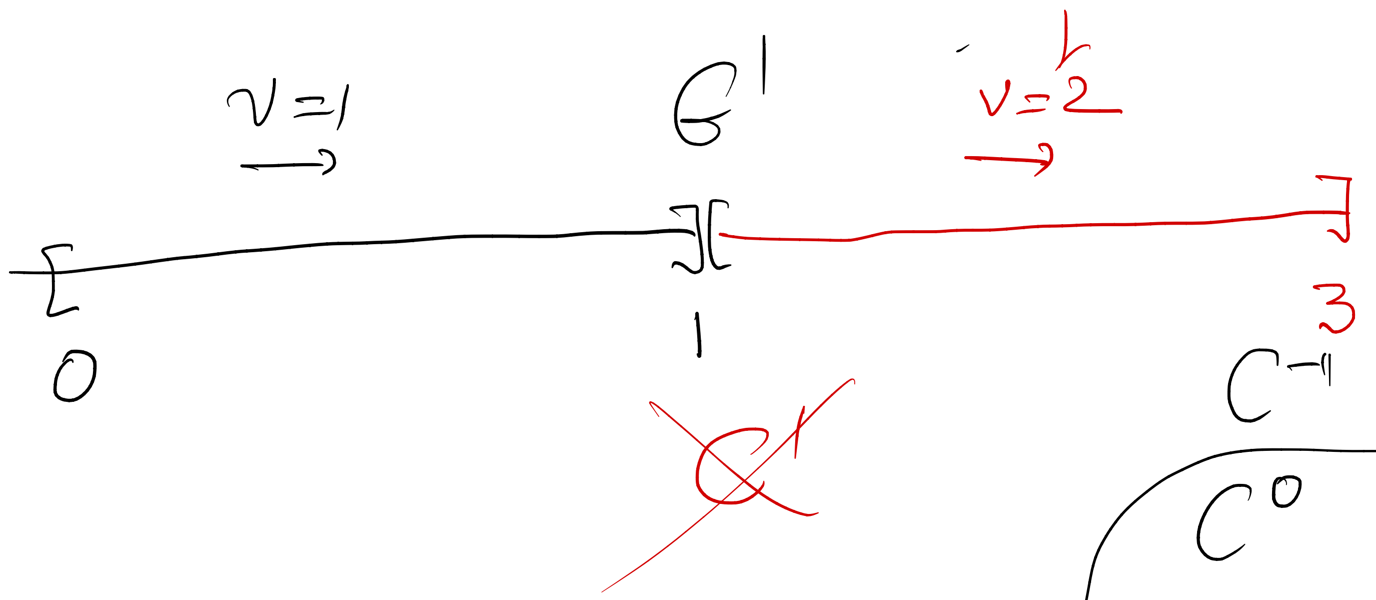
Continuity



Order of Continuity

Parametric   Geometric

$C^{(k)} \Rightarrow G^{(k)}$



$C^{-1} = G^{-1}$

$C^0 = G^0$ position

C^1 G^1 velocity

C^2 G^2 acceleration

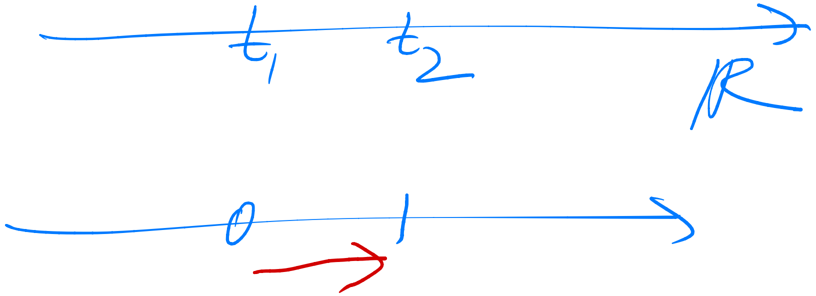
C^3 G^3 jerk

Derivadas

$[1,$

$$F(t) = (t, t^2) = [1, t, t^2]$$

$$[0, \ldots]$$



$$F(T) = (T, T^2)$$

$$T = \frac{t}{s} \qquad [s,t]$$

$$\boxed{\delta = (0,1)}$$

$$\overline{t+1} - \overline{t} = \delta$$

$$\boxed{\overline{t+h} - \overline{t} = h\delta}$$

$$\overline{t+h} = \overline{t} + h\delta$$

Homogenization & Polarization

$$G(T) = (1, T, T^2) \xrightarrow{P} g(T_1, T_2) = \left(1, \frac{T_1 + T_2}{2}, T_1 T_2\right)$$

