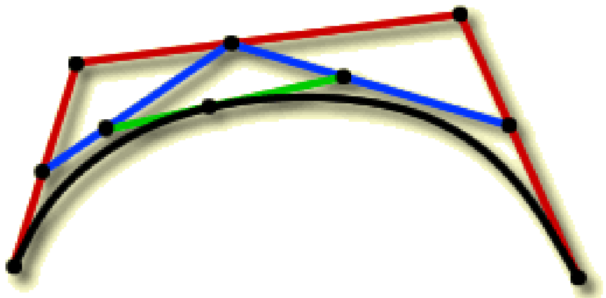
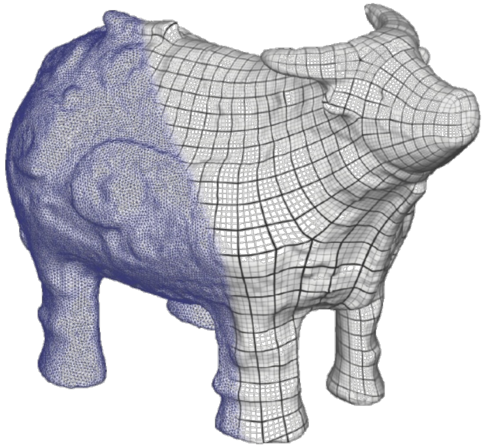
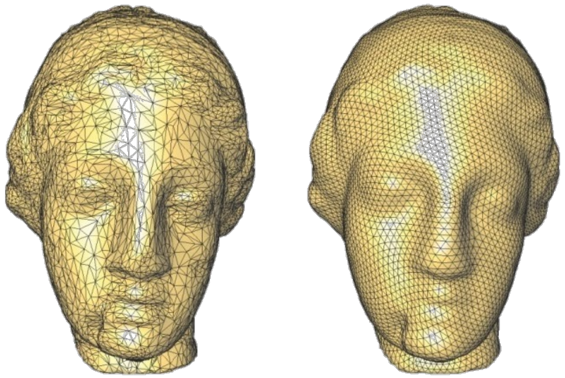
CS348a: Geometric Modeling and Processing



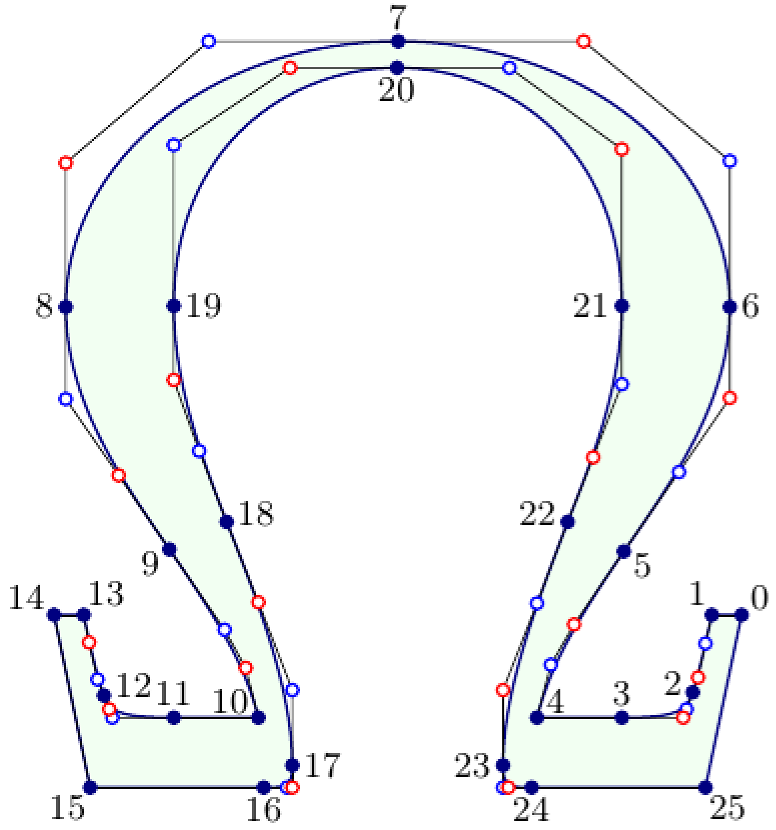
Leonidas Guibas
Computer Science Department
Stanford University





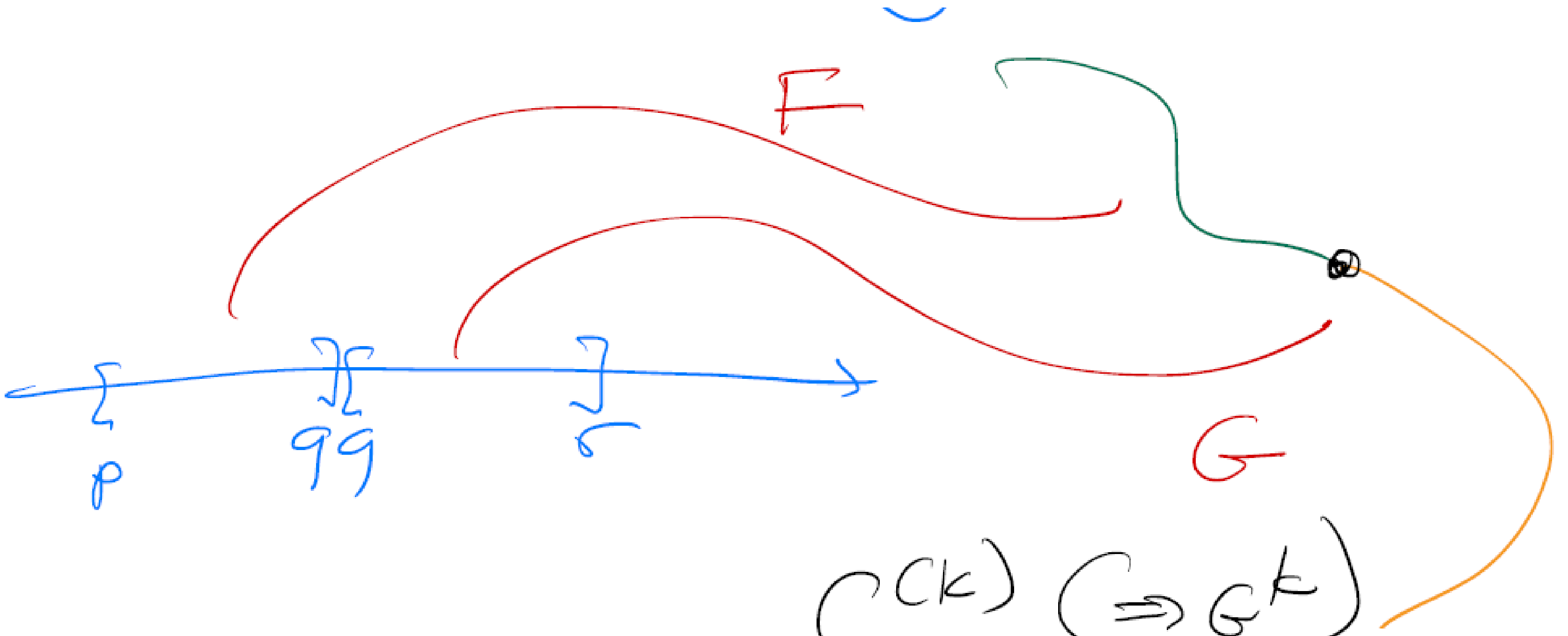
Last Time:
Continuity Constraints, Derivatives and Polar Forms

Modeling 2D Shapes with Spline Curves



The fonts we use …

Continuity of Joints Between Curves



$C^{(k)}$ $(\Rightarrow G^{k})$

$F([p..q])$ join $G([q..r])$ with $C^{(k)}$ cont

$F^{(0)} = G^{(0)}$ at q

$F^{(1)} = G^{(1)}$

$F^{(2)} = G^{(2)}$ $\quad F^{(3)} = G^{(3)}$

Derivatives and Polar Forms --

$$F^{(m)}(\bar{q}) = n(n-1)\cdots(n-q+1)f(\underbrace{\bar{q},\ldots,\bar{q}}_{n-m},\underbrace{\delta,\ldots,\delta}_{m})$$

$$G^{(m)}(\bar{q}) = n(n-1)\cdots(n-q+1)g(\underbrace{\bar{q},\ldots,\bar{q}}_{n-m},\underbrace{\delta,\ldots,\delta}_{m})$$

$$\delta = \bar{1} - \bar{0}$$

$$f(\underbrace{\bar{q},\ldots,\bar{q}}_{n-m},\bar{u}_1,\ldots,\bar{u}_m) = g(\underbrace{\bar{q},\ldots,\bar{q}}_{n-m},\bar{u}_1,\ldots,\bar{u}_m)$$

$$f(\underbrace{\bar{q},\ldots,\bar{q}}_{n-k},\underbrace{\delta,\ldots,\delta}_{k}) = g(\underbrace{\bar{q},\ldots,\bar{q}}_{n-k},\underbrace{\delta,\ldots,\delta}_{k}), \text{for } k \leq m$$

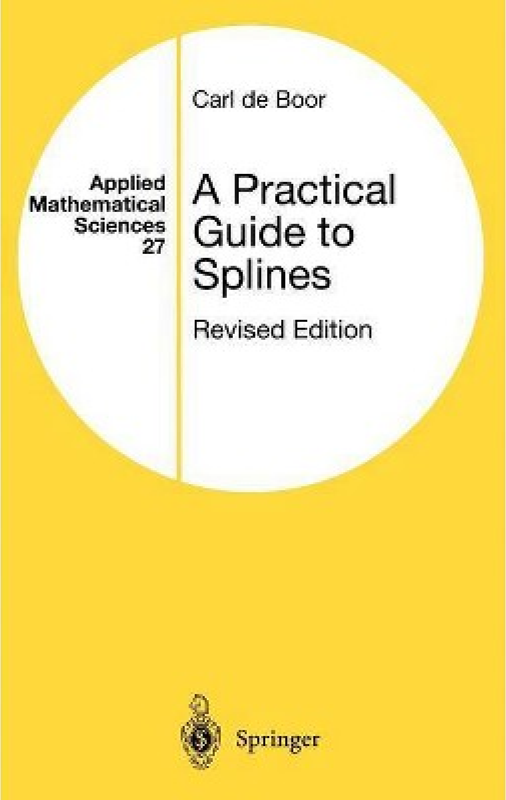
The Cubics Case

$$C^0 \leftrightarrow f(q,q,q) = g(q,q,q)$$
$$C^1 \leftrightarrow f(q,q,u) = g(q,q,u) \quad \forall u$$
$$C^2 \leftrightarrow f(q,u,v) = g(q,u,v) \quad \forall u,v$$
$$C^3 \leftrightarrow f(u,v,w) = g(u,v,w) \quad \forall u,v,w$$

$$C^0 \leftrightarrow f(q,q,q) = g(q,q,q)$$
$$C^1 \leftrightarrow f(q,q,\delta) = g(q,q,\delta)$$
$$C^2 \leftrightarrow f(q,\delta,\delta) = g(q,\delta,\delta)$$
$$C^3 \leftrightarrow f(\delta,\delta,\delta) = g(\delta,\delta,\delta)$$

