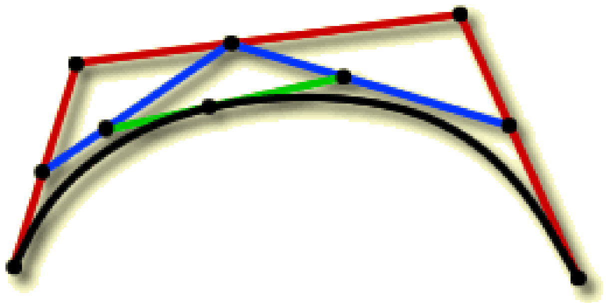
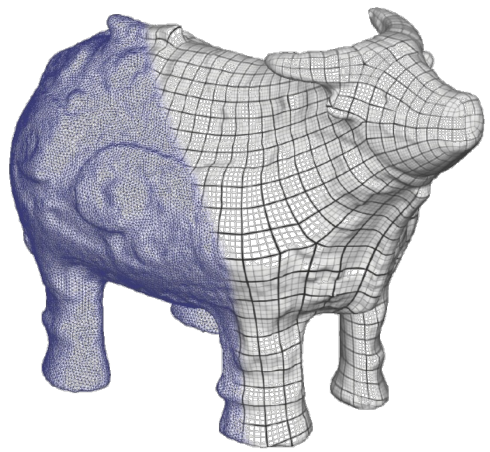
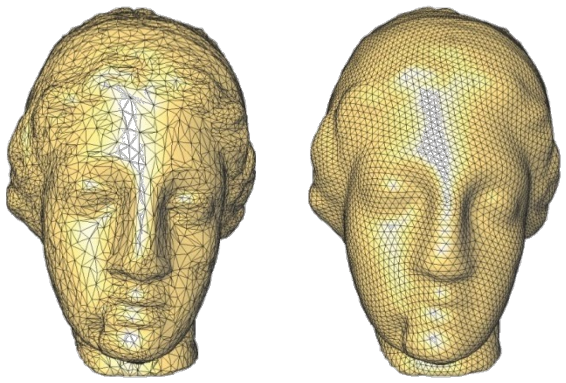
CS348a: Geometric Modeling and Processing







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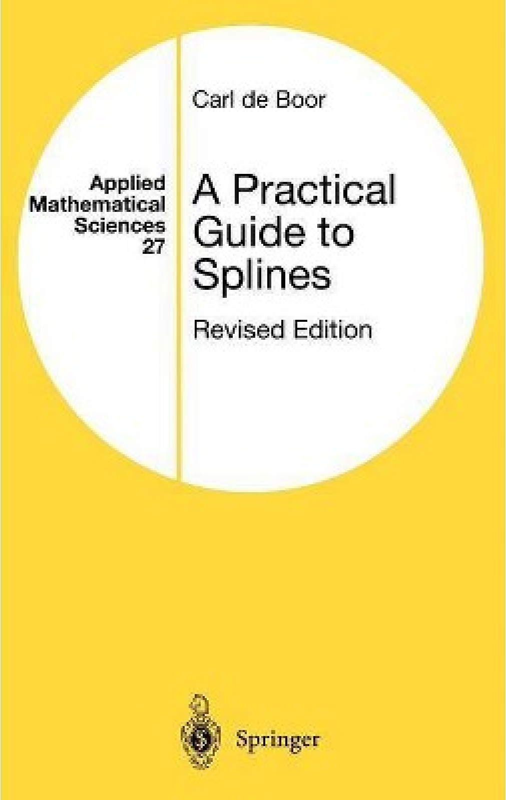
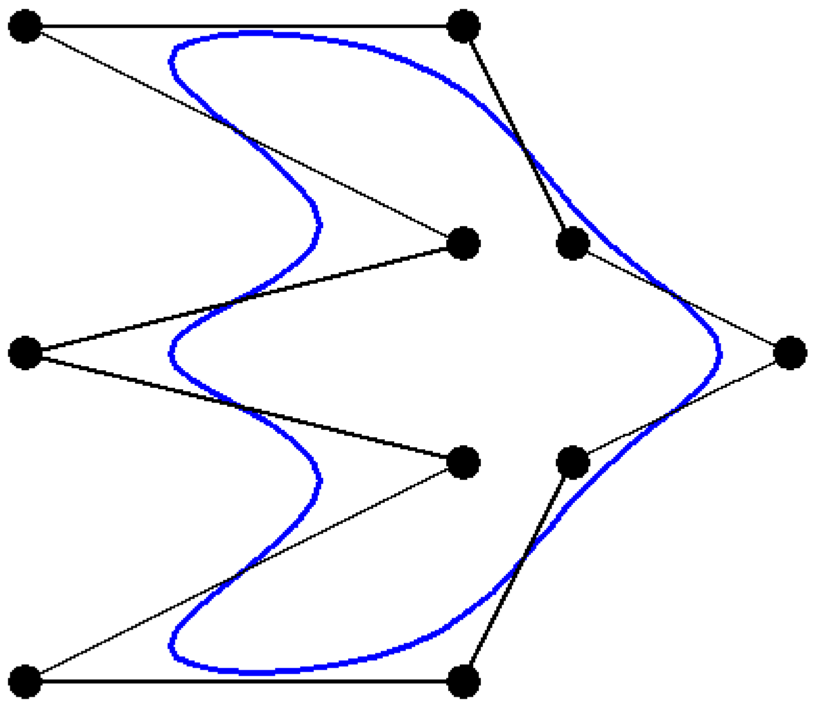
Last Time:
Splines and B-Splines

Carl de Boor


Carl de Boor
Applied
Mathematical
Sciences
27
A Practical
Guide to
Splines
Revised Edition
Springer

Cubic B-spline Curve Example

- Local Control
- C^2 continuity
- Non-interpolating



Cubic B-spline Curve Example

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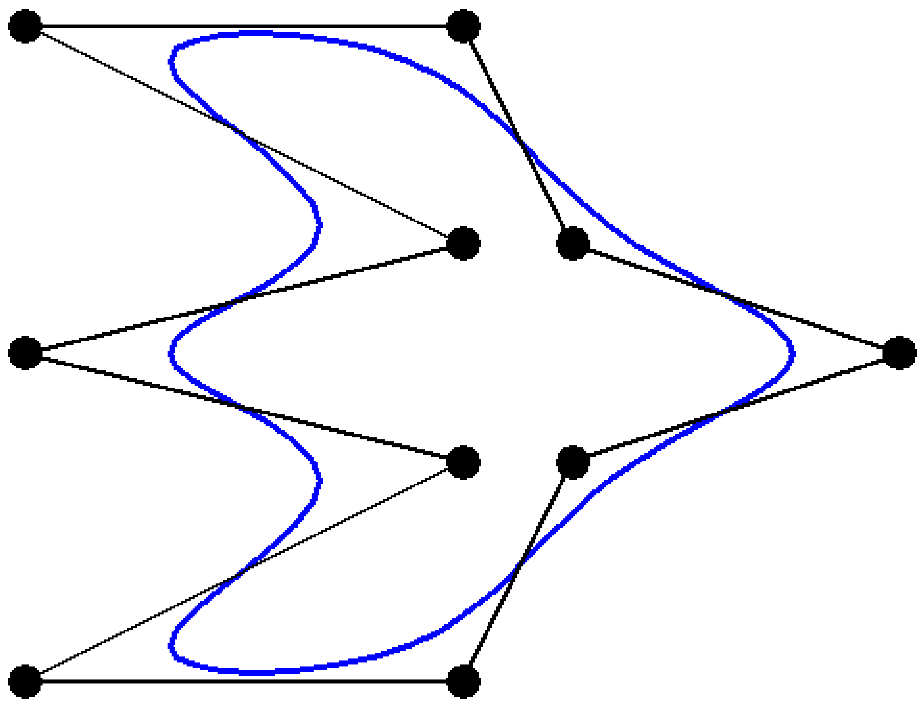

Cubic B-spline Curve Example

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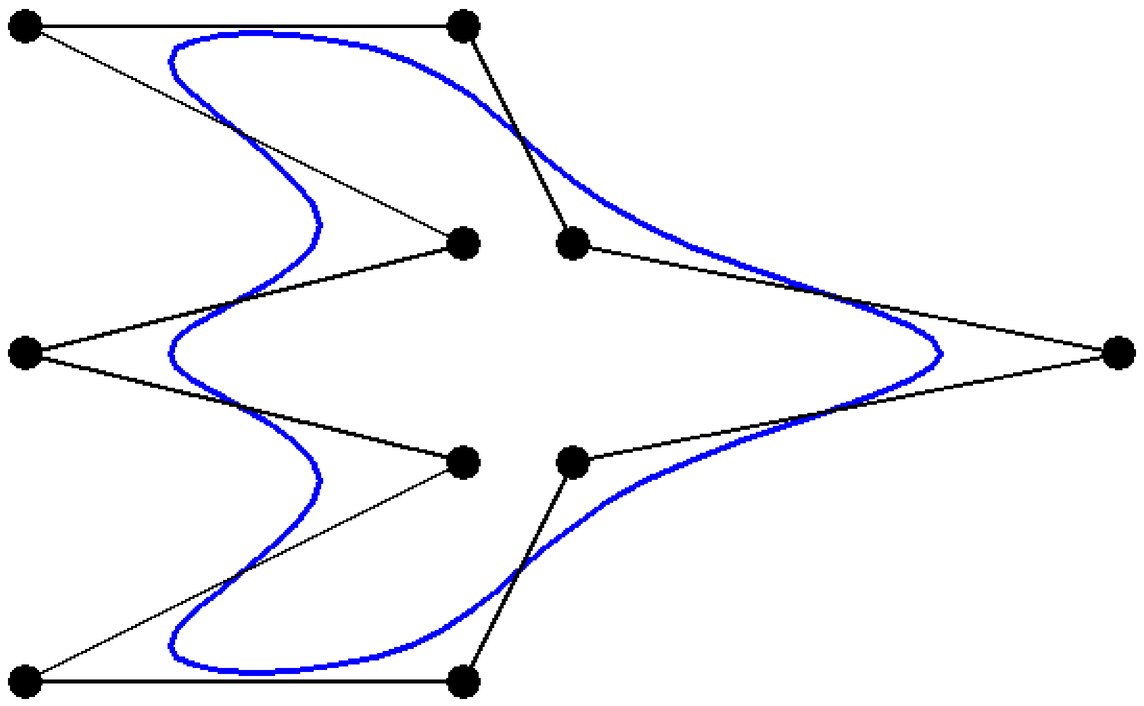


