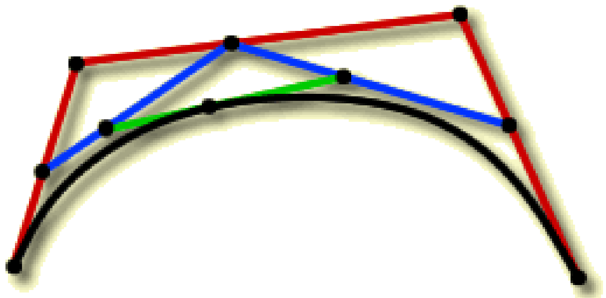
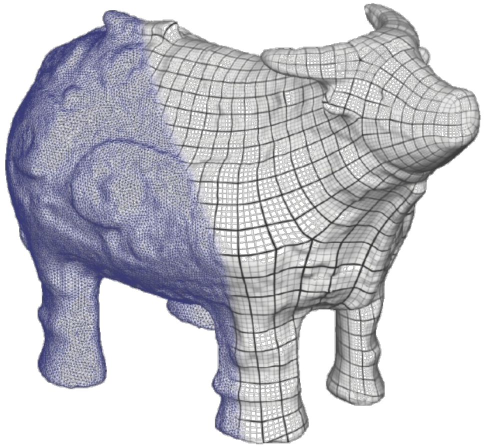
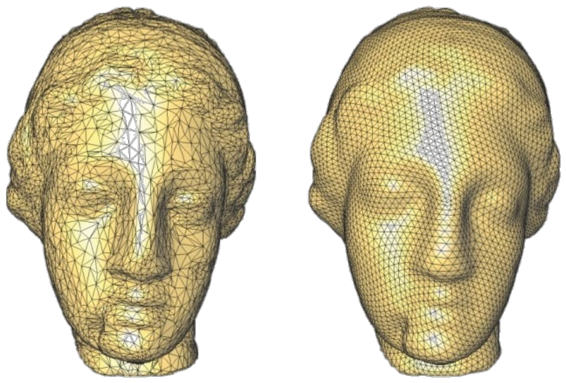
CS348a: Geometric Modeling and Processing