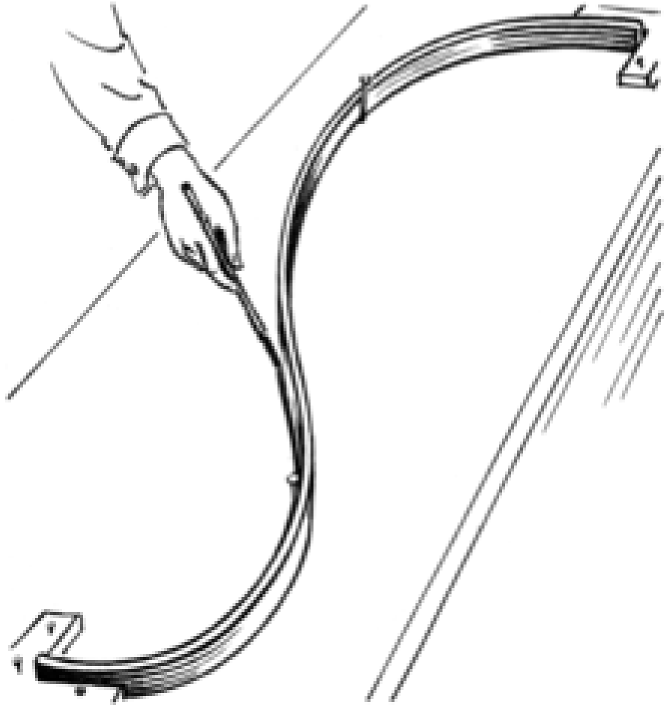
Today:
Splines and B-Splines

Carl de Boor


Carl de Boor
Applied
Mathematical
Sciences
27
A Practical
Guide to
Splines
Revised Edition
Springer

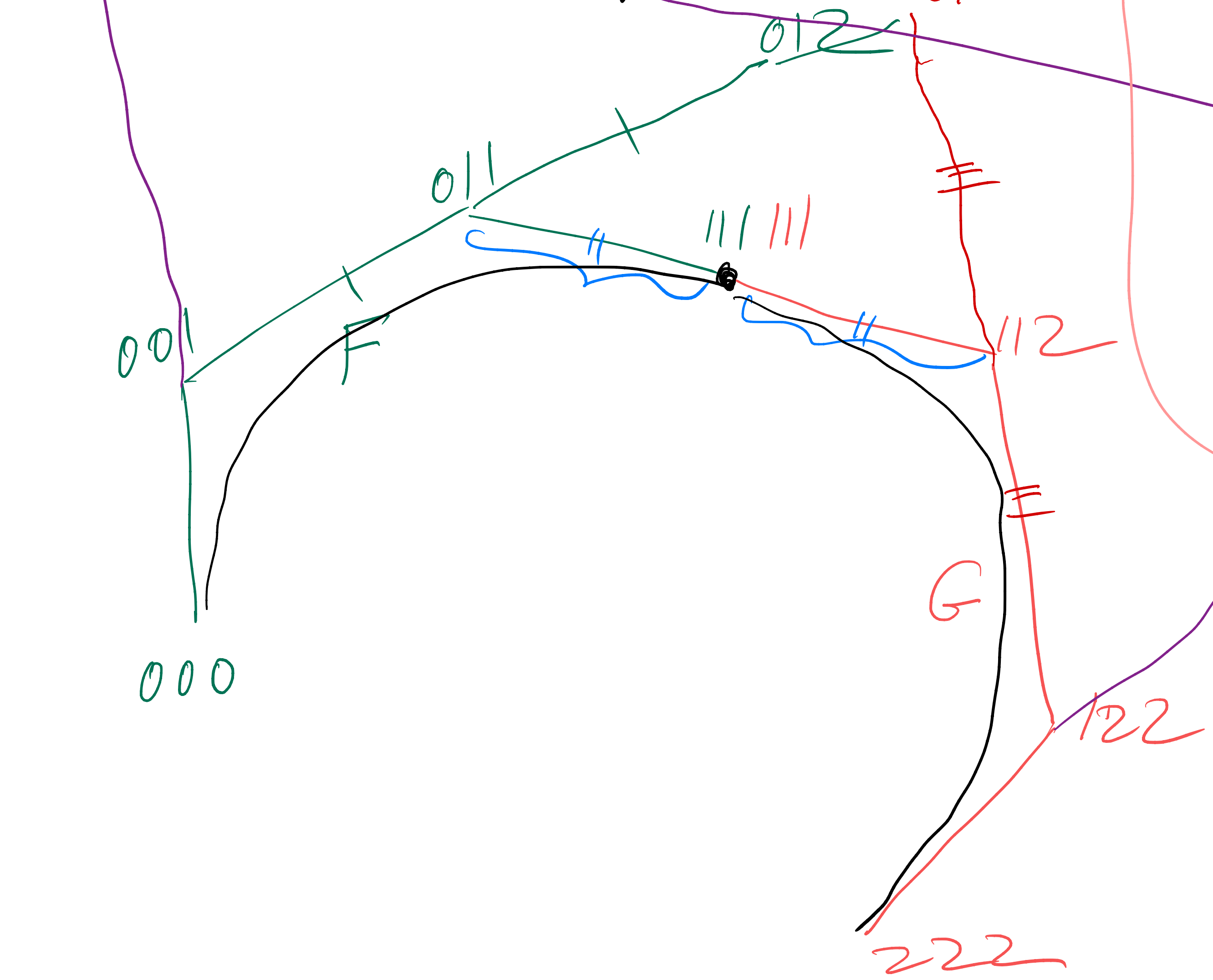
History of Splines

- Designed to create smooth curves
- Similar to physical process of bending thin wood plates



Whiteboard

Spline



$C^{(0)}$: $F(1)=G(1)$

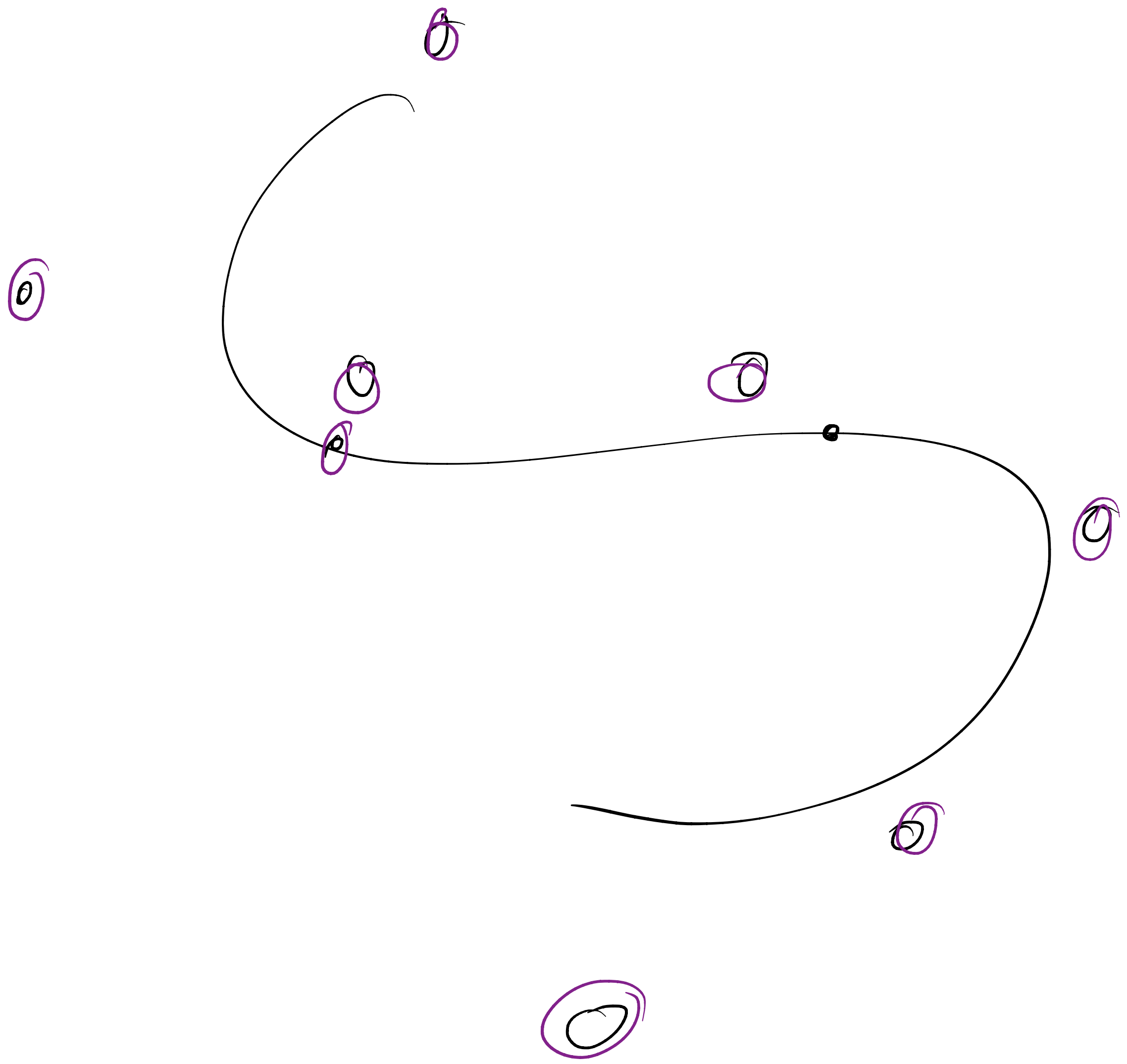
$$f(1,1,1)=g(1,1,1)$$

$C^{(1)}$ $f(1,1,\delta)=g(1,1,\delta)$

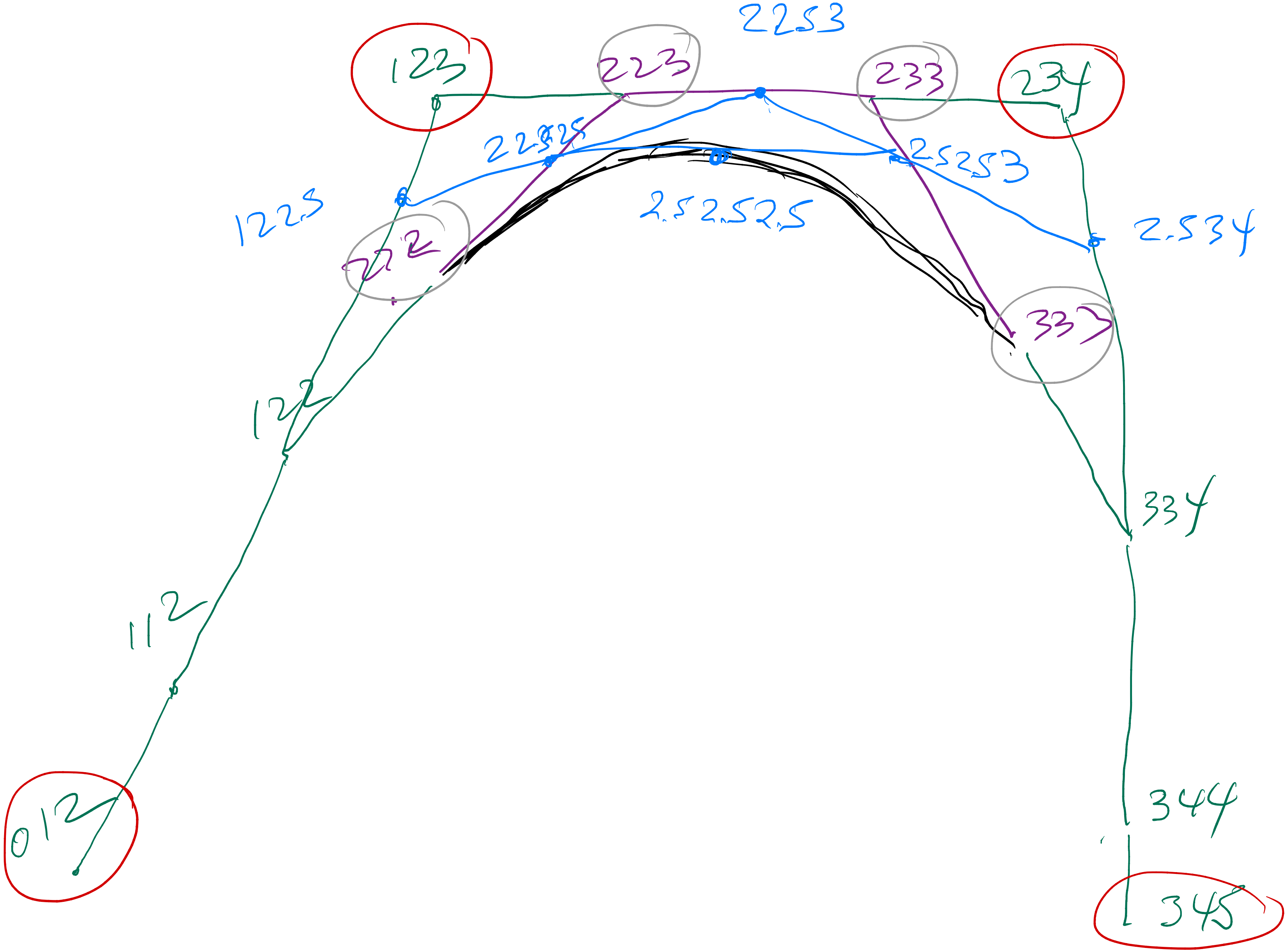
C^{2} $f(1,\delta,\delta)=g(1,\delta,\delta)$