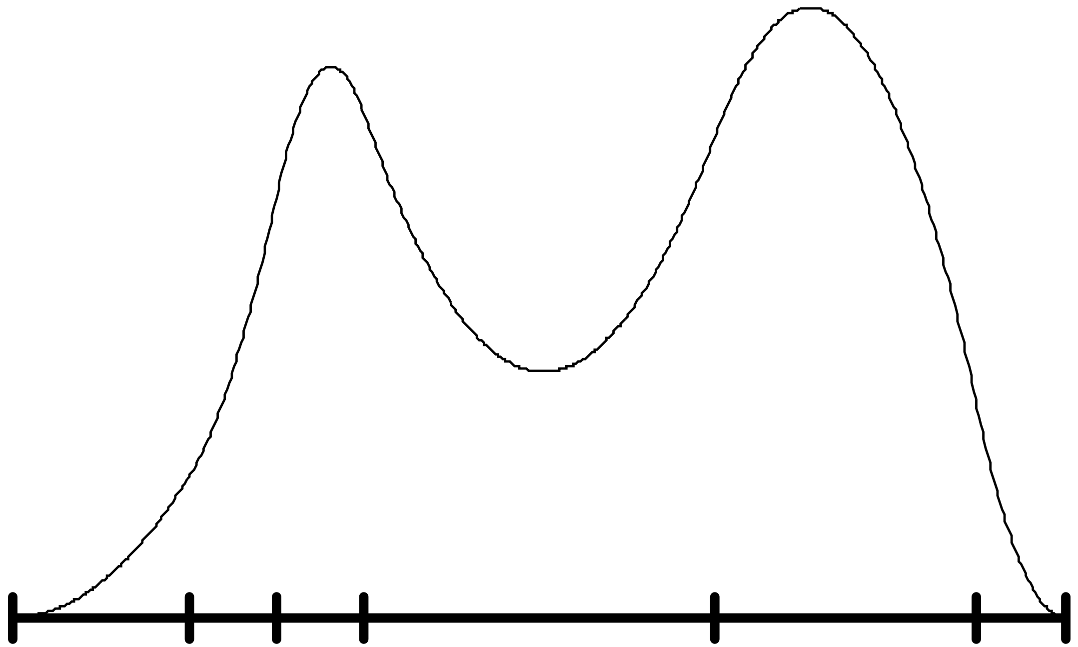
B-Spline Curves

$$p(t) = \sum_{i=1}^{m} p_i N_i^n(t)$$

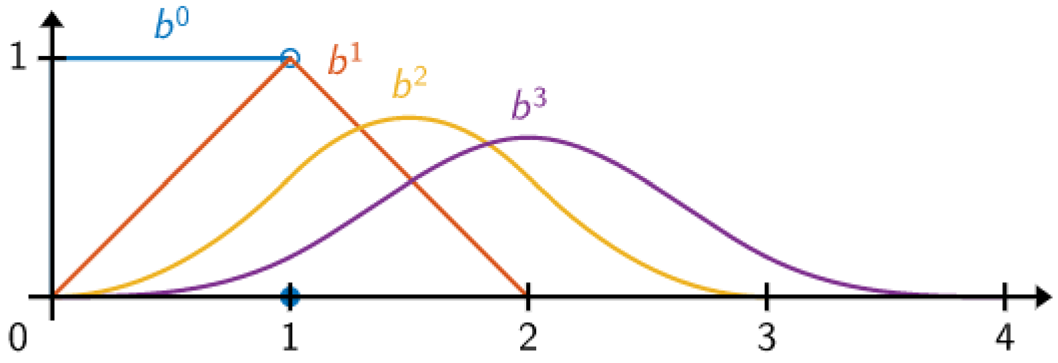
i control point index

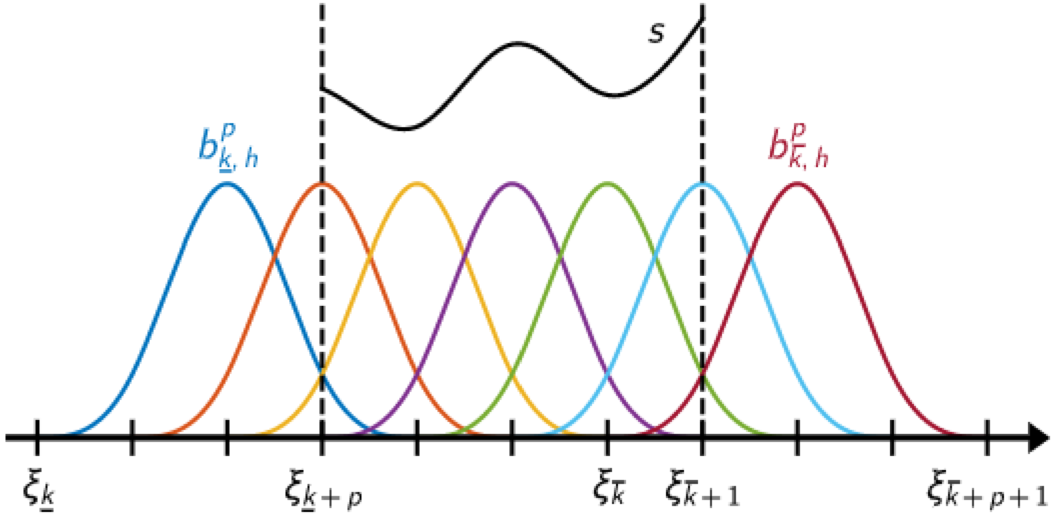
n degree

[order = degree + 1]