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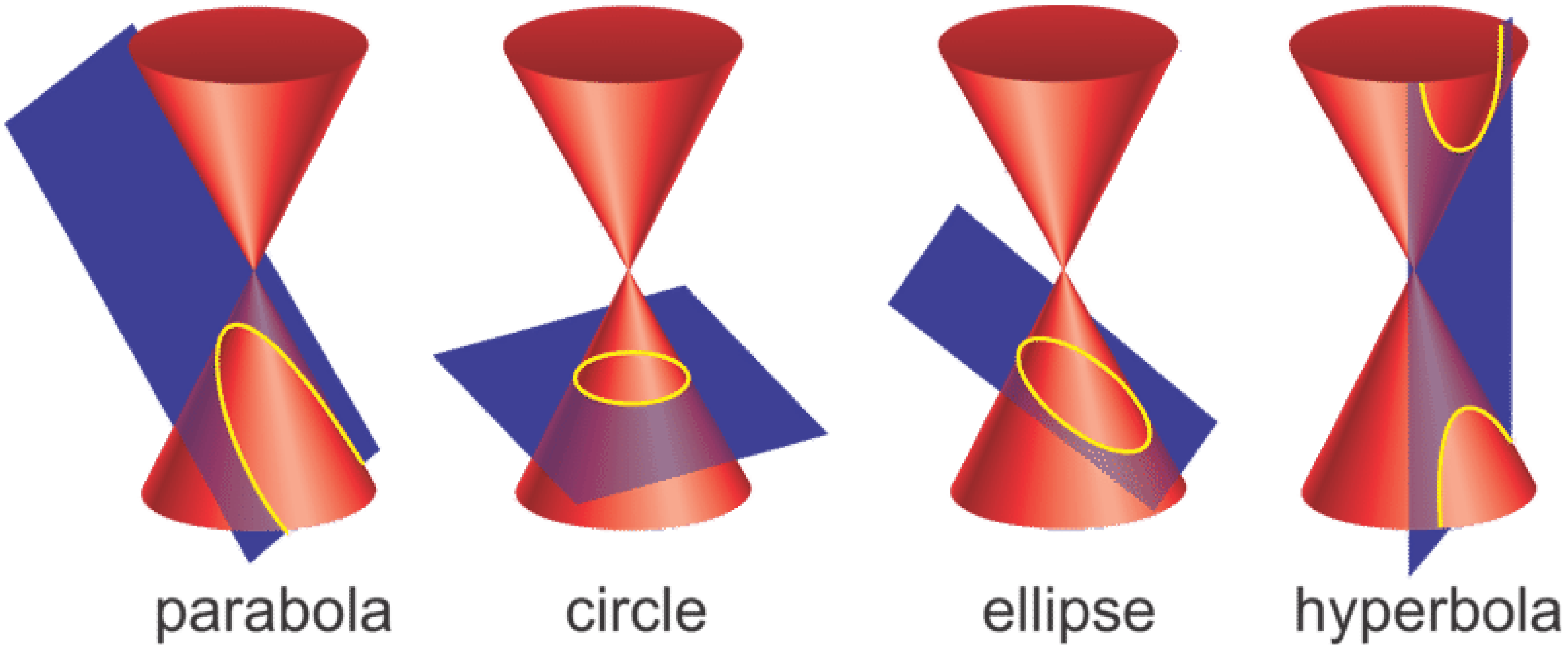
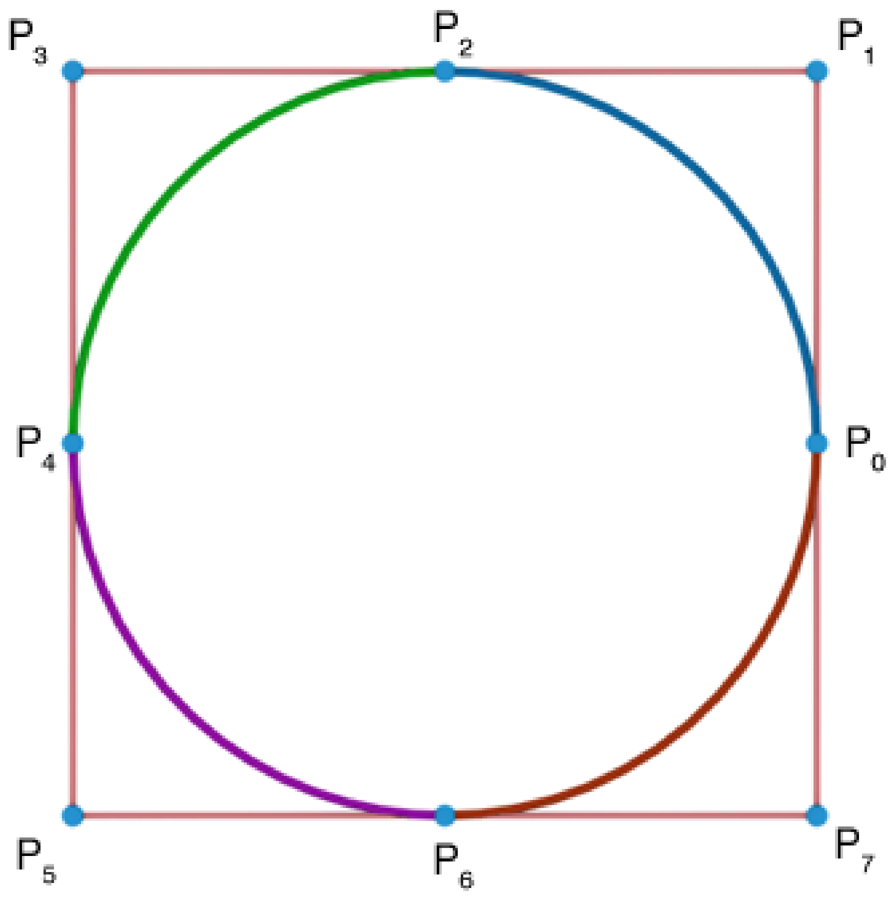
Geometric Computing