C^{3} $f(\delta,\delta,\delta)=g(\delta,\delta,\delta)$

B-Splines



Bézier Control Pts

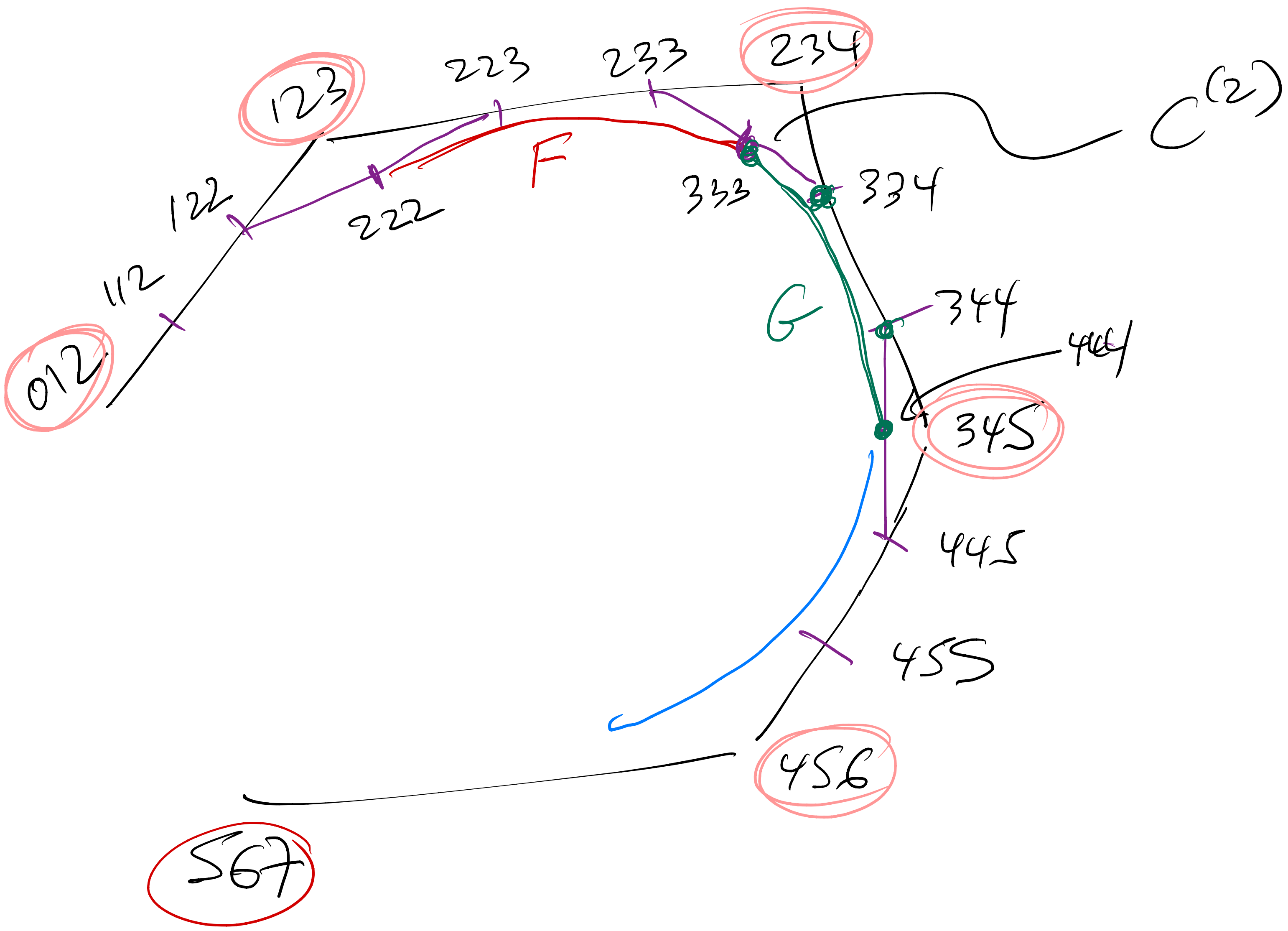


2.5 2.5 2.5

de Boor Control Pts

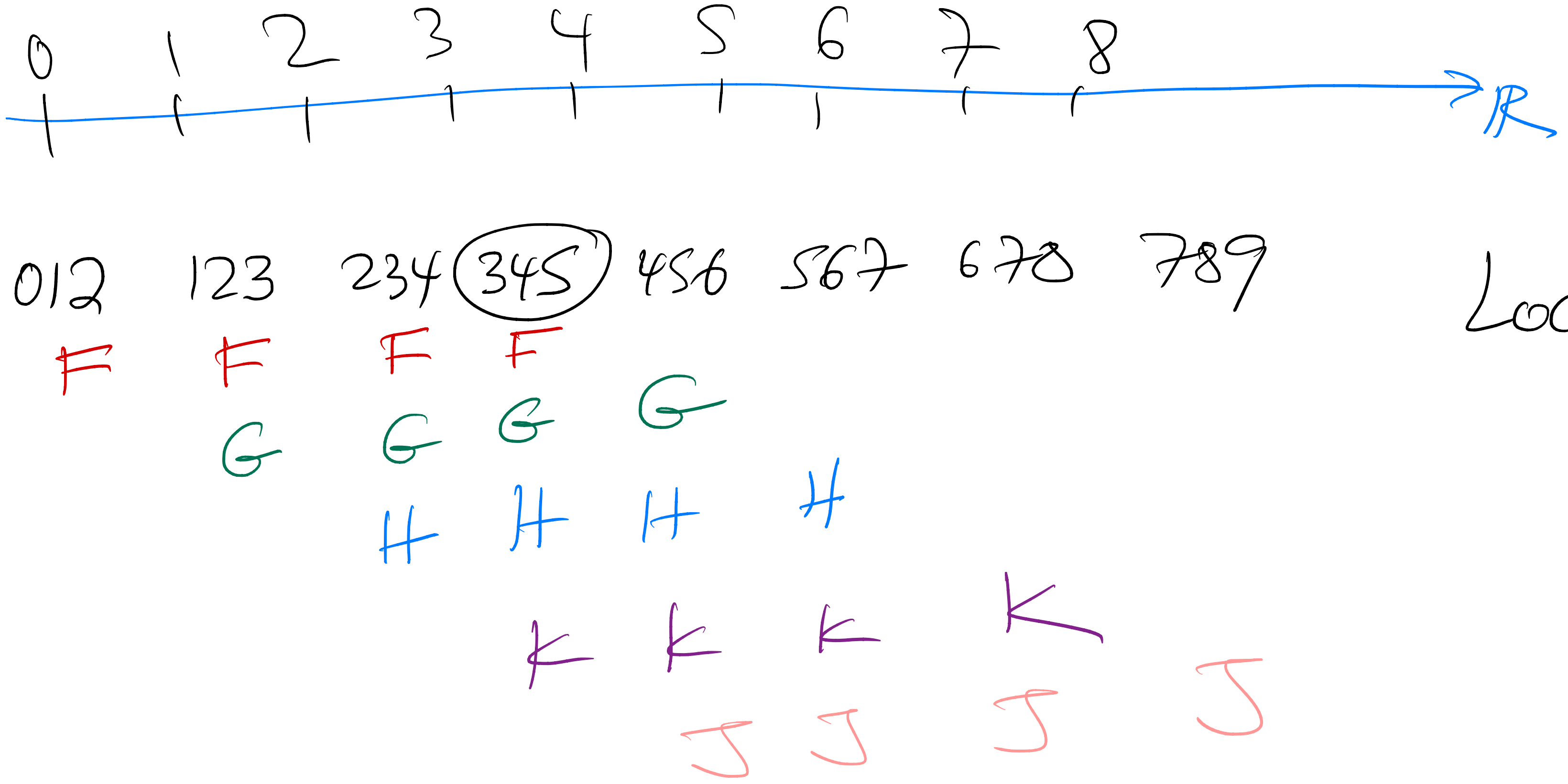
de Boor algorithm

de Boor Control Pts