Uniform B-splines



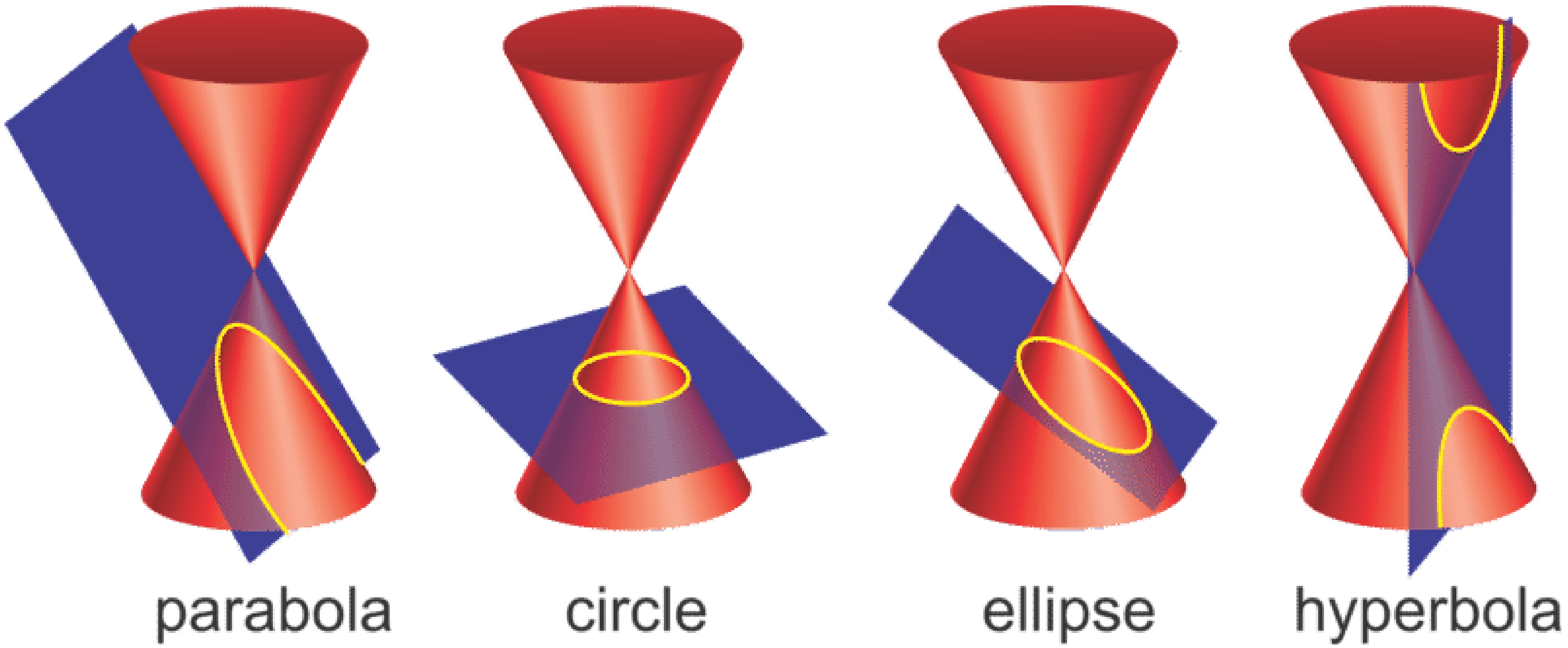


B-spline Properties

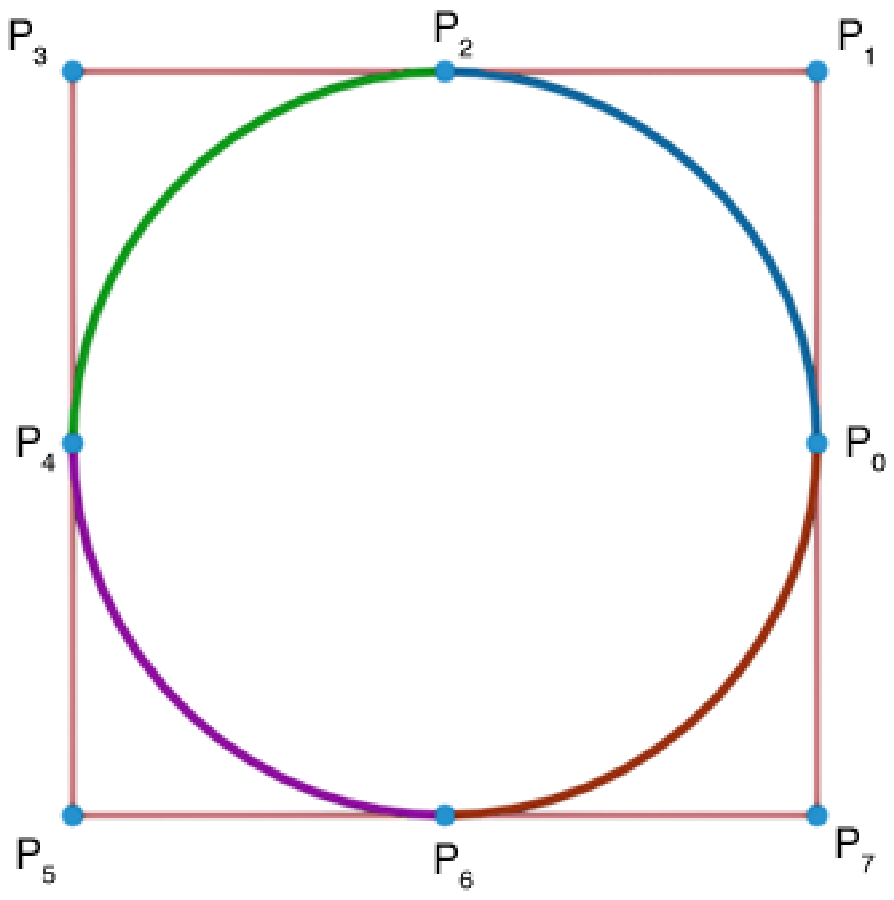
- Piecewise polynomial
- C^{n-u} continuity at knots of multiplicity *u*
- Compact support
- Non-negativity implies local convex hull property
- Variation diminishing

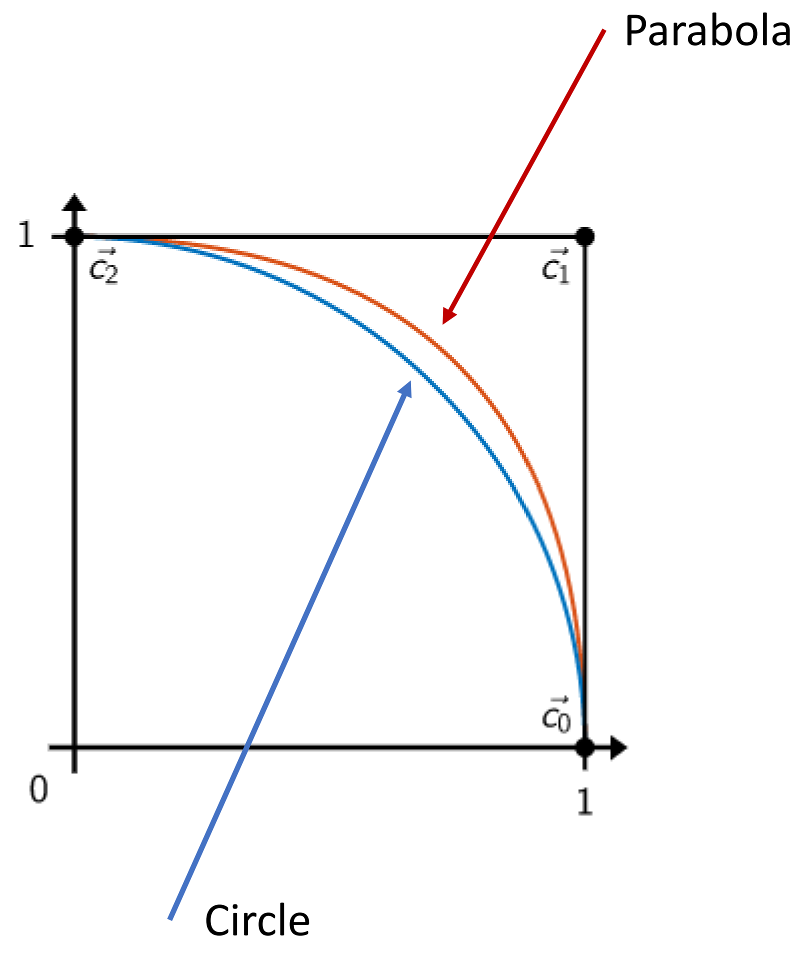
Today:
Rational Curves

We Want Conic Sections!