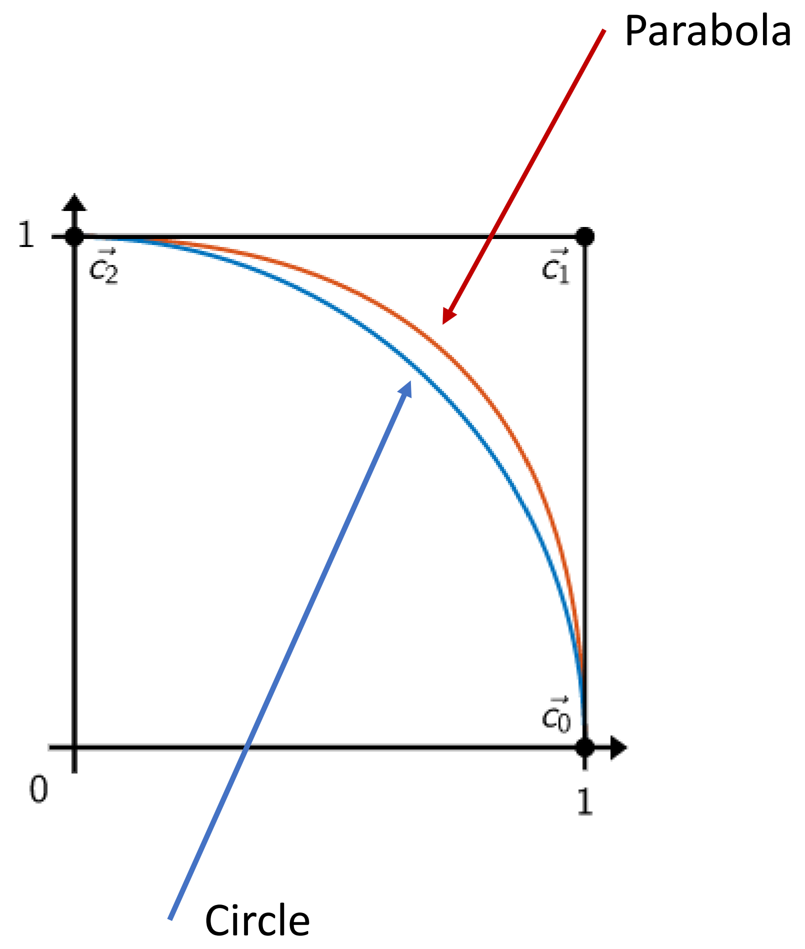
Last Time:
Rational Curves,
~~Subdivision Curves~~

We Want Conic Sections!