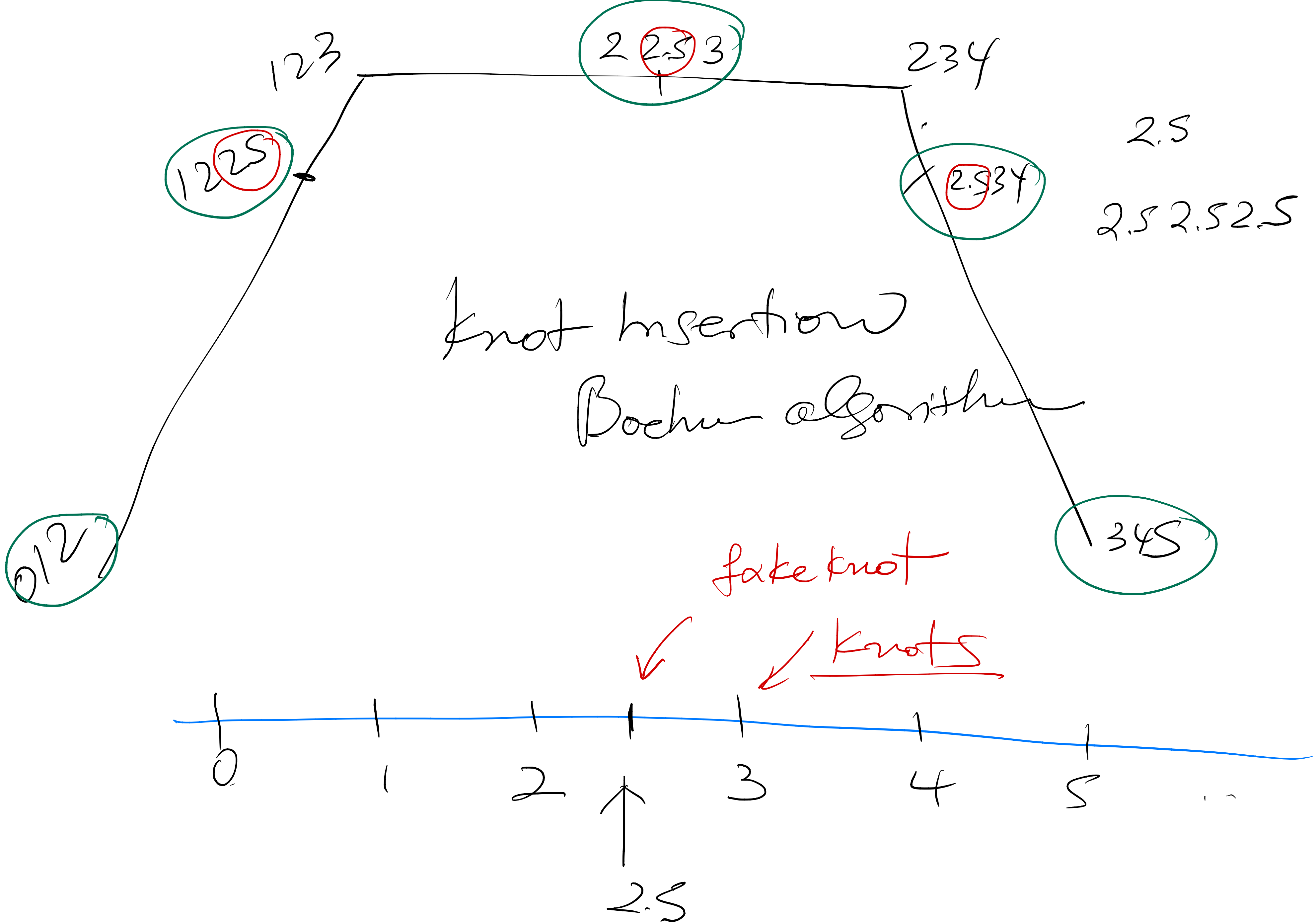


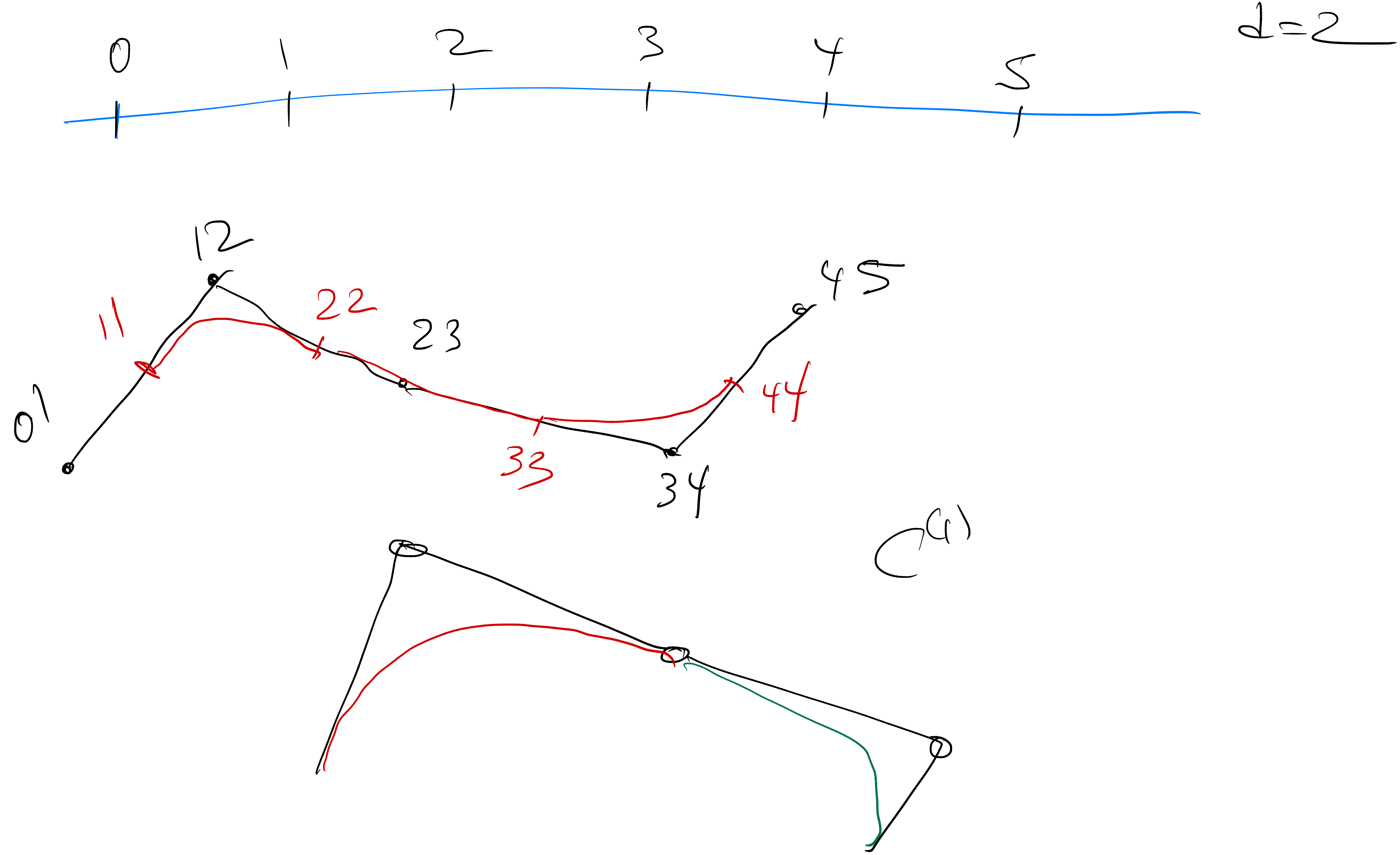
B-Spline

Local Control

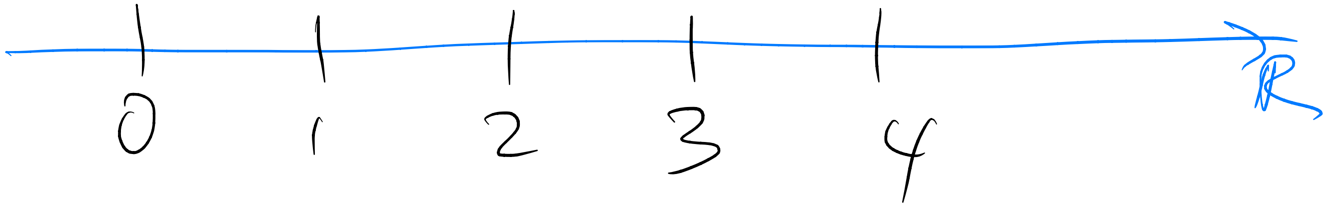


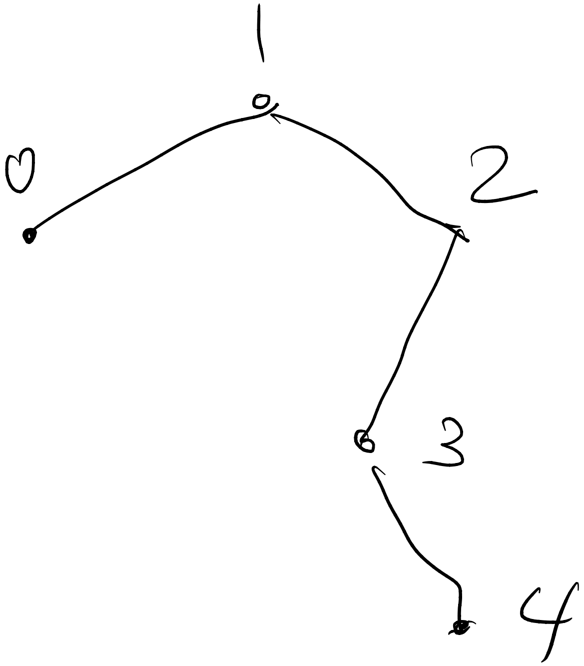
Knot Insertion
Boehm algorithm





d=2
0
1
2
3
4
5
12
11
22
23
01
33
34
44
45
C(1)