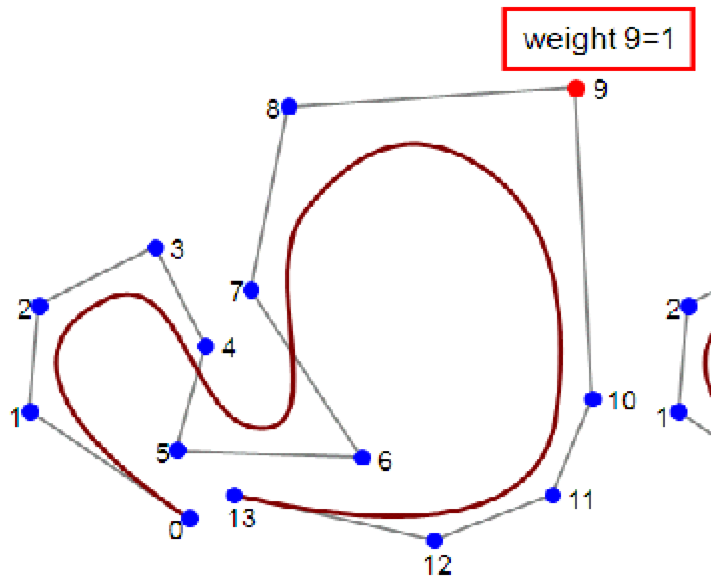


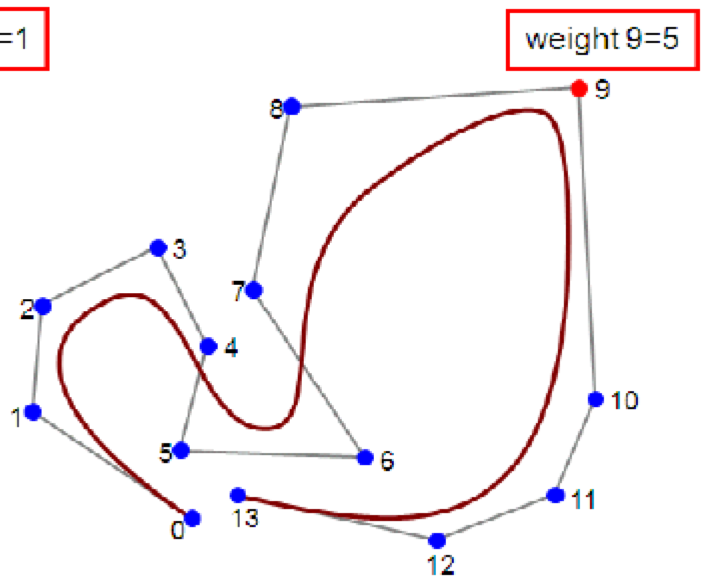
A Circle as a B-Spline

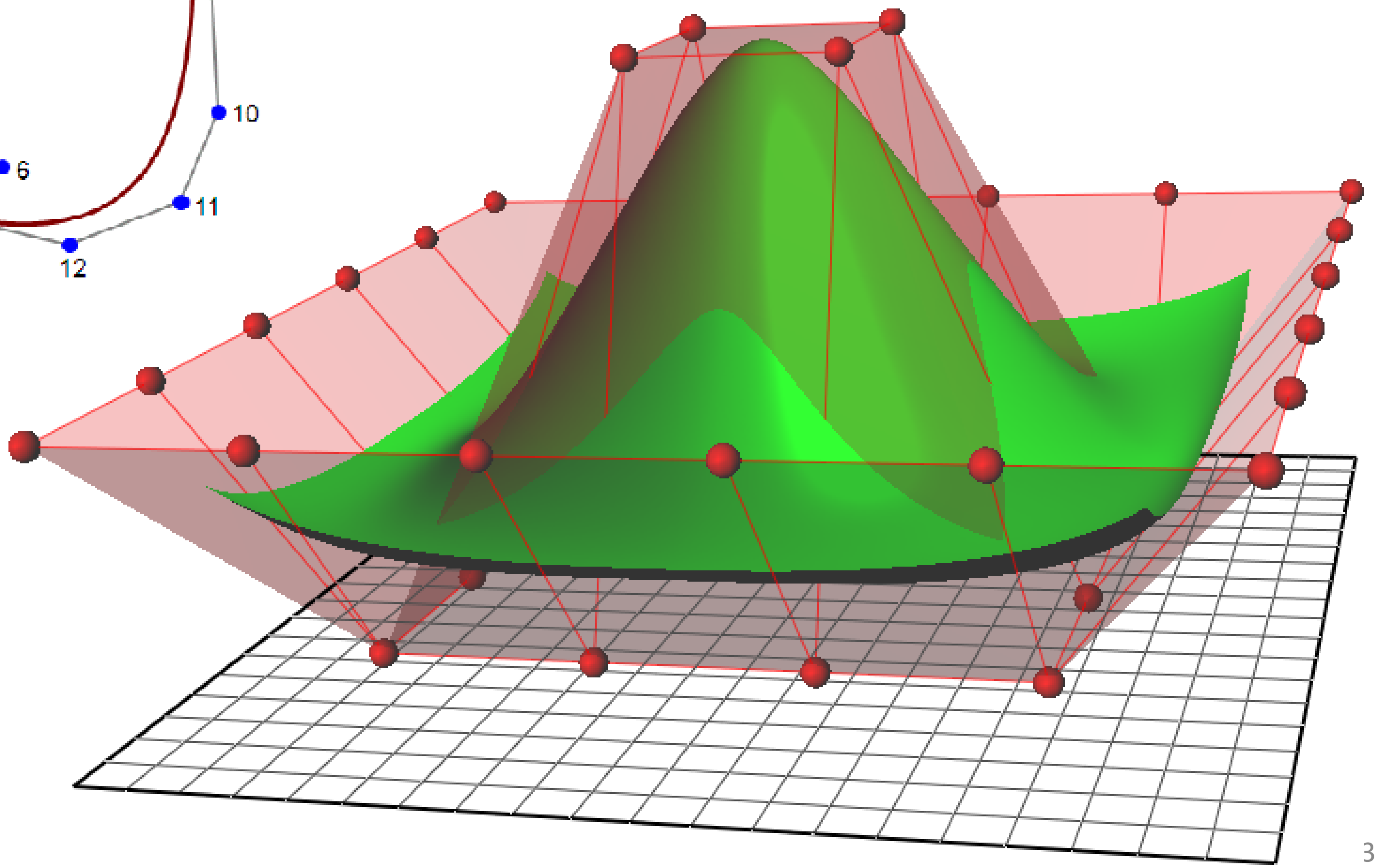




NURBS – Non-Uniform Rational B-Splines



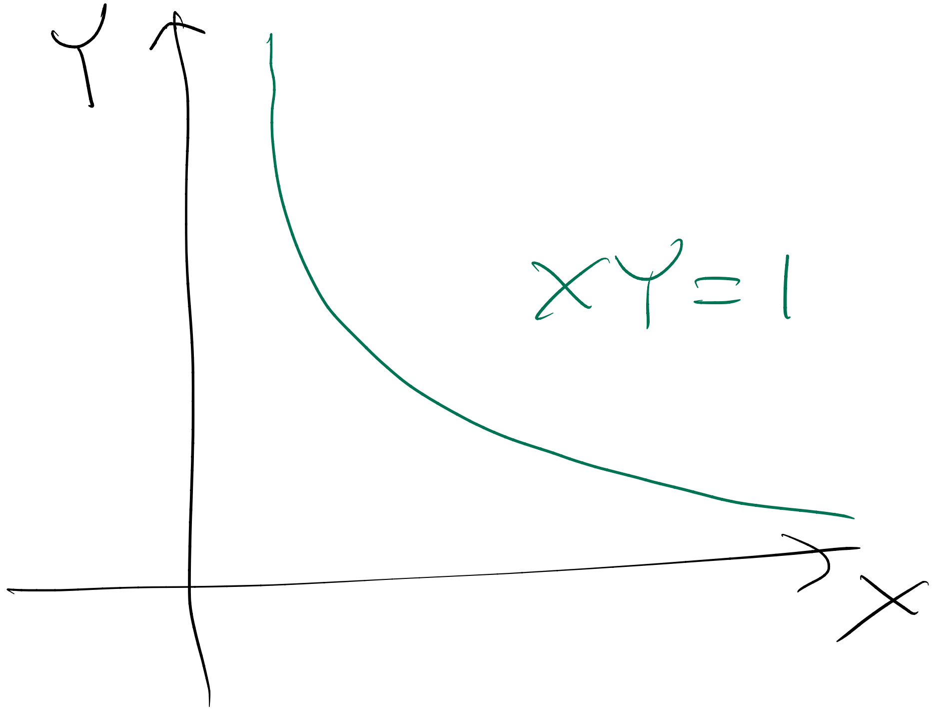




Whiteboard

Polynomials → Rationals

Hyperbola



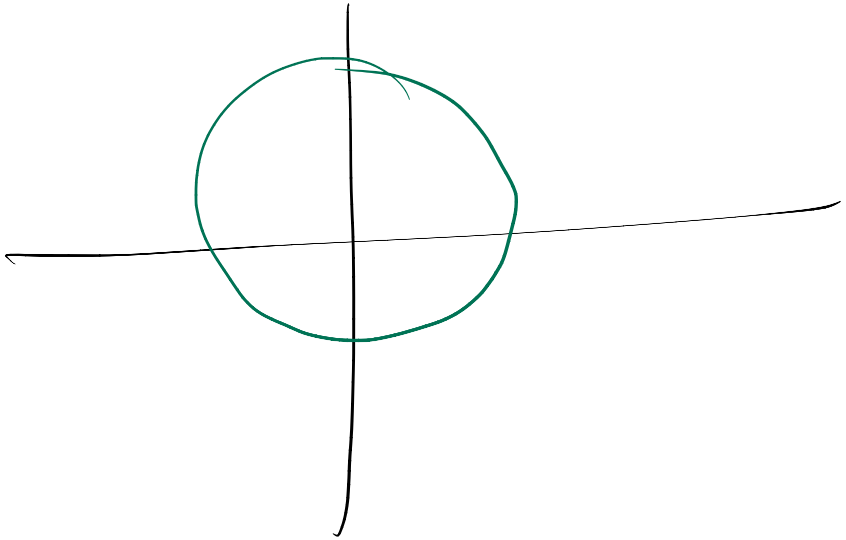
parametrized

$$X(T) = T$$
$$Y(T) = \frac{1}{T}$$

$$X(T) = \frac{T^2}{T}$$
$$Y(T) = \frac{1}{T}$$

$T, T^2, 1$

$$X^2 + Y^2 = 1$$



$$X(T) = \frac{1-T^2}{1+T^2}$$

$$Y(T) = \frac{2T}{1+T^2}$$