$$H \downarrow$$

$$G(s,t) = \left(1, \frac{t}{s}, \frac{t^2}{s^2}\right) = (s^2, st, t^2) \xrightarrow{P}$$

$$g\big((s_1,t_1),(s_2,t_2)\big) = \left(s_1 s_2, \frac{s_1 t_2 + s_2 t_1}{2}, t_1 t_2\right)$$

$$H$$

Differentiation ≡ Evaluation at δ

F is a cubic f

$F(\bar{t})$ $\qquad$ $f(\bar{t}_1, \bar{t}_2, \bar{t}_3)$

$$F'(\bar{t}) = \lim_{h \to 0} \frac{F(\overline{t+h}) - F(\bar{t})}{h} =$$

$$\lim_{h \to 0} \frac{f(\overline{t+h}, \overline{t+h}, \overline{t+h}) - f(\bar{t}, \bar{t}, \bar{t})}{h} =$$

$$\lim_{h \to 0} \frac{f(\bar{t} + h\delta, \bar{t} + h\delta, \bar{t} + h\delta) - f(\bar{t}, \bar{t}, \bar{t})}{h}$$

$$f(\bar{t}+h\delta,\ \bar{t}+h\Gamma,\ \bar{t}+h\delta) = f(\bar{t},\ \bar{t}+h\Gamma,\ \bar{t}+h\delta) + h\, f(\delta,\ \bar{t}+h\Gamma,\ \bar{t}+h\delta)$$

$$f(\bar{t}+h\delta,\ \bar{t}+h\delta,\ \bar{t}+h\delta) =$$

$$f(\bar{t},\bar{t},\bar{t}) + 3h\, f(\bar{t},\bar{t},\delta) + 3h^{2} f(\bar{t},\delta,\delta) + h^{3} f(\delta,\delta,\delta)$$