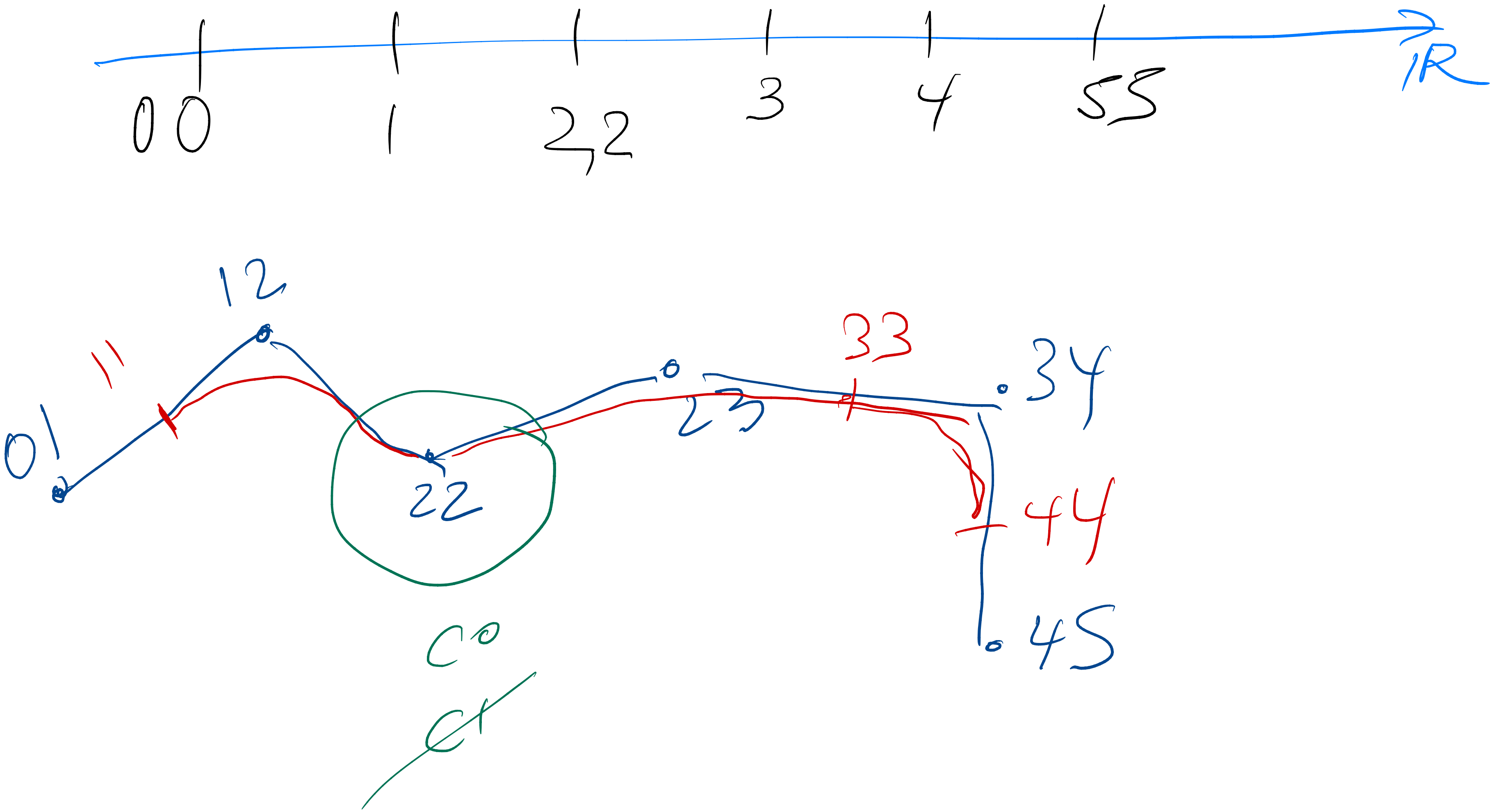
deg 1 B-splines

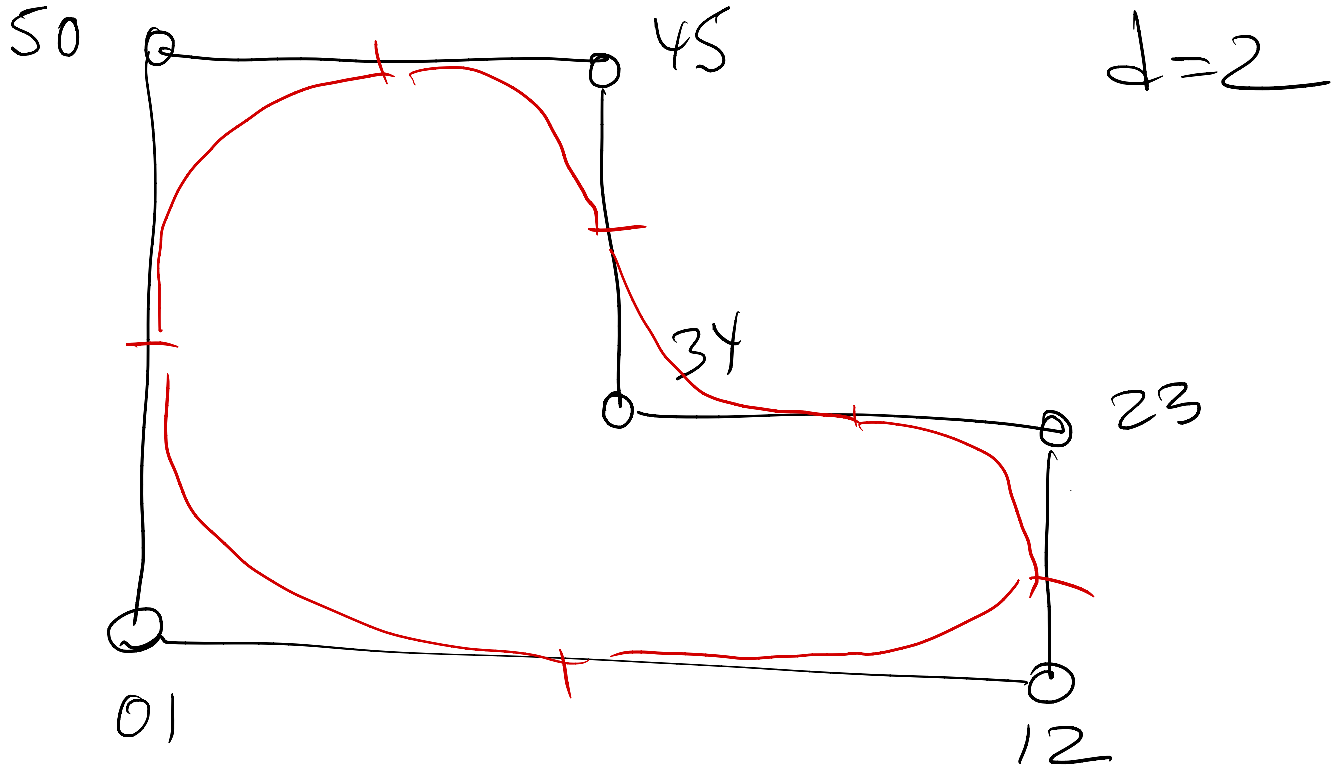
d = 1
d = 2
d = 3

degree

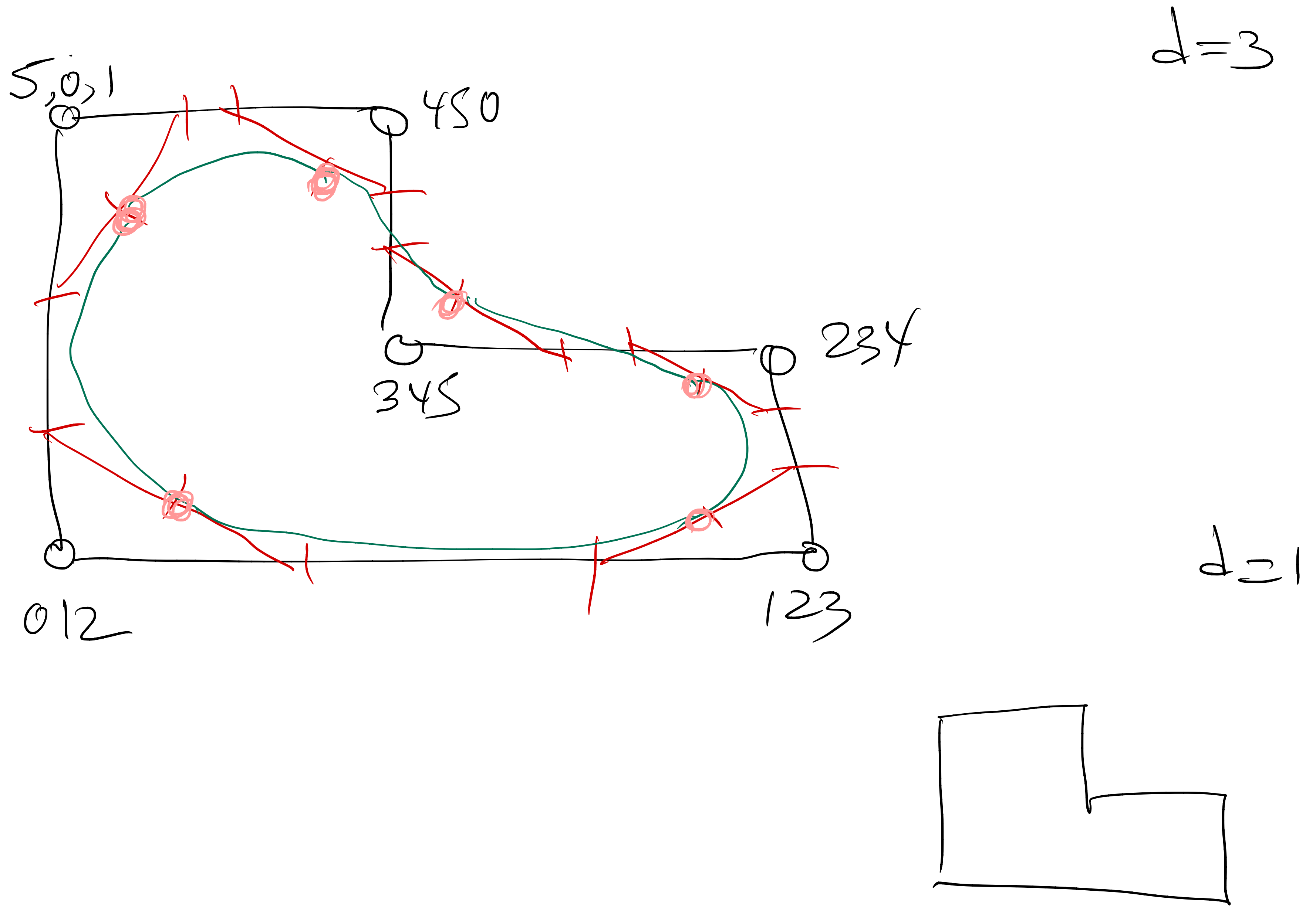
order = degree + 1

Repeated Knots





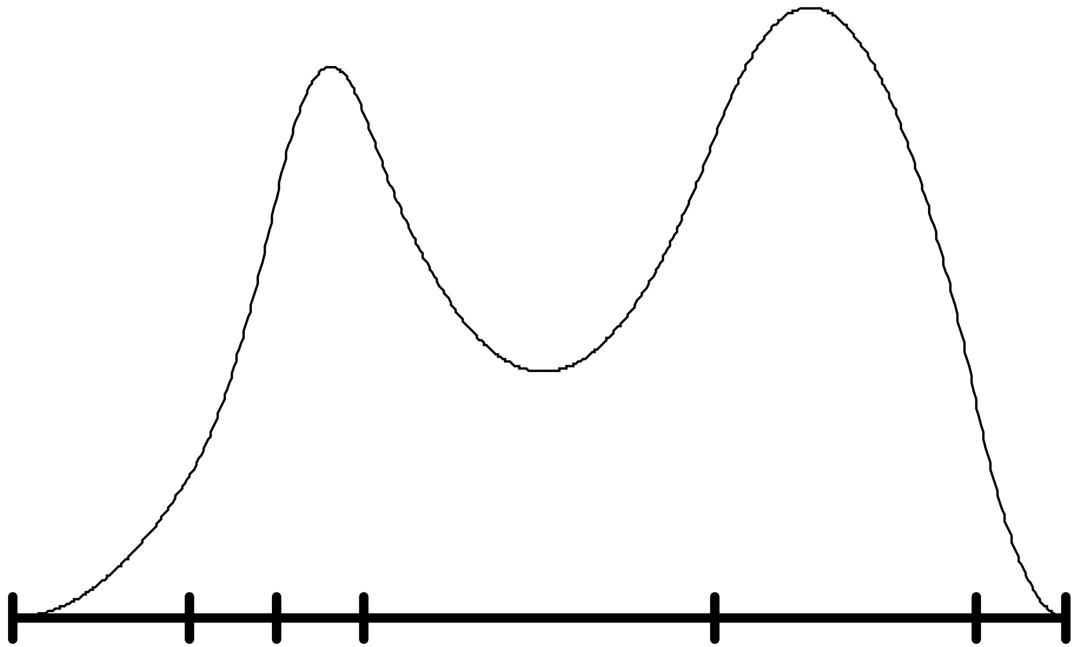
50
45
d=2
34
23
01
12



5,0,1
450
234
345
012
123
d=3
d=1

B-Spline Curves

$$p(t) = \sum_{i=1}^{m} p_i N_i^n(t)$$



i control point index

n degree

[order = degree + 1]