$$X(T) = \frac{x(T)}{w(T)}$$

$$Y(T) = \frac{y(T)}{w(T)}$$

Poly: $(x(T), y(T))$

Rational $(w(T), x(T), y(T))$

↑ homogeneous

Poly $\mathbb{R} \to \mathbb{A}^2$

Rational $\mathbb{R} \to \mathbb{P}^2$

$F(T) = T^2/T$

$Y(T) = 1/T$

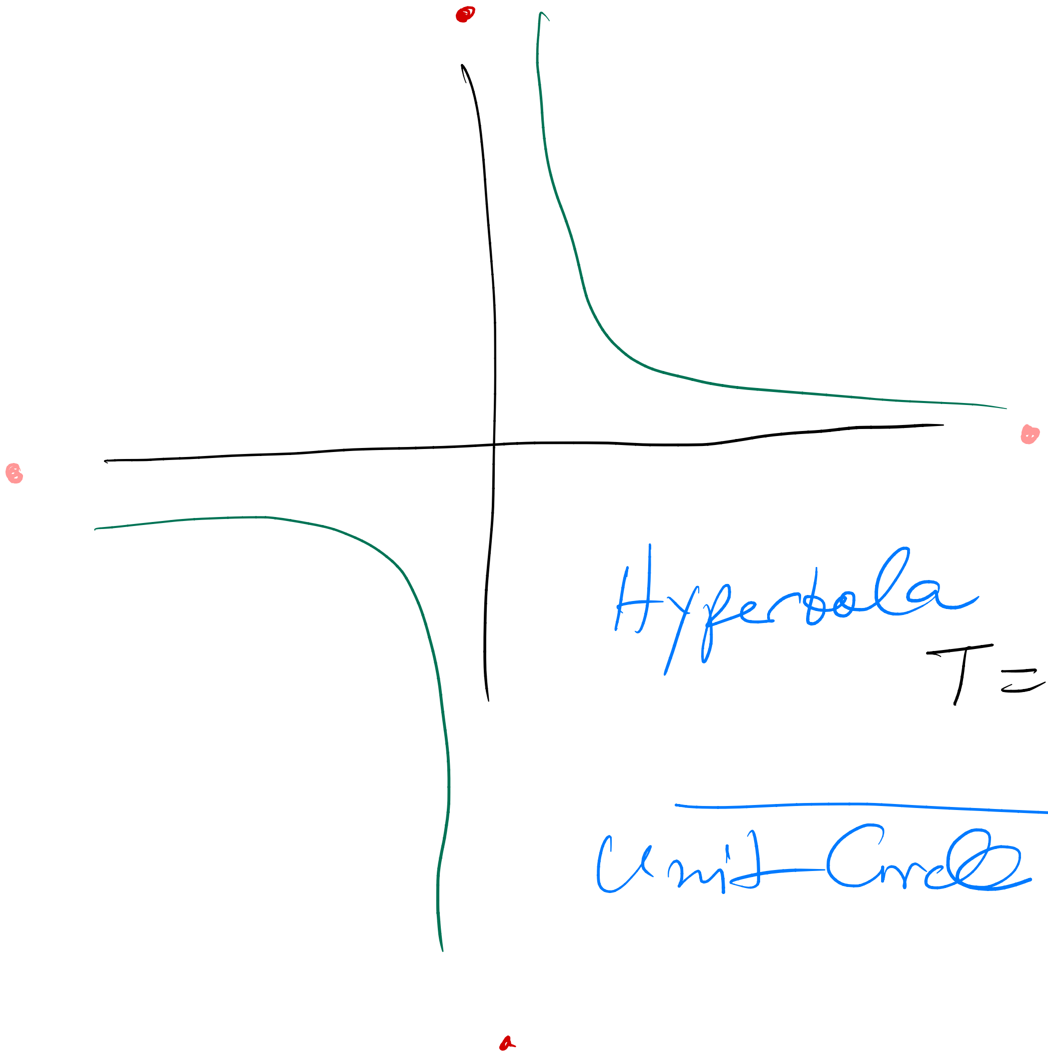
$\to [T, T^2, 1]$ $\quad$ $T = 0$ $\quad$ $[0, 0, 1]$

$=$

$[1, T, \frac{1}{T}]$ $\quad$ $T = \infty$

$=$

$[\frac{1}{T}, 1, \frac{1}{T^2}]$ $\quad$ $[0, 1, 0]$



Hyperbola

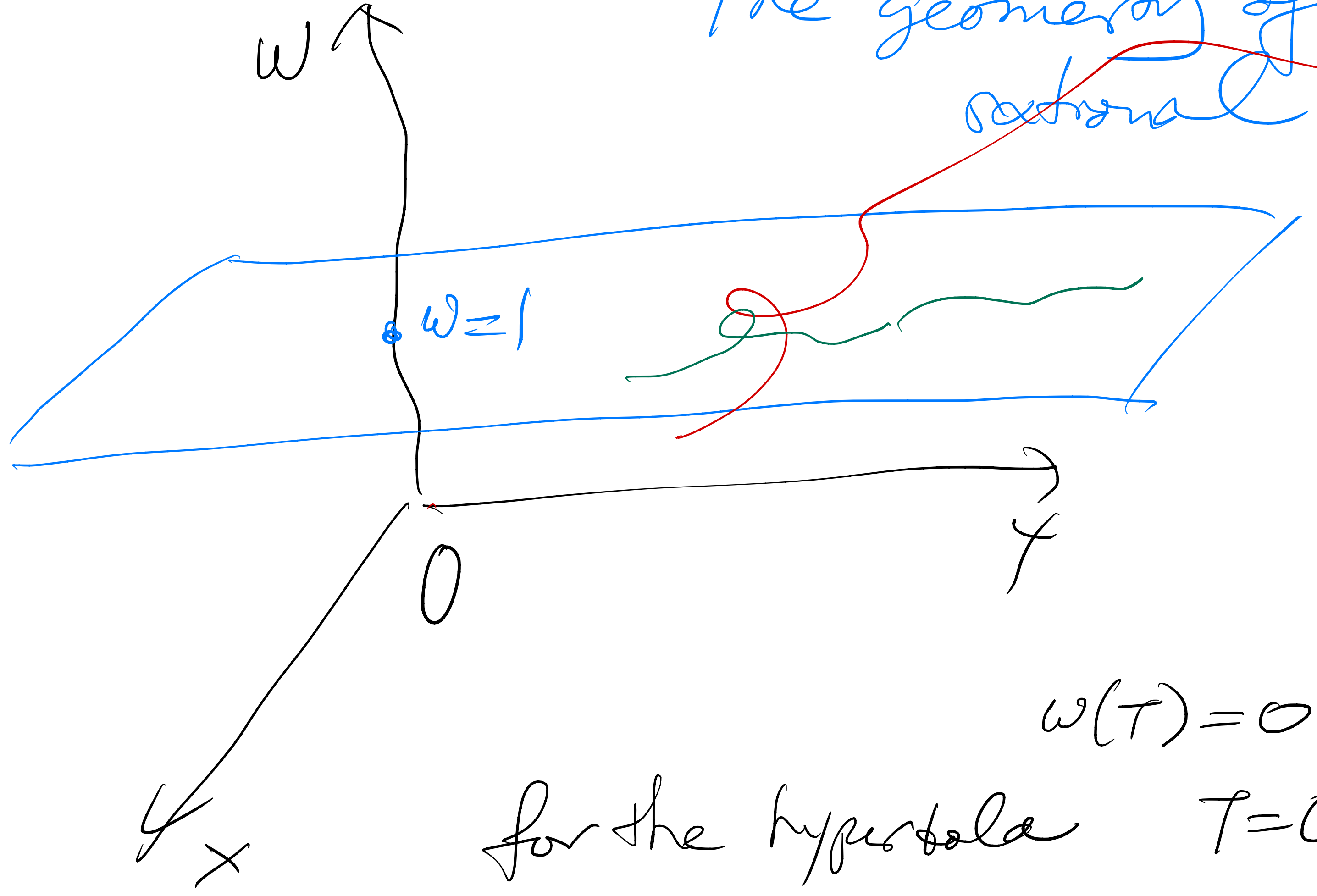
$T = \frac{t}{s}$ $\quad$ $F((s,t)) = [st, t^2, s^2]$