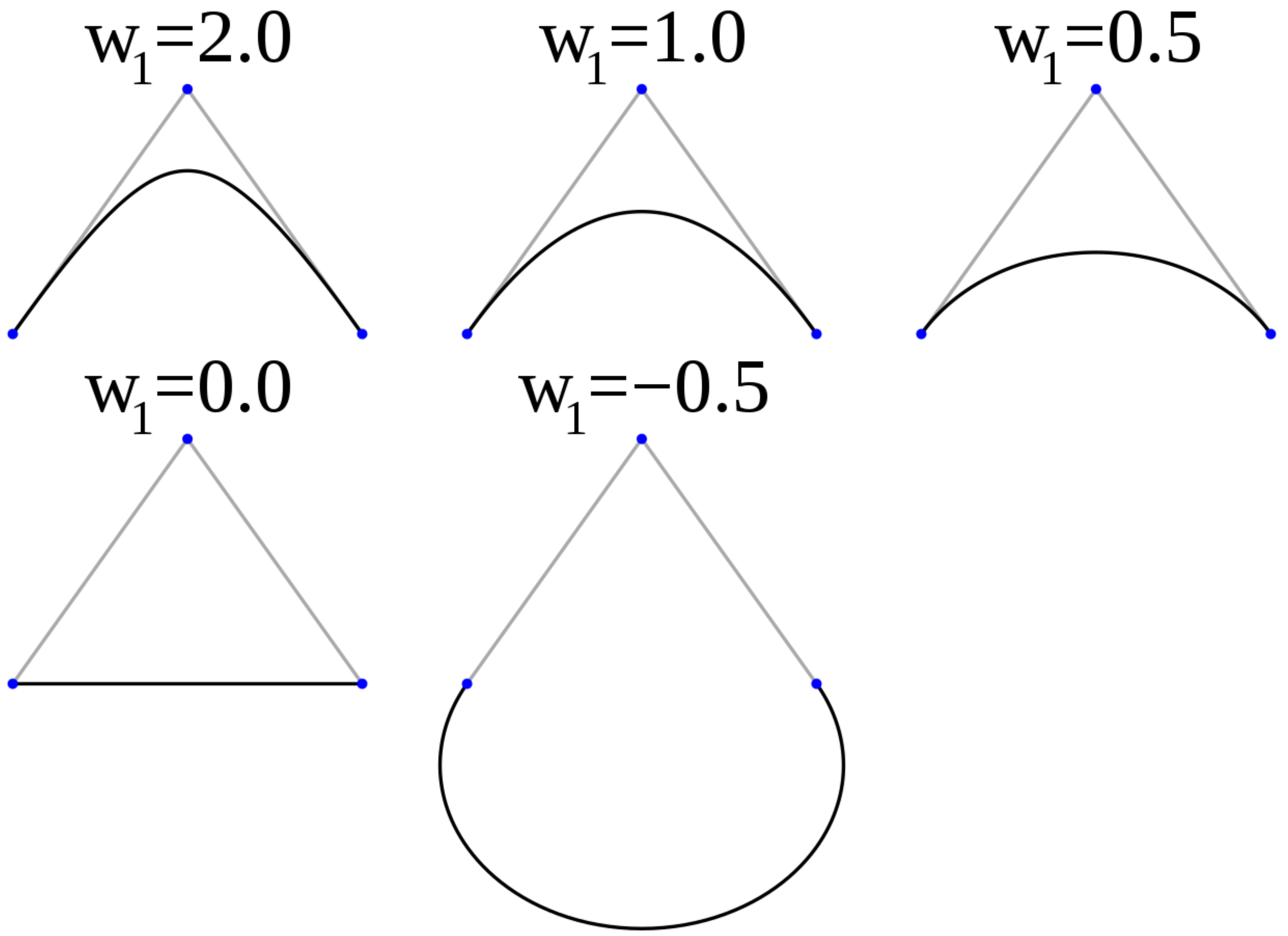
A Circle as a B-Spline





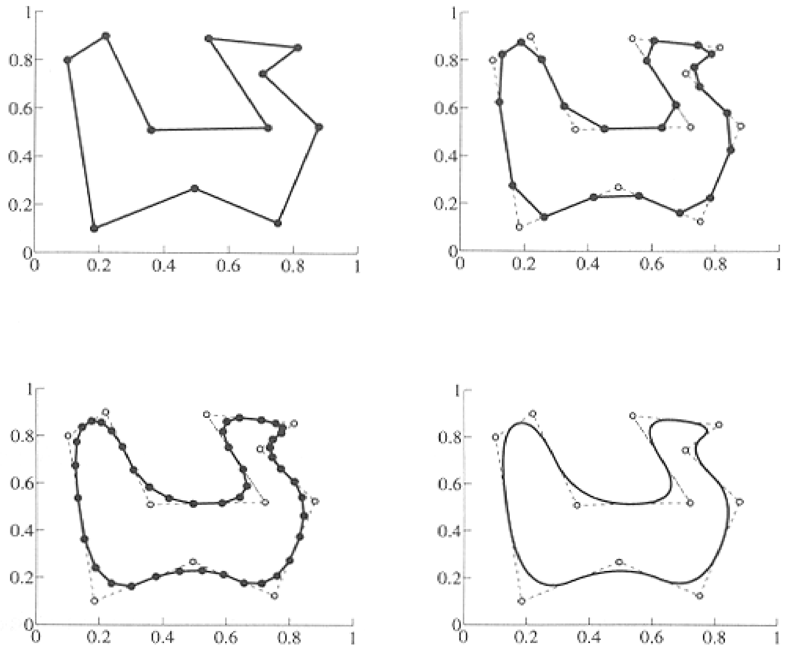
Rational Bézier Arcs

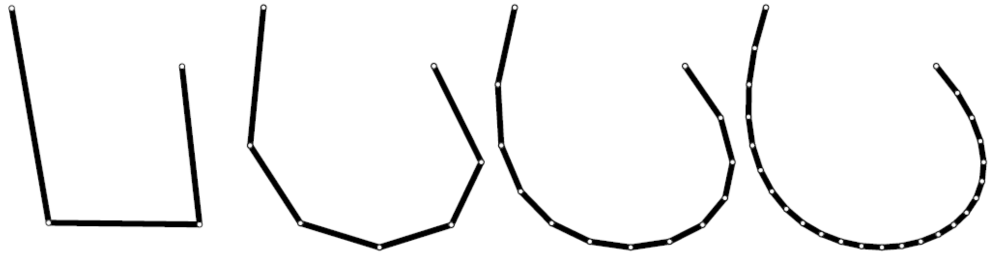