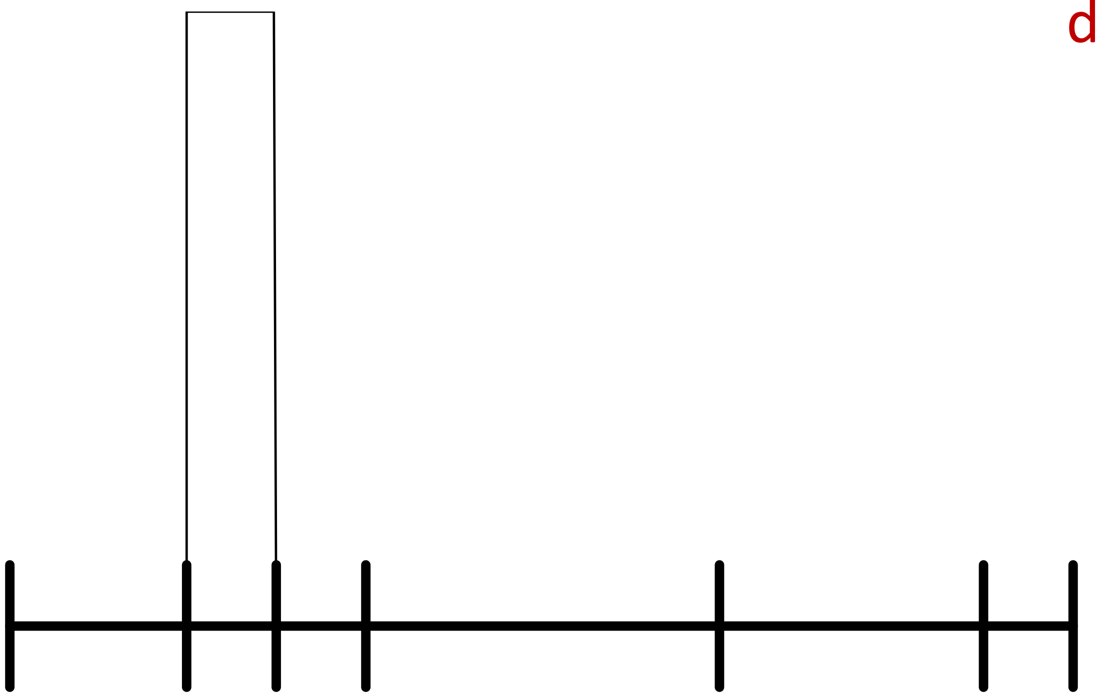
B-Spline Basis Functions

$$N_i^0(t) = \begin{cases} 1 & t_{i-1} \le t < t_i \\ 0 & o.w. \end{cases}$$

$$N_i^n(t) = \frac{t - t_{i-1}}{t_{i+n-1} - t_{i-1}} N_i^{n-1}(t) + \frac{t_{n+i} - t}{t_{i+n} - t_i} N_{i+1}^{n-1}(t)$$

de Boor recurrence

$N_3^0(t) =$



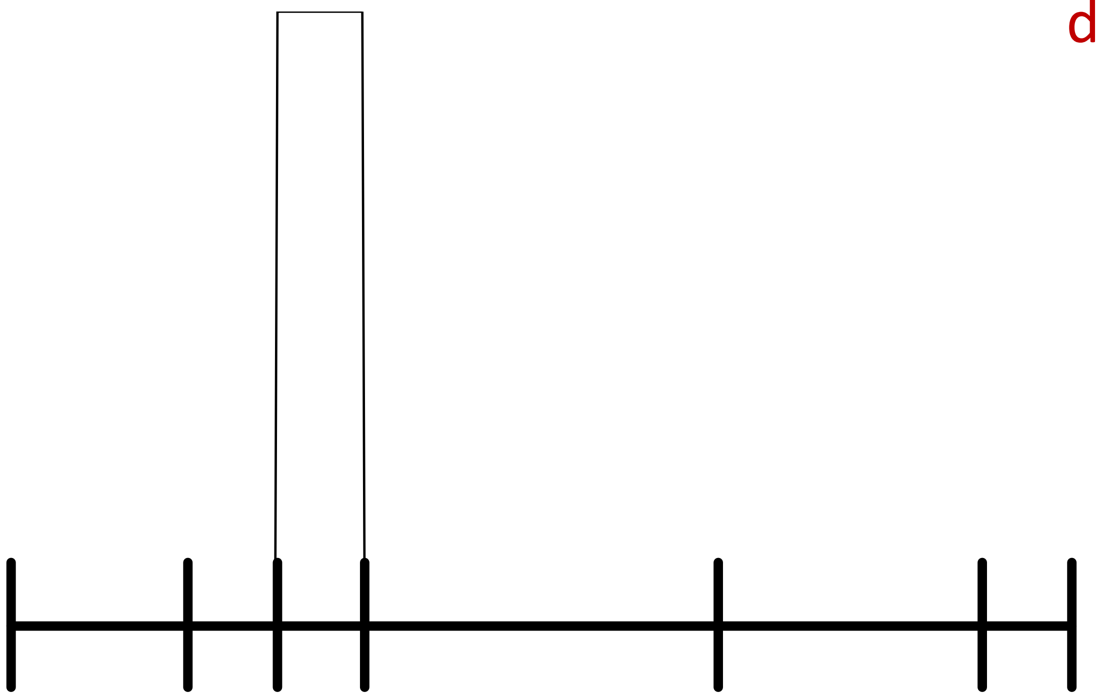
B-Spline Basis Functions

$$N_i^0(t) = \begin{cases} 1 & t_{i-1} \le t < t_i \\ 0 & o.w. \end{cases}$$

$$N_i^n(t) = \frac{t - t_{i-1}}{t_{i+n-1} - t_{i-1}} N_i^{n-1}(t) + \frac{t_{n+i} - t}{t_{i+n} - t_i} N_{i+1}^{n-1}(t)$$

de Boor recurrence

$$N_4^0(t) =$$



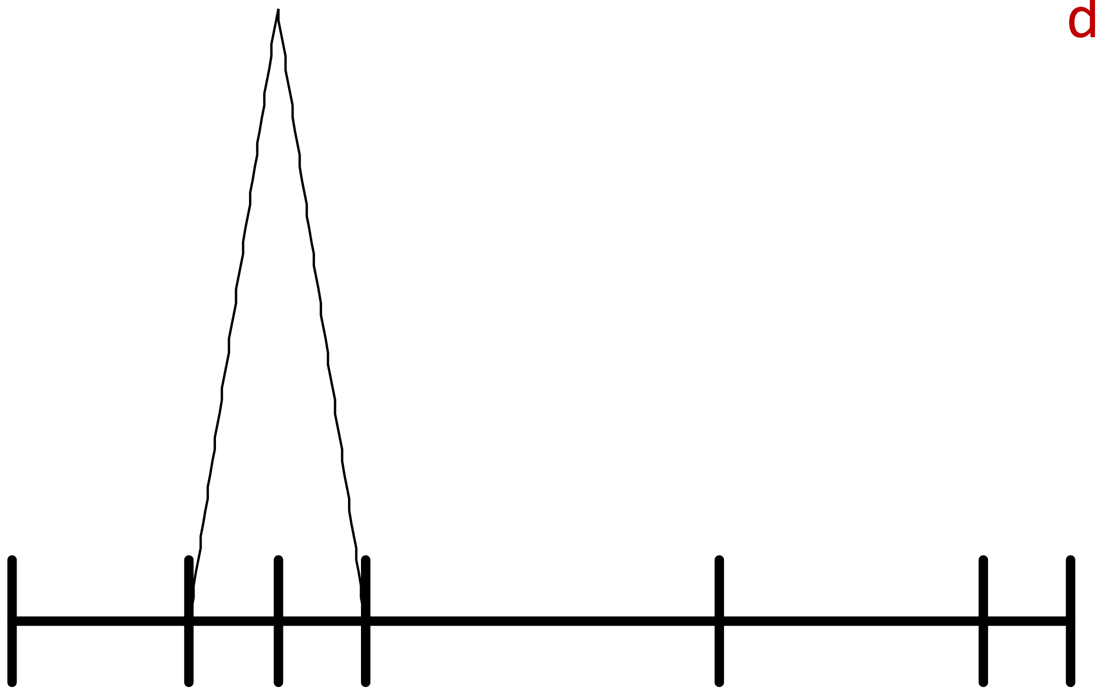
B-Spline Basis Functions

$$N_i^0(t)=\begin{cases}1 & t_{i-1}\le t<t_i\\ 0 & o.w.\end{cases}$$

$$N_i^n(t)=\frac{t-t_{i-1}}{t_{i+n-1}-t_{i-1}}N_i^{n-1}(t)+\frac{t_{n+i}-t}{t_{i+n}-t_i}N_{i+1}^{n-1}(t)$$

de Boor recurrence

$$N_3^1(t)=$$



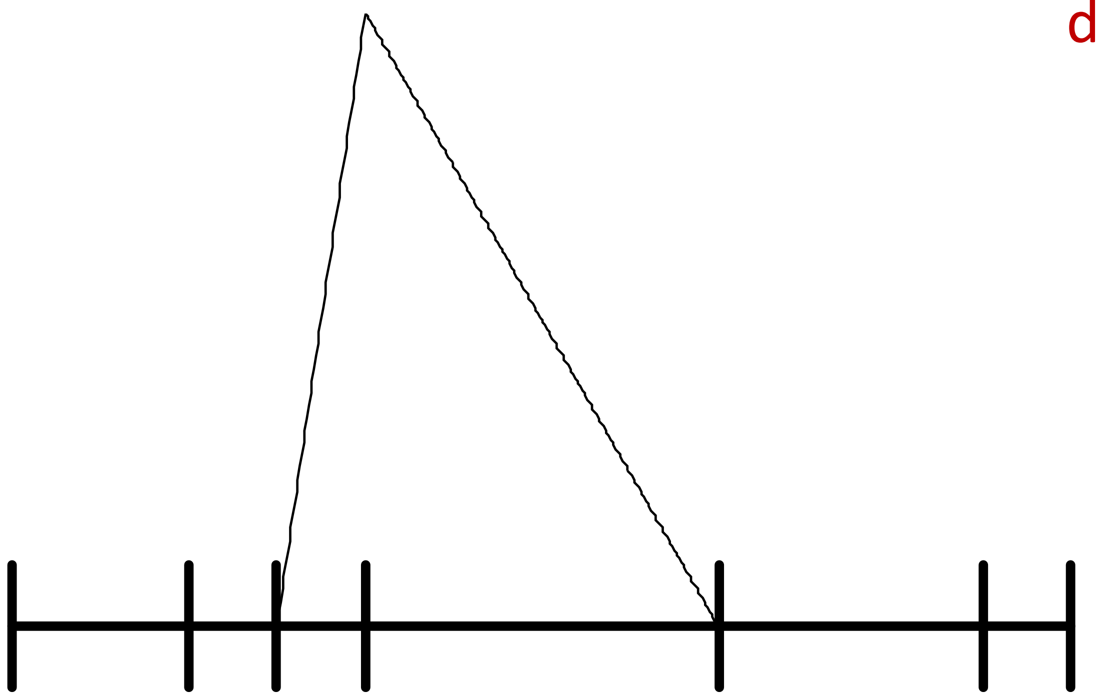
B-Spline Basis Functions

$$N_i^0(t) = \begin{cases} 1 & t_{i-1} \le t < t_i \\ 0 & o.w. \end{cases}$$

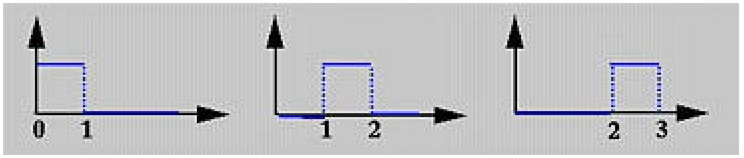
$$N_i^n(t) = \frac{t - t_{i-1}}{t_{i+n-1} - t_{i-1}} N_i^{n-1}(t) + \frac{t_{n+i} - t}{t_{i+n} - t_i} N_{i+1}^{n-1}(t)$$

de Boor recurrence

$$N_4^1(t) =$$

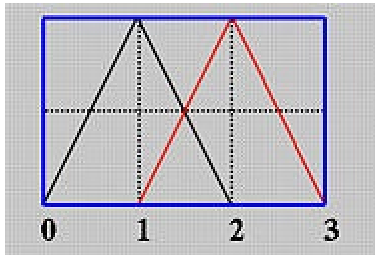


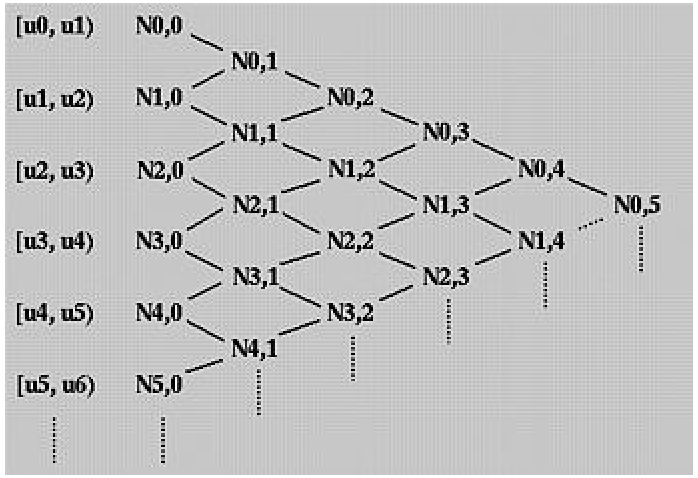
Uniform Case



$$N_{0,1}(u) = \frac{u - u_0}{u_1 - u_0} N_{0,0}(u) + \frac{u_2 - u}{u_2 - u_1} N_{1,0}(u)$$