Unit Circle

$F(T) = [1+T^2, 1-T^2, 2T]$

$F((s,t)) = [s^2+t^2, s^2-t^2, 2st]$

The geometry of rational curves



$$F(T) = [w(T), x(T), y(T)]$$

$$\Downarrow$$

$$F(T) = \left[\frac{x(T)}{w(T)}, \frac{y(T)}{w(T)}\right]$$

$$w(T) = 0$$

for the hyperbola

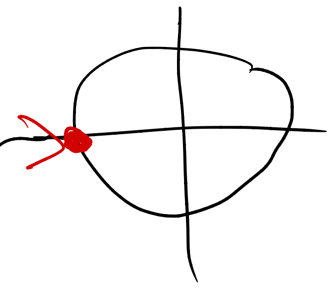
$T=0 \rightarrow [0,0,1]$ $+y\ \infty$

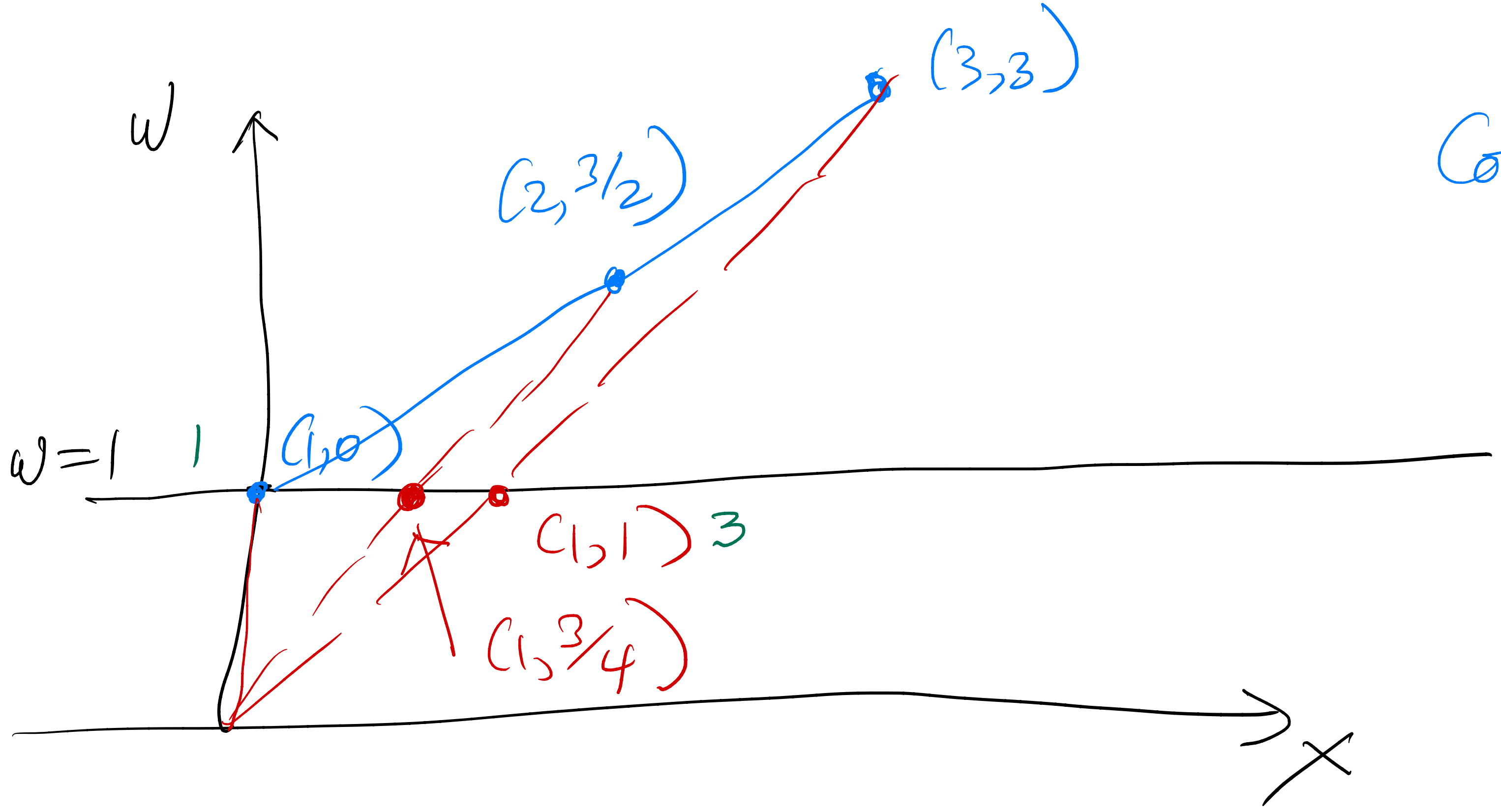
$T=\infty \rightarrow [0,1,0]$ $+x\ \infty$

for the circle

$$\left[\frac{1-T^2}{1+T^2}, \frac{2T}{1+T^2}\right]$$

$T=\infty \rightarrow [1,-1,0]$





Control Pts $\longrightarrow$ Control Sites

$$H(T) = \frac{(1-T)\, w_A A + T\, w_B B}{(1-T)\, w_A + T\, w_B}$$

Weighted control pt interpolations.

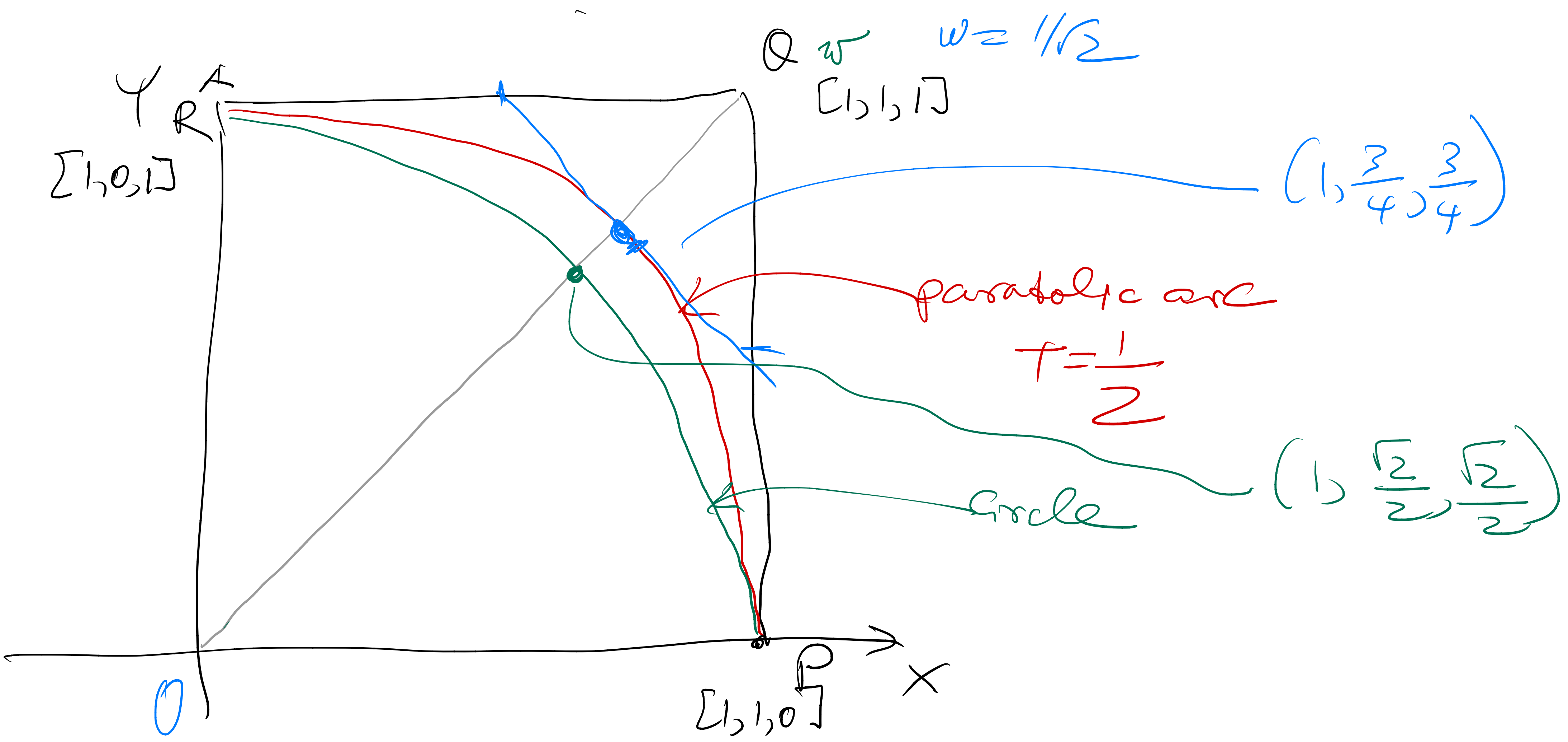
$$A: \quad (1, a_x, a_y) \quad \underline{\underline{u}} \quad \longrightarrow \quad (u, ua_x, ua_y)$$

$$B: \quad (1, b_x, b_y) \quad \underline{\underline{v}} \quad \longrightarrow \quad (v, vb_x, vb_y)$$

$$(1-T)\,(u, ua_x, ua_y) + T\,(v, vb_x, vb_y)$$

$$x(T) = \frac{(1-T)ua_x + Tvb_x}{(1-T)u + Tv}$$

$$y(T) = \frac{(1-T)ua_y + Tvb_y}{(1-T)u + Tv}$$



$$F(T) = \frac{(1-T)^2 P + 2T(1-T) Q\omega + T^2 R}{(1-T)^2 + 2T(1-T)\omega + T^2}$$

$T = 1/2 \qquad \left[1, \frac{2\omega+1}{2\omega+2}, \frac{2\omega+1}{2\omega+2}\right] = \left[1, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$

$$\frac{2\omega+1}{2\omega+2} = \frac{\sqrt{2}}{2} \qquad \omega = 1/\sqrt{2} = \sqrt{2}/2 < 3/4$$

$$F(T) := \frac{(1-T)^2 P + 2T(1-T)\theta\omega\lambda + T^2 R \lambda^2}{(1-T)^2 + 2T(1-T)\omega\lambda + T^2\lambda^2}$$

$$= \frac{P + \frac{2T\lambda}{1-T}\theta\omega + \frac{T^2\lambda^2}{(1-T)^2} R}{1 + \frac{2T\lambda}{1-T}\omega + \frac{T^2\lambda^2}{(1-T)^2}}$$

S

$$\frac{S}{1-S} = \frac{T\lambda}{1-T}$$

$$S = \lambda T / 1-(1-\lambda)T$$

$$\frac{P + 2\frac{S}{1-S}\theta\omega + \frac{S^2}{(1-S)^2} R}{1 + 2\frac{S}{1-S}\omega + \frac{S^2}{(1-S)^2}}$$

$$\frac{(1-S)^2 P + 2S(1-S)\theta\omega + S^2 R}{(1-S)^2 + 2S(1-S)\omega + S^2}$$

$\lambda = \sqrt{2}$

$$\frac{(1-T)^2 P + 2T(1-T) Q + 2T^2 R}{\cancel{(1-T)^2 + 2T(1-T) + 2T^2}\ \ 1+T^2}$$