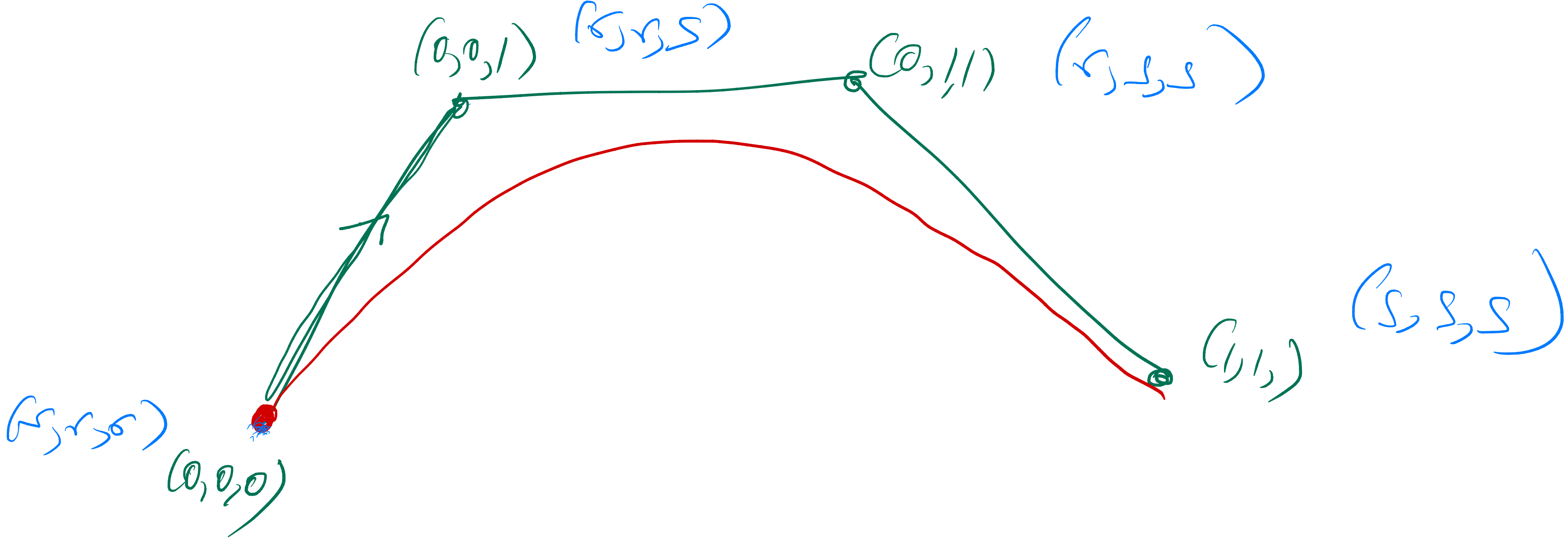
$h \to 0$

$$F'(\bar{t}) = 3 f(\bar{t},\bar{t},\delta)$$

$$F'(t) = n\, f(\underbrace{t, t, \ldots, t}_{n-1}, \delta)$$

$$F''(t) = n(n-1)\, f(\underbrace{t, t, \ldots, t}_{n-2}, \delta, \delta)$$

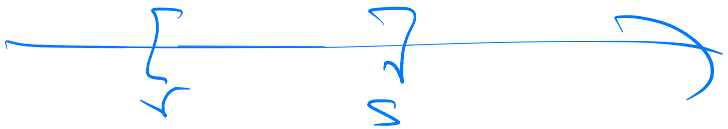
$$F^{(k)}(t) = n(n-1)\cdots(n-k+1)\, f(\underbrace{t \ldots t}_{n-k}, \underbrace{\delta, \delta, \delta, \ldots}_{k})$$



$$\frac{1}{s-r}$$

$$F'(0) = f'(\bar{0},\bar{0},\bar{0}) = 3f(\bar{0},\bar{0},\delta) = 3f(\bar{0},\bar{0},\bar{1}-\bar{0}) = 3\left[f(0,0,1) - f(0,0,0)\right]$$