$$N_{0,1}(u) = uN_{0,0}(u) + (2 - u)N_{1,0}(u)$$





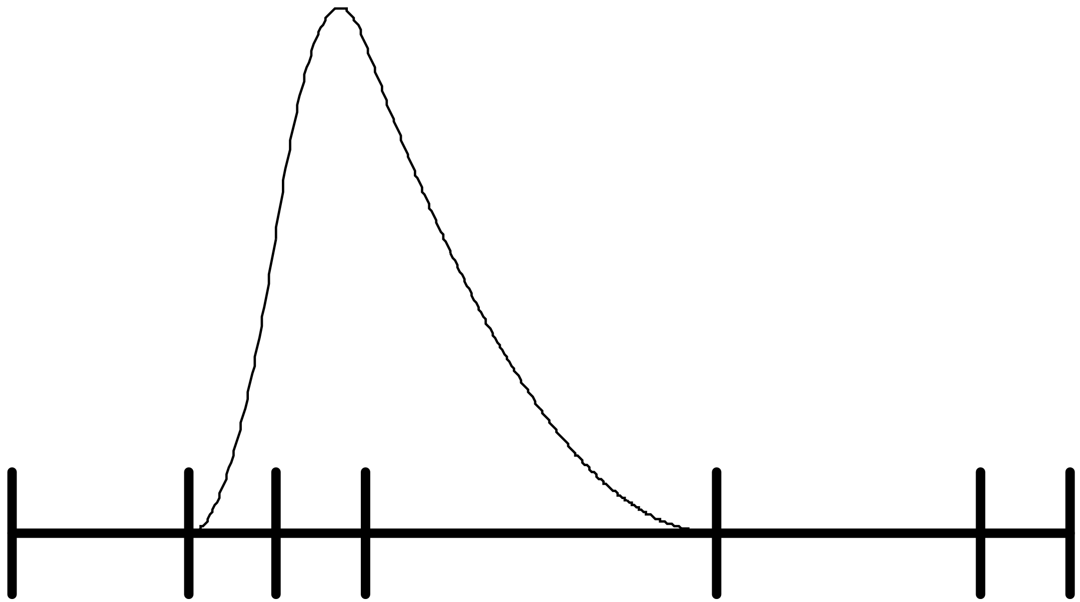
B-Spline Basis Functions

$$N_i^0(t)=\begin{cases}1 & t_{i-1}\le t<t_i\\ 0 & o.w.\end{cases}$$

$$N_i^n(t)=\frac{t-t_{i-1}}{t_{i+n-1}-t_{i-1}}N_i^{n-1}(t)+\frac{t_{n+i}-t}{t_{i+n}-t_i}N_{i+1}^{n-1}(t)$$

de Boor recurrence

$$N_3^2(t)=$$



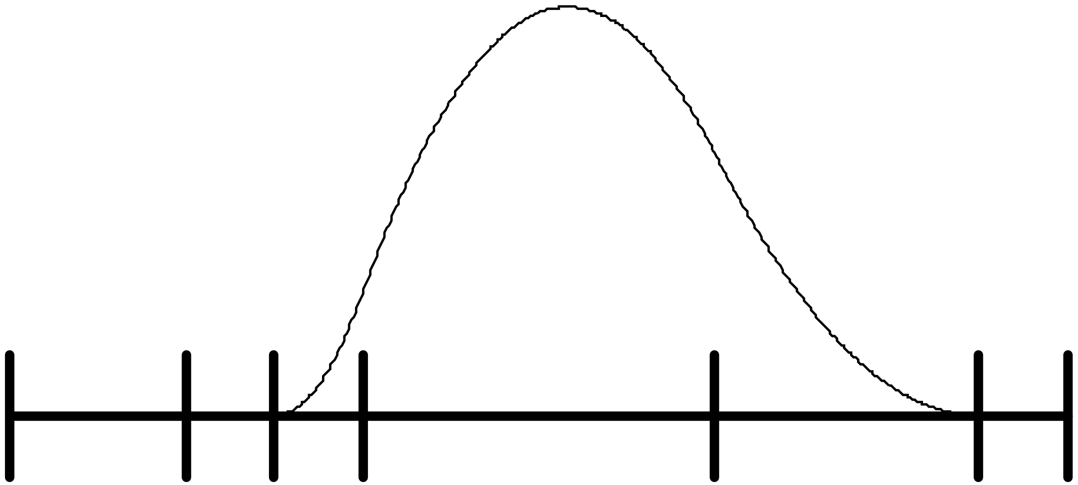
B-Spline Basis Functions

$$N_i^0(t) = \begin{cases} 1 & t_{i-1} \le t < t_i \\ 0 & o.w. \end{cases}$$

$$N_i^n(t) = \frac{t - t_{i-1}}{t_{i+n-1} - t_{i-1}} N_i^{n-1}(t) + \frac{t_{n+i} - t}{t_{i+n} - t_i} N_{i+1}^{n-1}(t)$$

de Boor recurrence

$$N_4^2(t) =$$



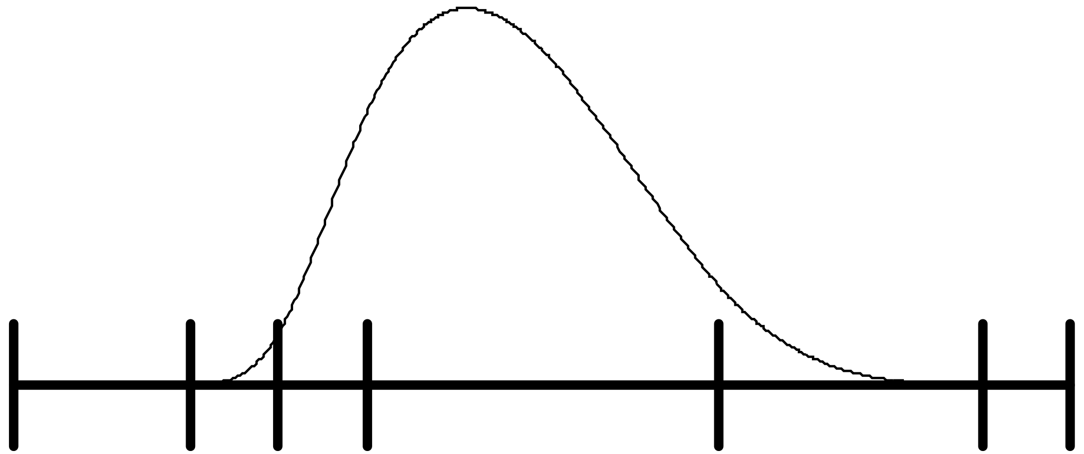
B-Spline Basis Functions

$$N_i^0(t) = \begin{cases} 1 & t_{i-1} \le t < t_i \\ 0 & o.w. \end{cases}$$

$$N_i^n(t) = \frac{t - t_{i-1}}{t_{i+n-1} - t_{i-1}} N_i^{n-1}(t) + \frac{t_{n+i} - t}{t_{i+n} - t_i} N_{i+1}^{n-1}(t)$$