$P = [1,1,0]$, $Q = [1,1,1]$, $R = [1,0,1]$

$$x(T) = \frac{1-T^2}{1+T^2} \qquad y(T) = \frac{2T}{1+T^2}$$

Bézier Control Sites

	P	Q	R
Rational	$P\ \omega_P$	$Q\ \omega_Q$	$R\ \omega_R$
Unif Scaling	$P\ \lambda\omega_P$	$Q\ \lambda\omega_Q$	$R\ \lambda\omega_R$
Geometric Scaling	$P\ \omega_P$	$Q\ \lambda\omega_Q$	$R\ \lambda^2\omega_R$

$\omega_P > 0$

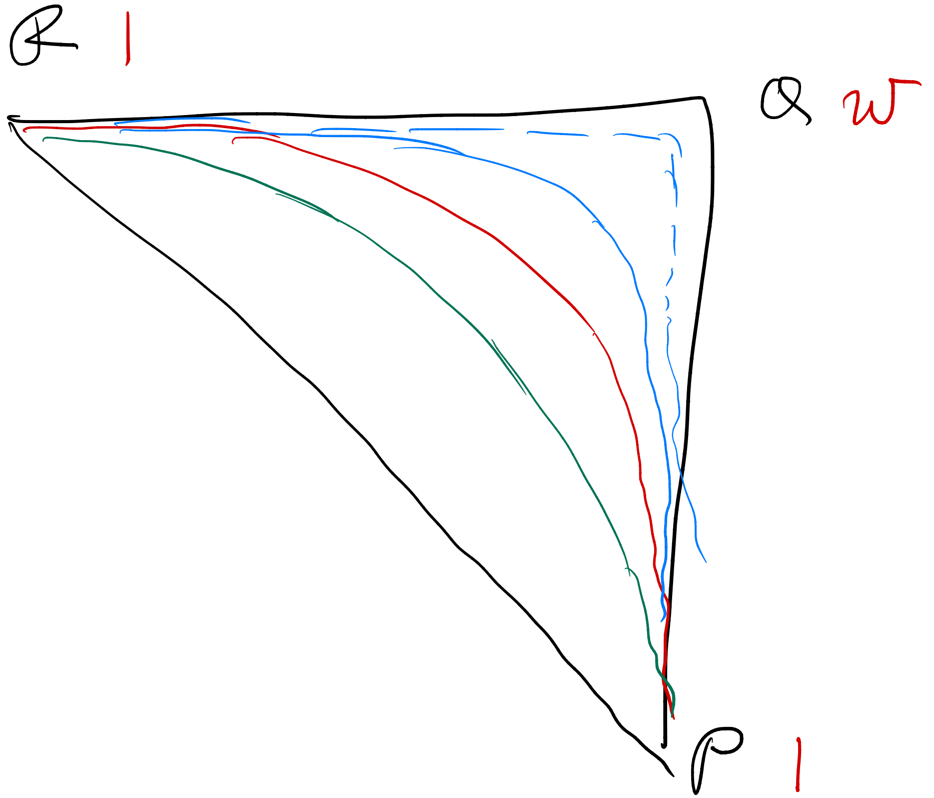
$\omega_Q > 0$

$\omega_P = 1$

$$\frac{\omega_R}{\omega_P}$$

$$\lambda = \sqrt{\frac{\omega_P}{\omega_R}}$$

P	Q	R
1	$\frac{w_1}{\sqrt{w_0 w_2}}$	1
1		-1



$w = 1$ → parabola

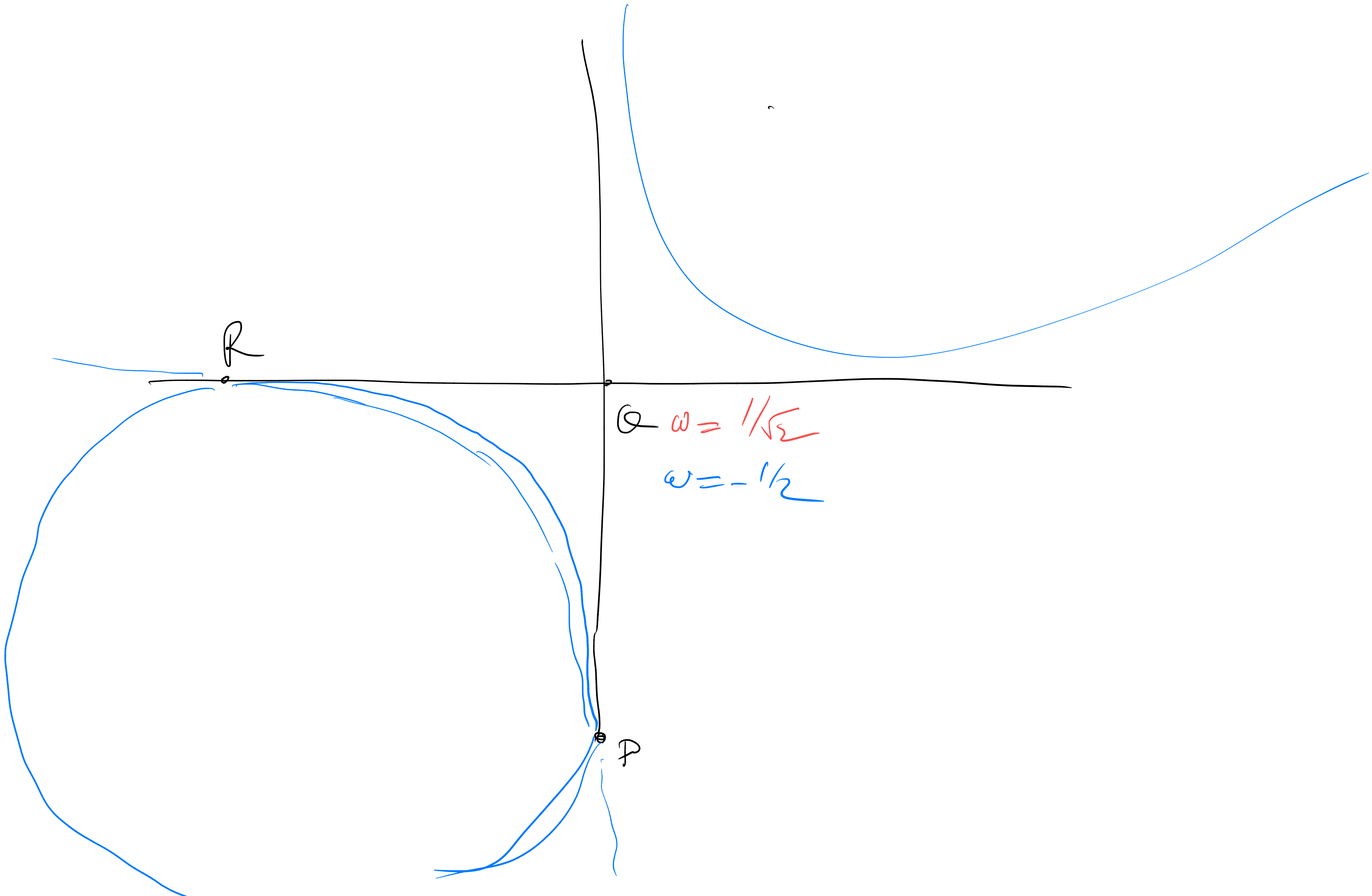
$w \Uparrow 1$ → hyperbolic

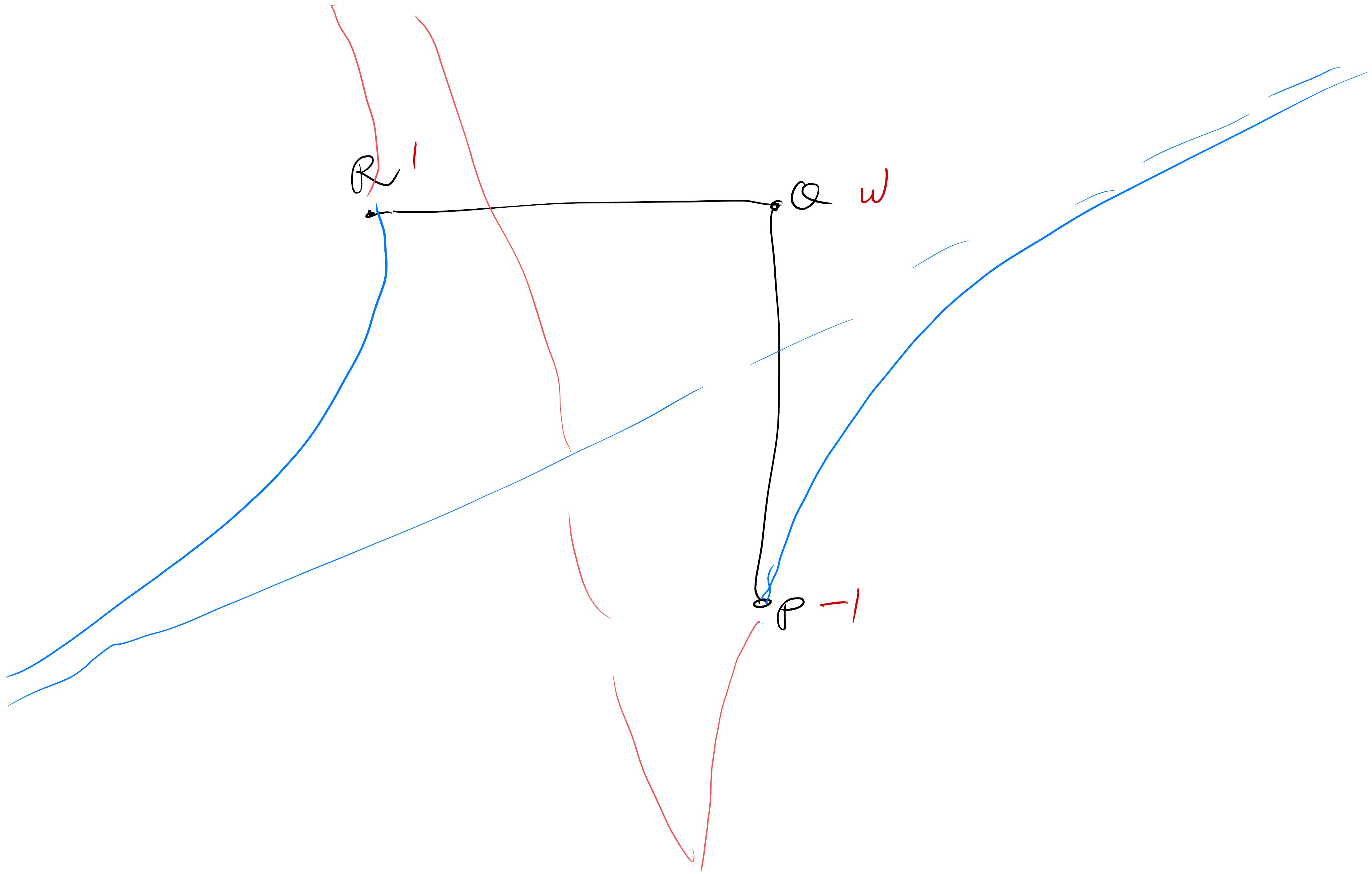
$w \Downarrow$ → ellipse

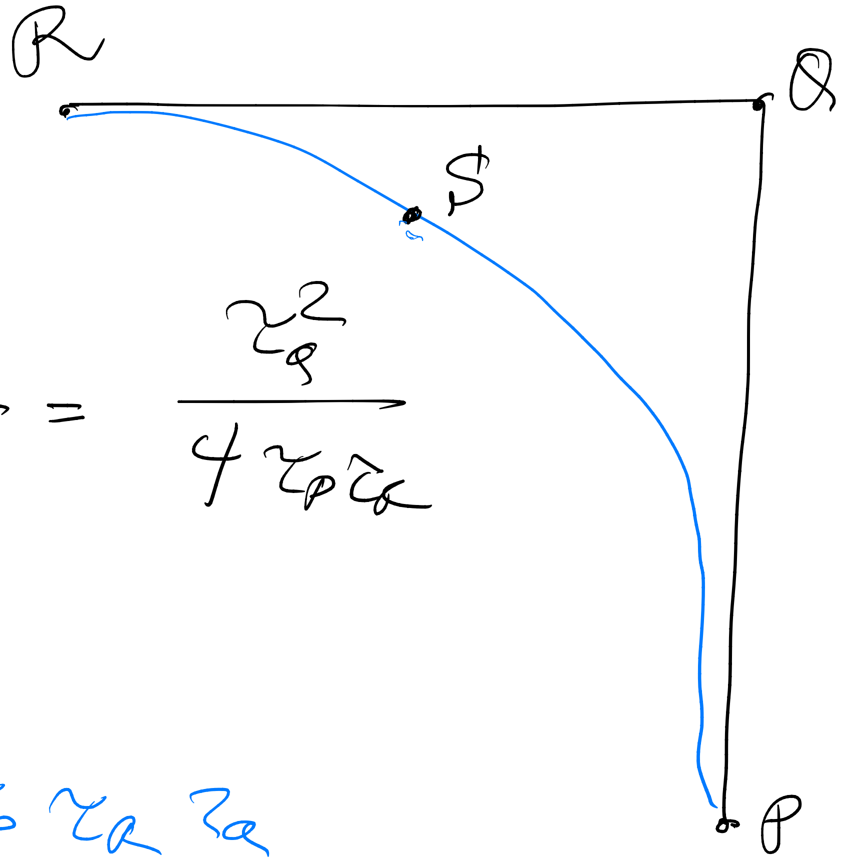