From Control Points
to Control Sites



Subdivision Curves

$$P^1 \to P^2 \to P^2 \to \cdots$$





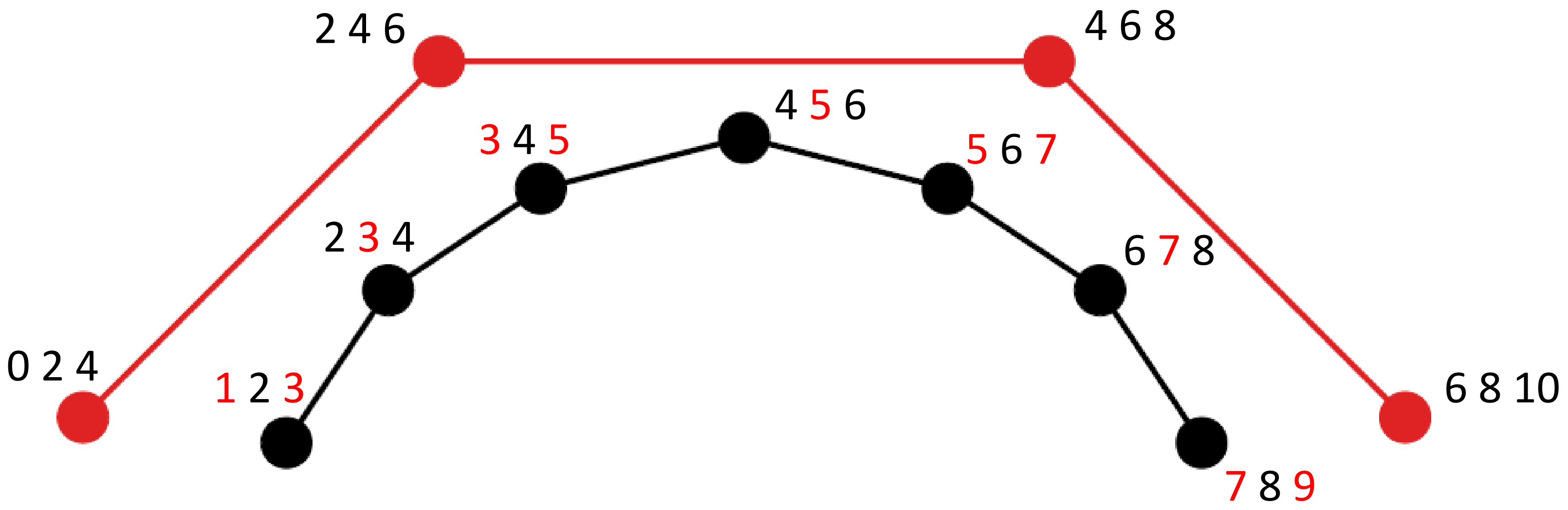
$$Q = \lim_{j \to \infty} P^j$$