$\uparrow$ $\bar{1}-\bar{0}$

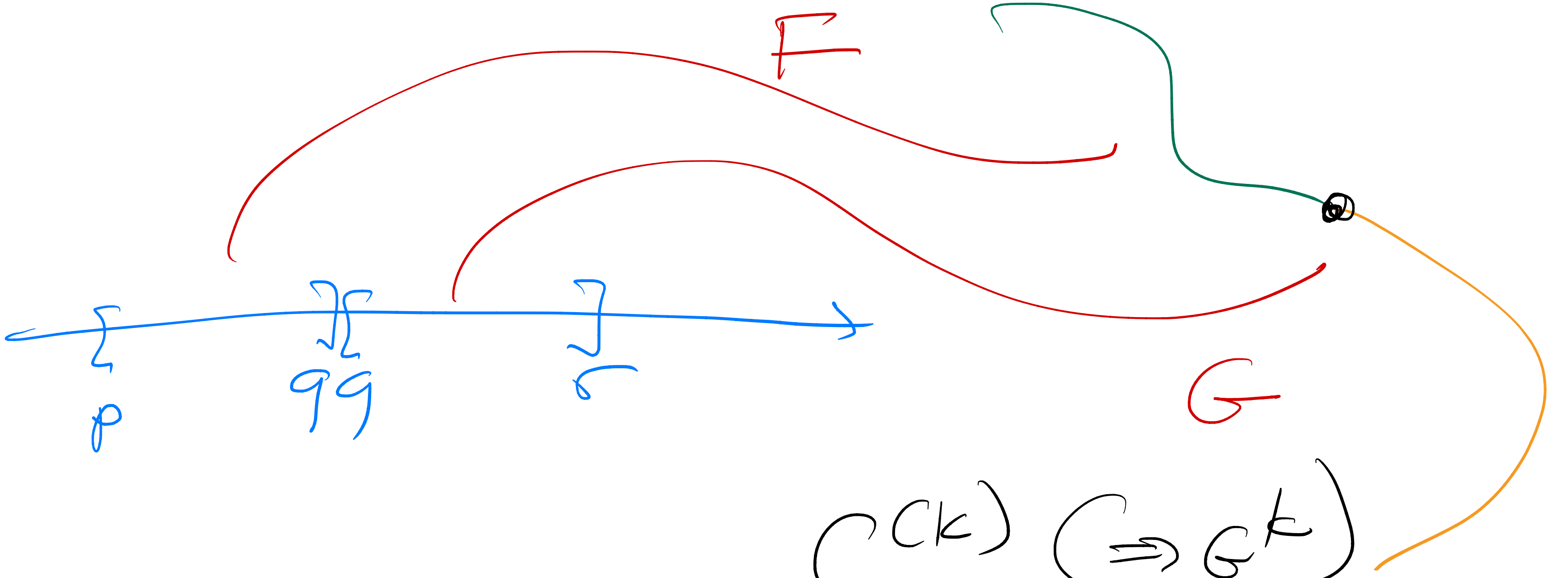


$$\delta = \frac{1}{r-s}\left[\bar{r}-\bar{s}\right]$$

$$\frac{1}{s-r}\left[\bar{s}-\bar{r}\right]$$

$$F''(0) = 6\left[f(\bar{0},\bar{1},\bar{1}) - 2f(\bar{0},\bar{0},\bar{1}) + f(\bar{0},\bar{0},\bar{0})\right]$$

Joints Between Polynomial Curves



$C^{(k)}$ ($\Rightarrow G^{k}$)