$w = 1/\sqrt{2}$ → circle

$w = 0$ → line

$w < 0$


R
O
P
ω = 1/√2
ω = -1/2


R
1
Q
ω
P
−1



$$\frac{\omega_Q^2}{\omega_P \omega_R} = \frac{\tau_Q^2}{4\tau_P \tau_R}$$

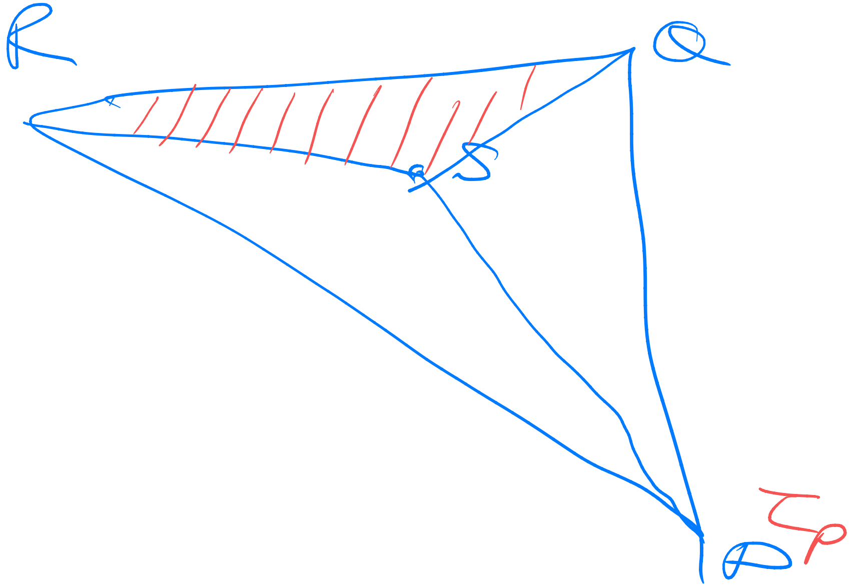
$$S = \tau_P P + \tau_Q Q + \tau_R R$$

$$\tau_P, \tau_Q, \tau_R \geq 0$$

$$\tau_P + \tau_Q + \tau_R = 1$$

$\tau_P \quad \tau_R \quad \tau_Q$

$$\tau_P = \begin{vmatrix} x_R & y_R & 1 \\ x_Q & y_Q & 1 \\ x & y & 1 \end{vmatrix}$$



$$F(T) = (1+T^2, 1-T^2, 2T)$$

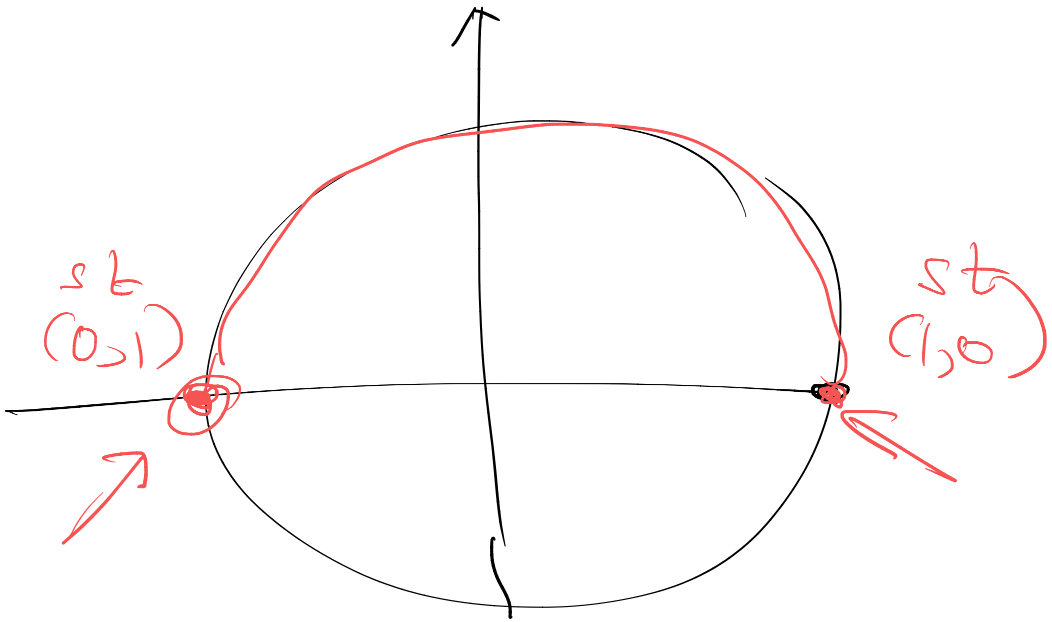
$$F((T_1, T_2)) = (1+T_1T_2, 1-T_1T_2, T_1+T_2)$$

$$F(((s_1,t_1),(s_2,t_2))) = (s_1s_2+t_1t_2, s_1s_2-t_1t_2, s_2t_1+s_1t_2)$$

$$F((1,0),(1,0)) = [1,1,0]$$

$$F((1,0),(0,1)) = [0,0,1]$$

$$F((0,1),(0,1)) = [1,-1,0]$$



That's All