B-Spline as a Subdivision Curve

- Massive knot insertion

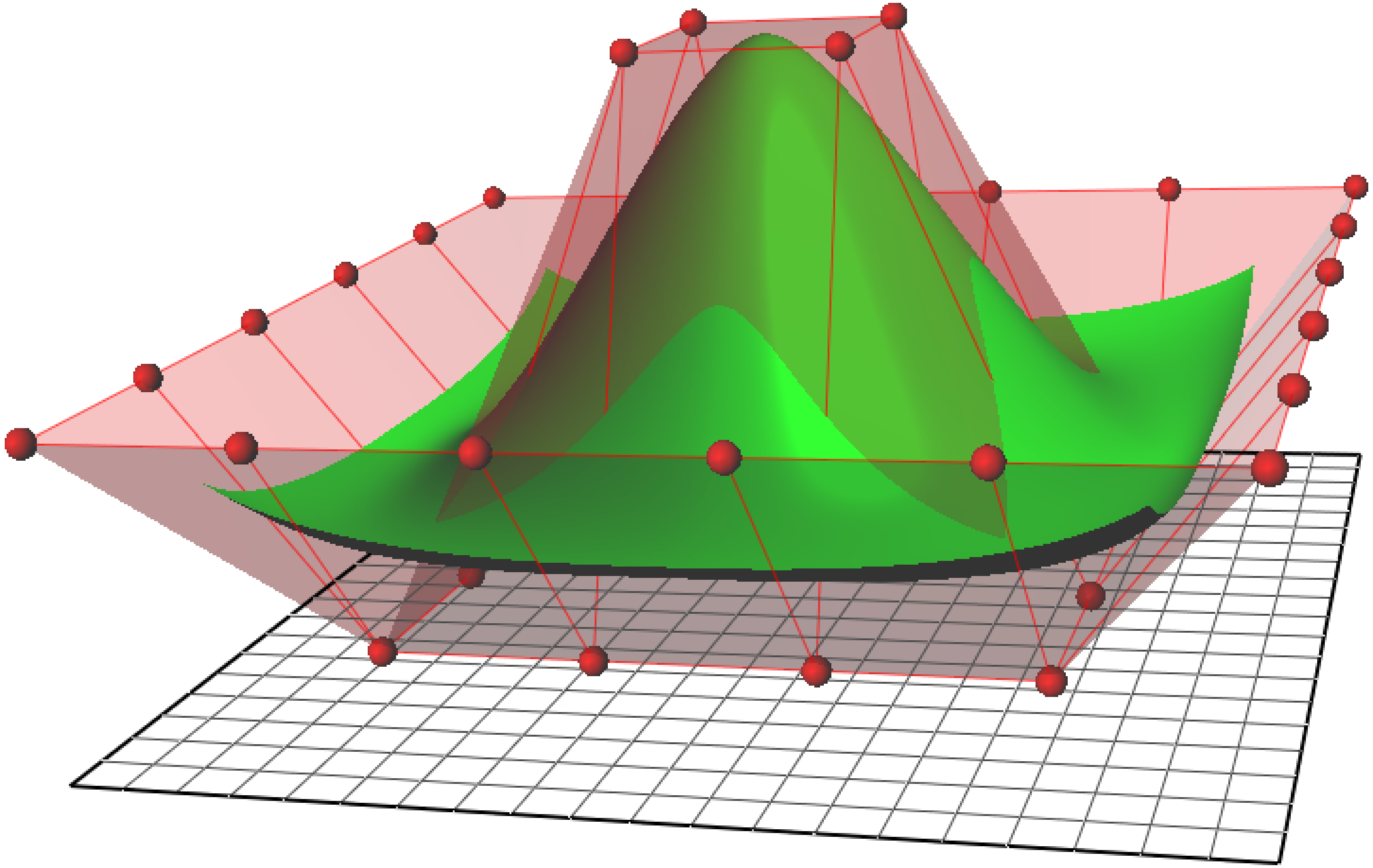
$$p_{2i+1}^{j+1} = \frac{1}{2}(p_i^j + p_{i+1}^j)$$