$F([p..q])$ join $G([q..r])$ with $C^{(k)}$ cont

$F^{(0)} = G^{(0)}$ at q

$F^{(1)} = G^{(1)}$

$F^{(2)} = G^{(2)}$ $\qquad F^{(3)} = G^{(3)}$

$C^{(0)}$ $$f(q,q,q) = g(q,q,q)$$

$$F(q) = G(q)$$

$C^{(1)}$ $$f(q,q,\delta) = g(q,q,\delta)$$

equivalent

$$f(q,q,x) = g(q,q,x) \quad \forall x$$

$$x = \bar{q} + (\bar{x} - \bar{q})\delta$$

$$f(q,q,q) + (\bar{x} - \bar{q}) f(q,q,\delta)$$

$$\parallel$$

$$f(q,q,x)$$

$$\downarrow$$

$$g(q,q,q) + (\bar{x} - \bar{q}) g(q,q,\delta)$$

$$\parallel$$

$$g(q,q,x)$$

$C^{(2)}$

$$f(q,x,y) = g(q,x,y) \quad \forall x \forall y$$

Summary) $t=3$

$C^{(0)}$ $\quad f(q,q,q) = g(q,q,q)$

$C^{(1)}$ $\quad f(q,q,x) = g(q,q,x)$

$C^{(2)}$ $\quad f(q,x,x) = g(q,x,x)$

$C^{(3)}$ $\quad f(x,y,z) = g(x,y,z)$

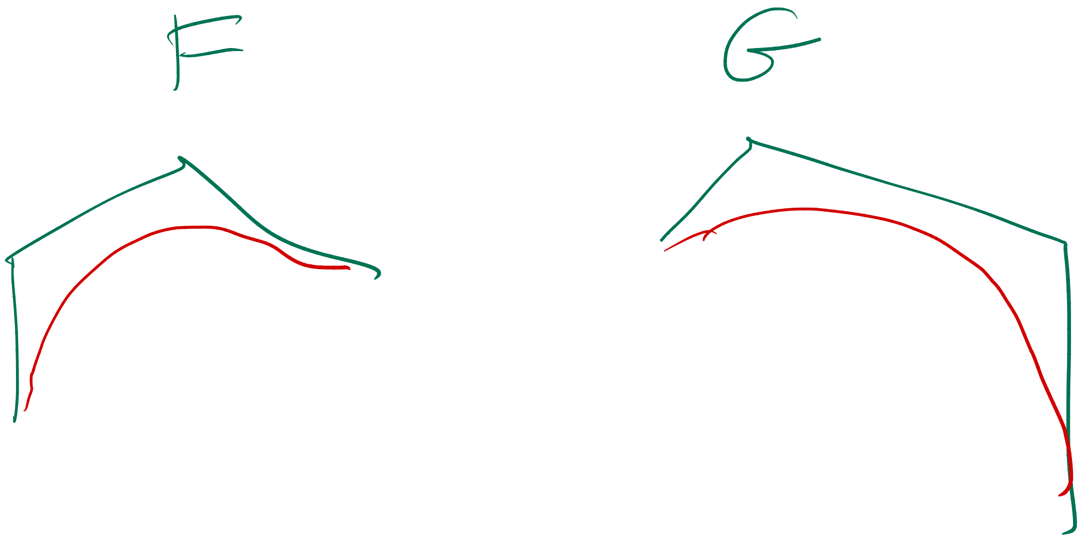
Summary

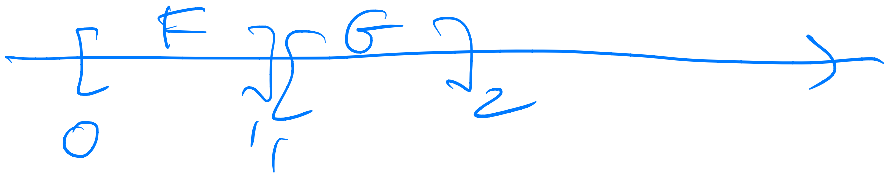
$$C^{(0)} \equiv f(q,q,q) = g(q,q,q)$$

$$C^{(1)} = f(q,q,\bar{\delta}) = g(q,q,\delta)$$

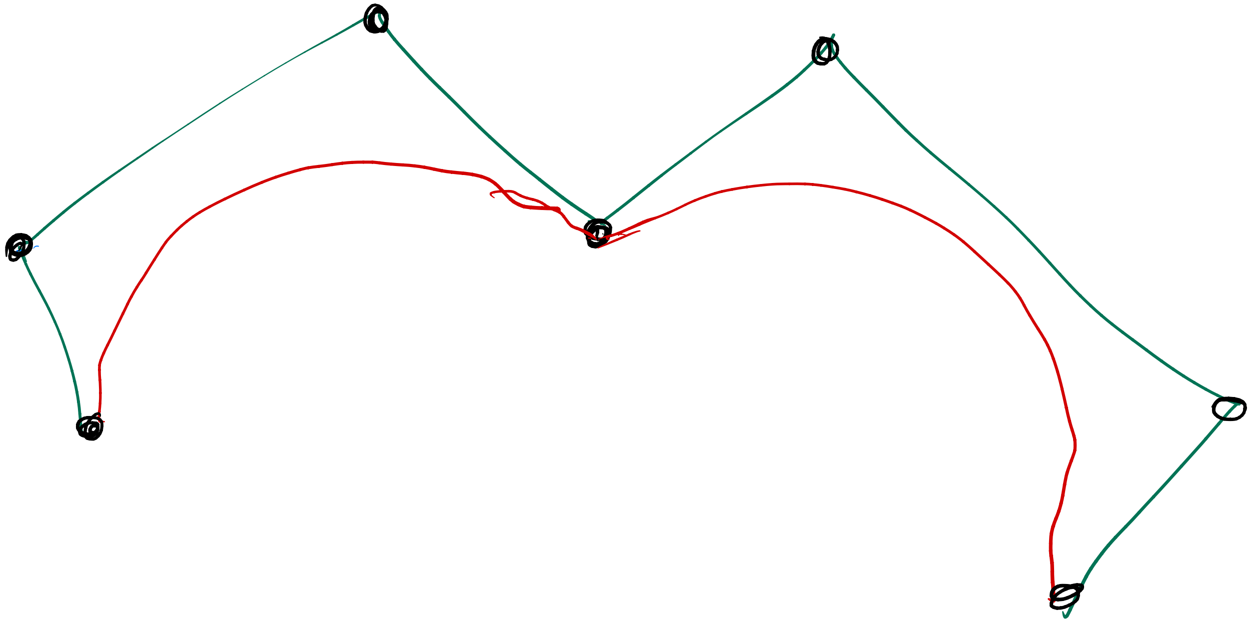
$$C^{(2)} \equiv f(q,\delta,\delta) = g(q,\delta,\delta)$$