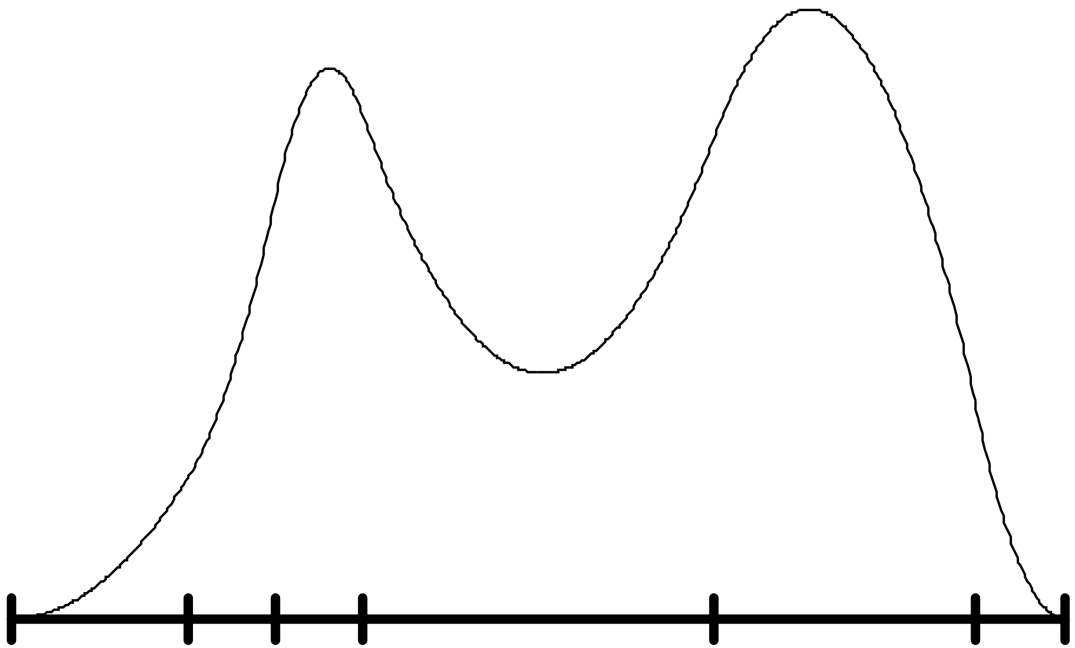
de Boor recurrence

$$N_3^3(t) =$$

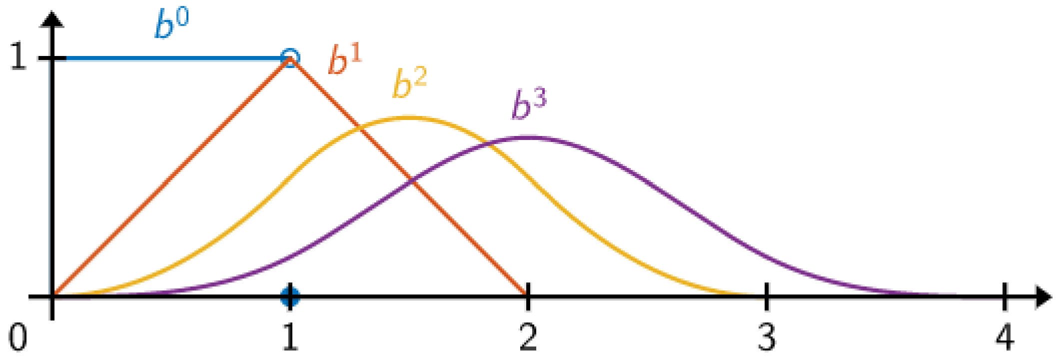


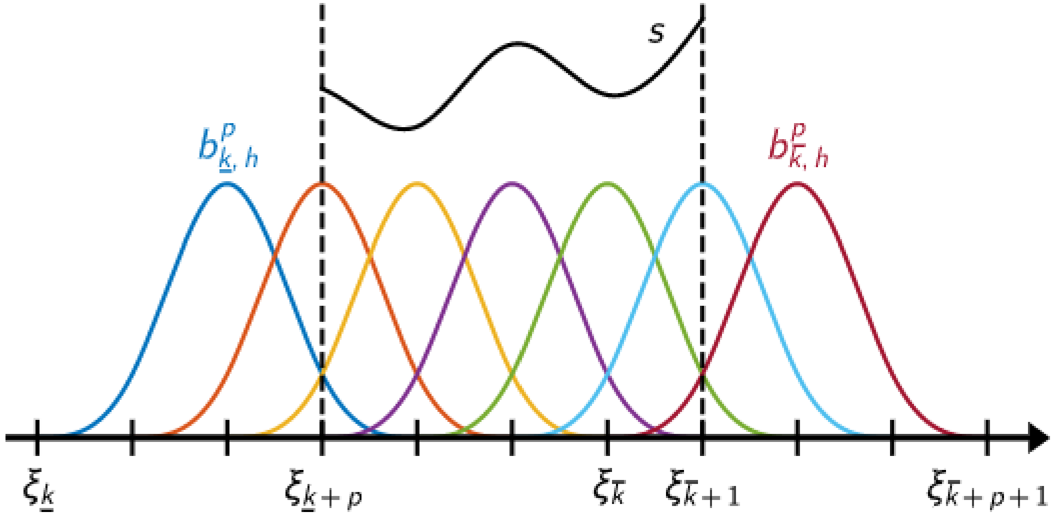
B-Spline Curves

$$p(t) = \sum_{i=1}^{m} p_i N_i^n(t)$$



Uniform B-splines

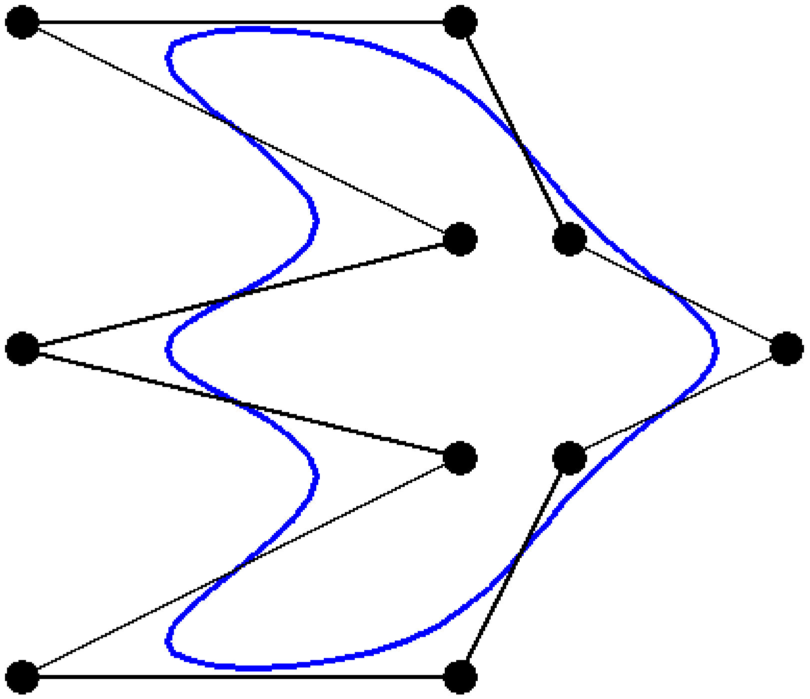




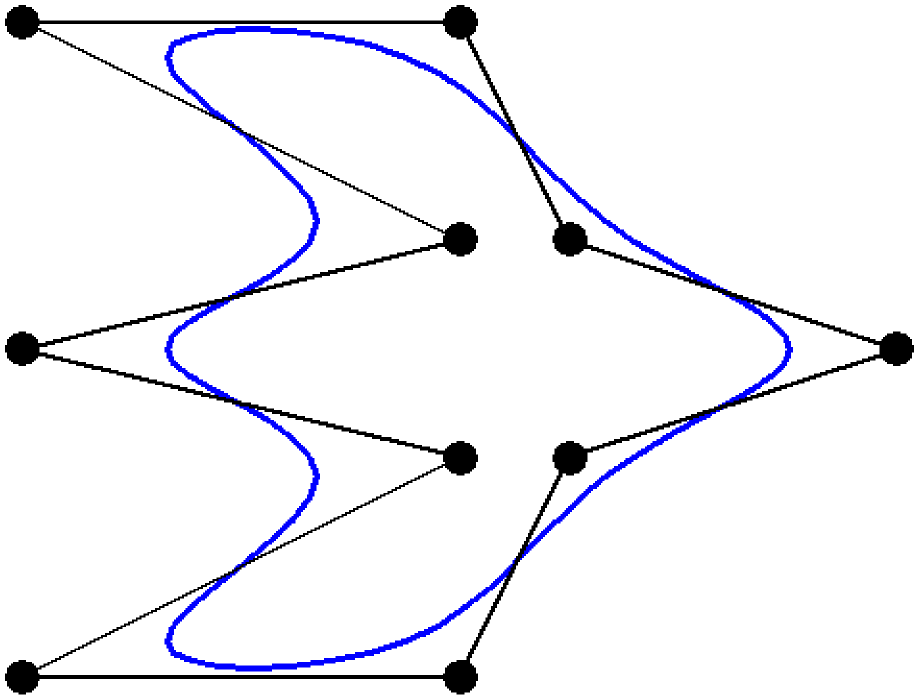
B-spline Properties

- Piecewise polynomial
- C^{n-u} continuity at knots of multiplicity u
- Compact support
- Non-negativity implies local convex hull property
- Variation diminishing

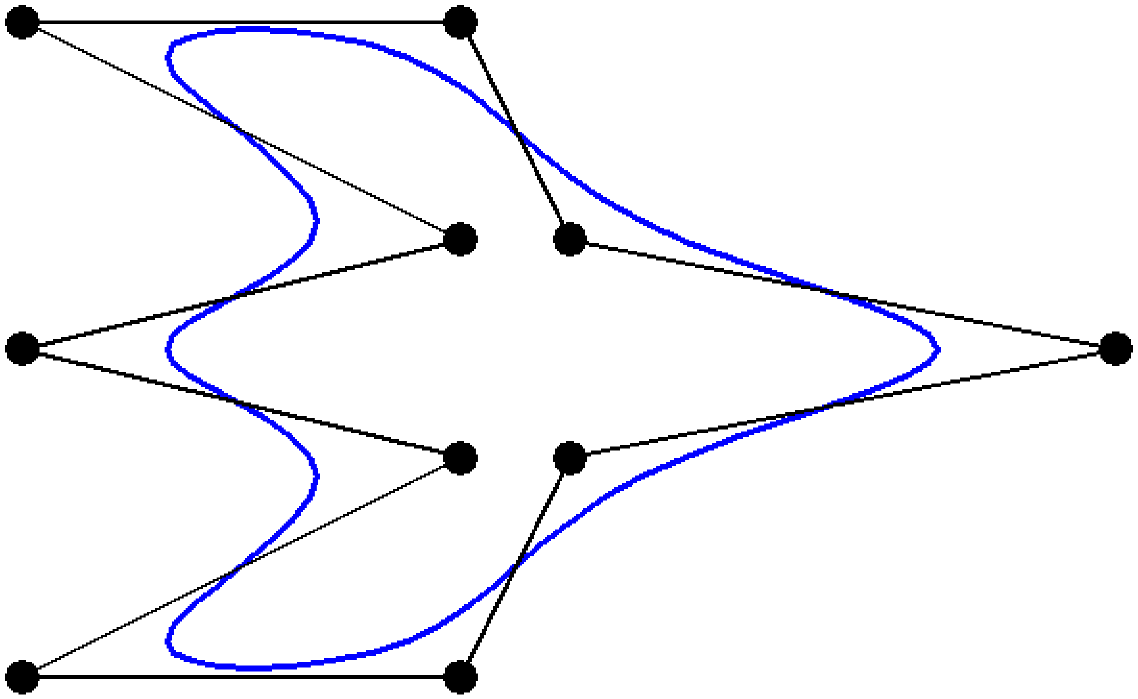
B-spline Curve Example



B-spline Curve Example



B-spline Curve Example



Whiteboard

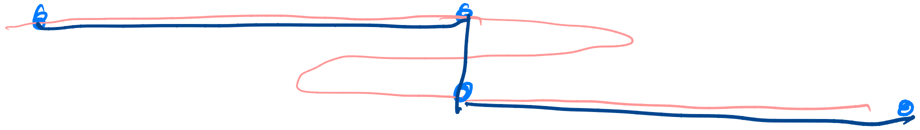
Desiderata for Splines

	B-Splines	Natural
– $C^{(2)}$ continuity	✓	✓
– Local control	✓	✗
– Interpolation	✗	✓

Parametrize

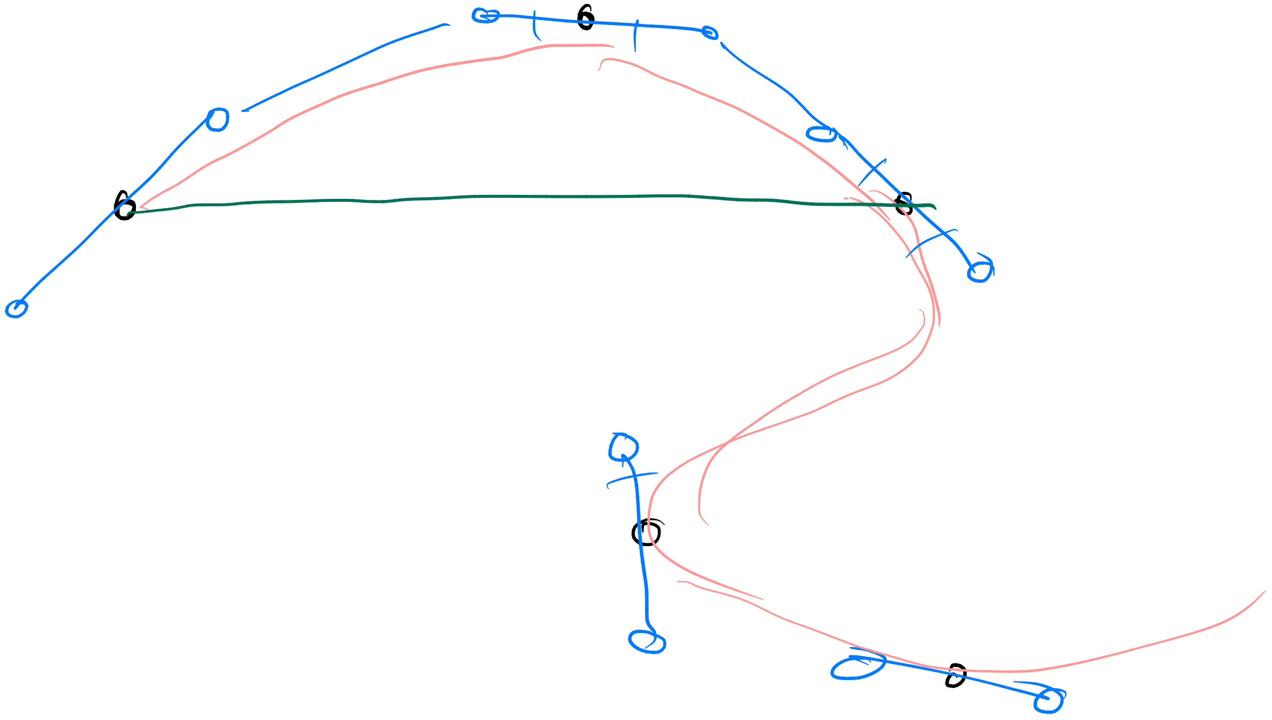
"Chord length"

parametrization



B-splines

Catmull-Rom



That’s All