$$p_{2i}^{j+1} = \frac{1}{8}p_{i-1}^j + \frac{3}{4}p_i^j + \frac{1}{8}p_{i+1}^j$$

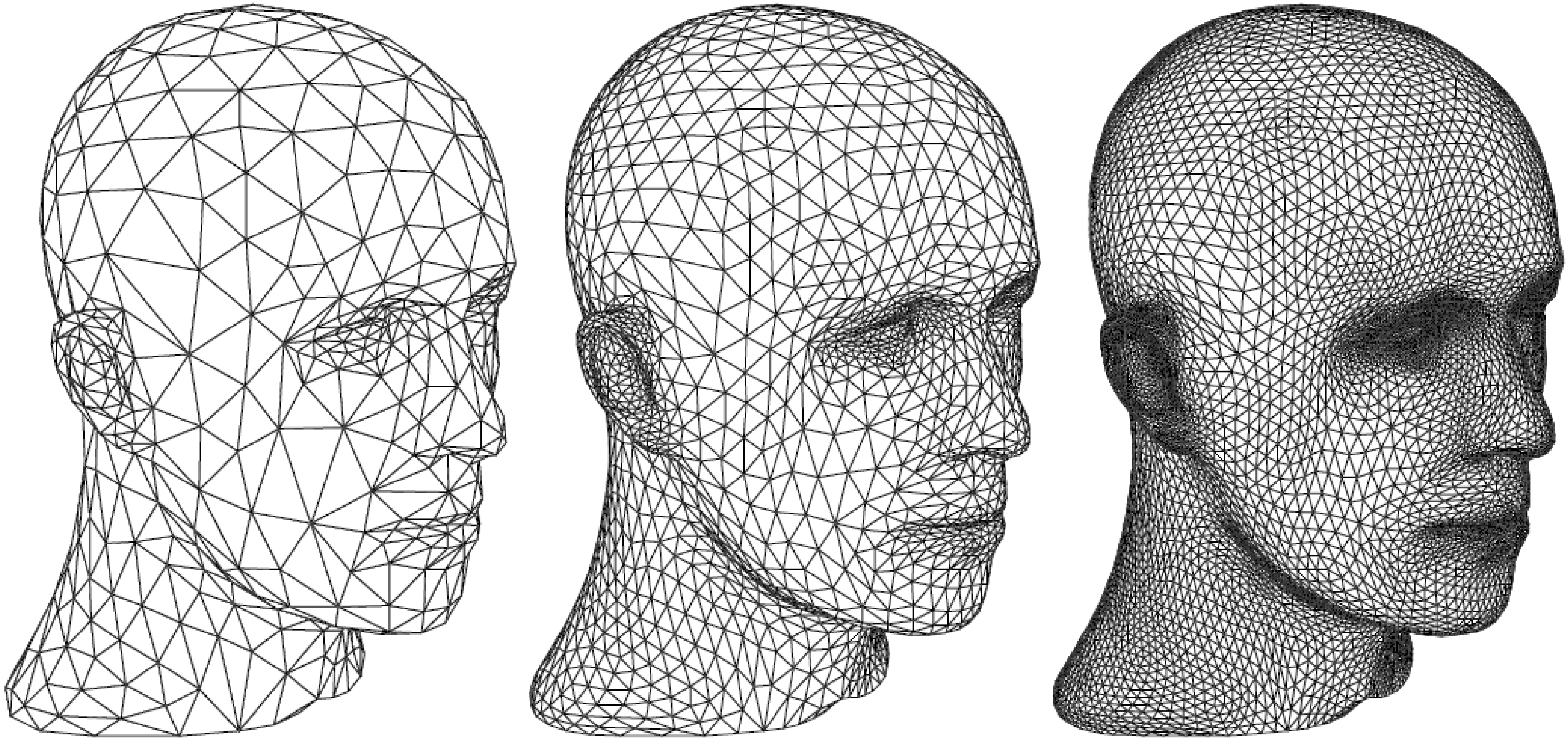


Today:
Parametric Surfaces,
Subdivision Surfaces

NURBS – Non-Uniform Rational B-Splines

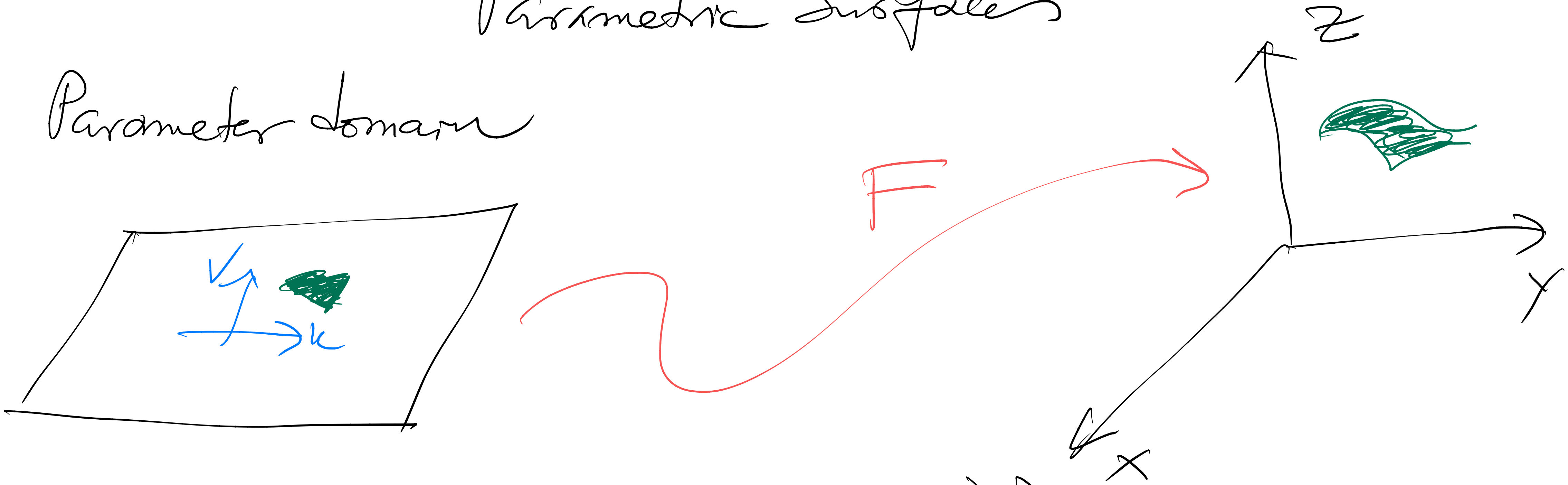


Subdivision Surfaces



Whiteboard

Parametric Surfaces



$$F(u,v) = (X(u,v), Y(u,v), Z(u,v))$$

Polynomial

$$X(u,v) = 2uv + 3uv^2 + 7u^2$$

$$Y =$$

$$Z =$$