$$C^{(3)} = f(\delta,\delta,\delta) = g(\delta,\delta,\delta)$$





$C^{(-1)}$



$C^{(0)}$

$F(1) = G(1)$

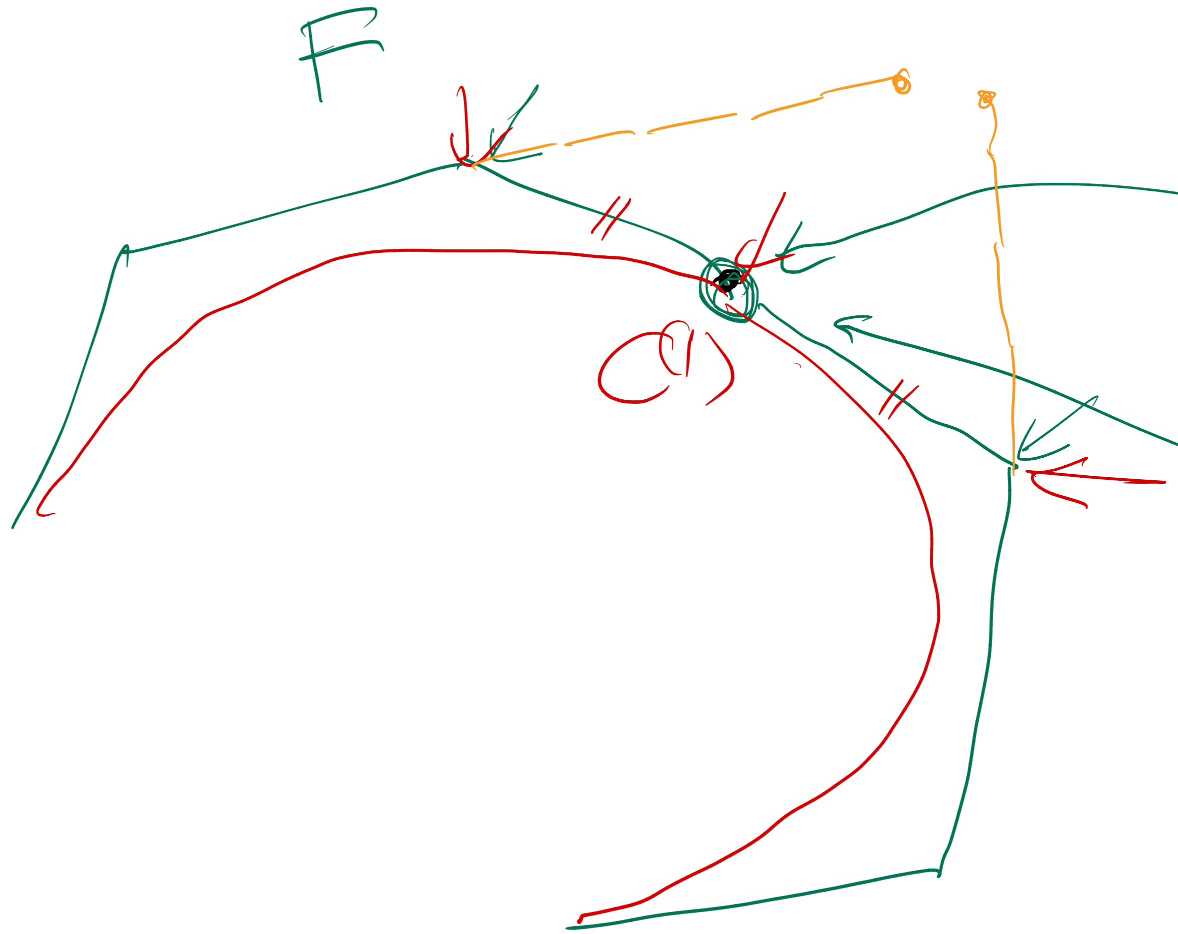
$f(1,1,1) = g(1,1,1)$

$C^{(1)}$

$F(1) = G(1)$

$f(1,1,1) = g(1,1,1)$

$f(1,1,0) = g(1,1,0)$



$f(1,1,1) - f(1,1,0) =$

$g(1,1,2) - g(1,1,1)$

That’s All