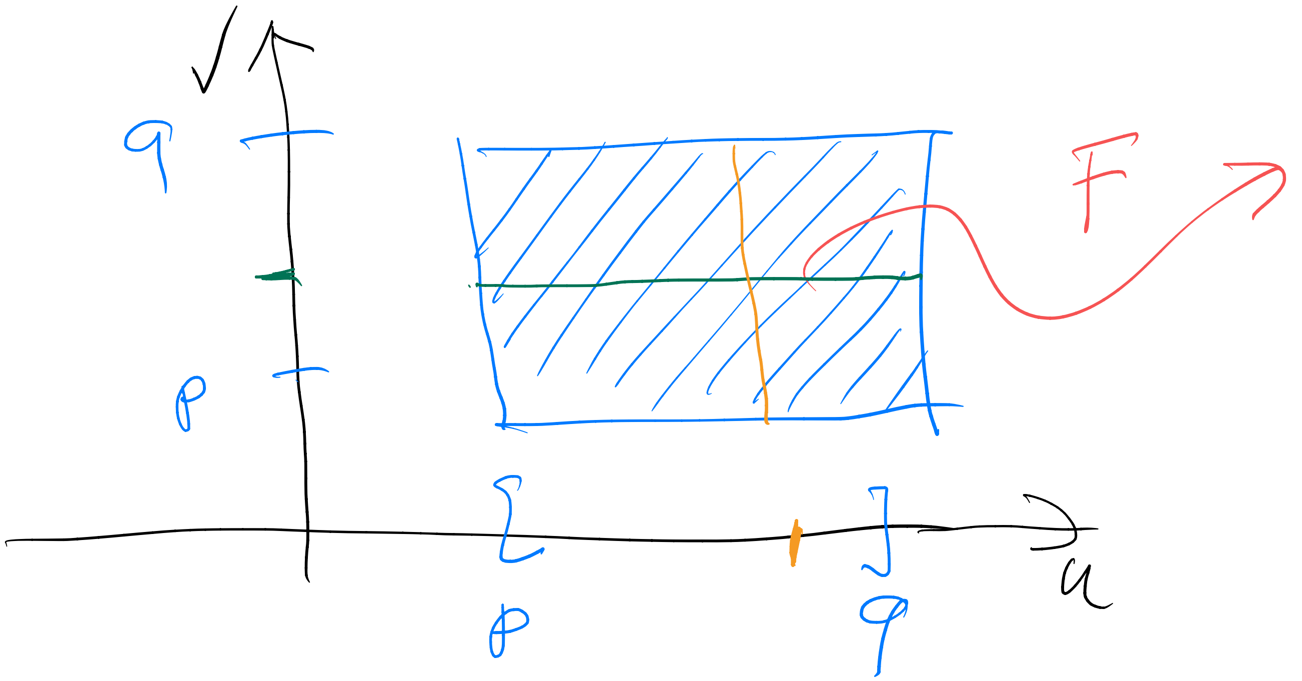
$$X(u,v) = 2uv + 3uv^2 + 7u^2$$

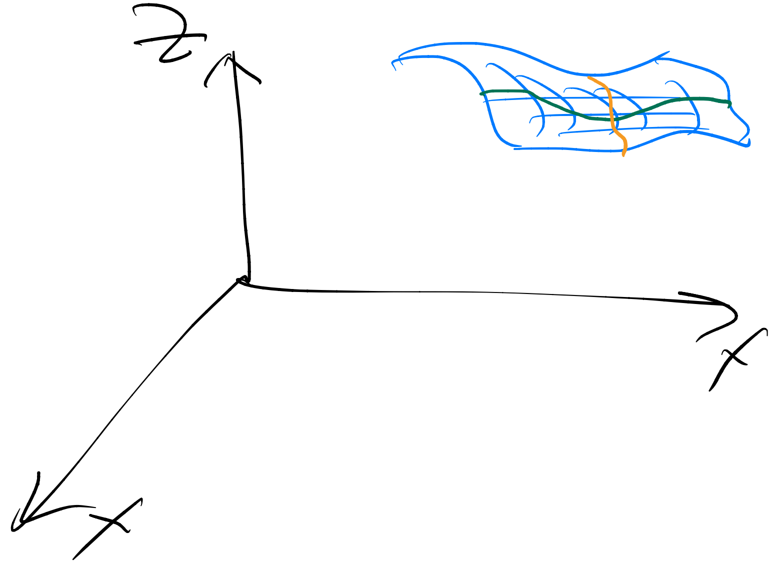
$(2,2)$

$u \to u_1, u_2 \qquad v \to v_1, v_2$

Tensor Product Surfaces

$$X((u_1,u_2);(v_1,v_2)) = 2\left(\frac{u_1+u_2}{2}\right)\left(\frac{v_1+v_2}{2}\right) + 3\left(\frac{u_1+u_2}{2}\right) v_1 v_2 + 7 u_1 u_2$$





Control Points of

F ([p,q], [r,s])

pp pq qq

rr rs ss

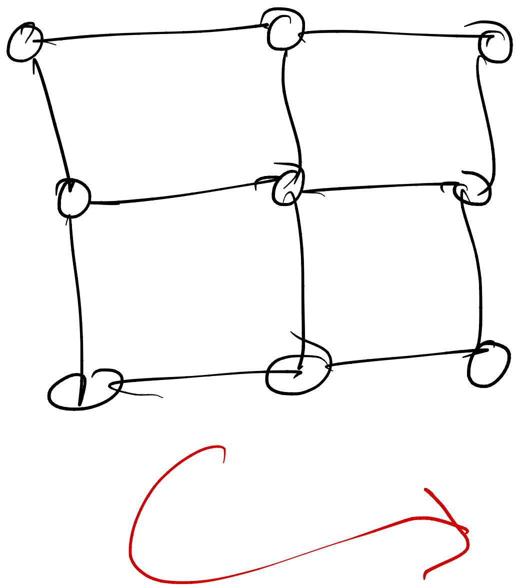
d = 2

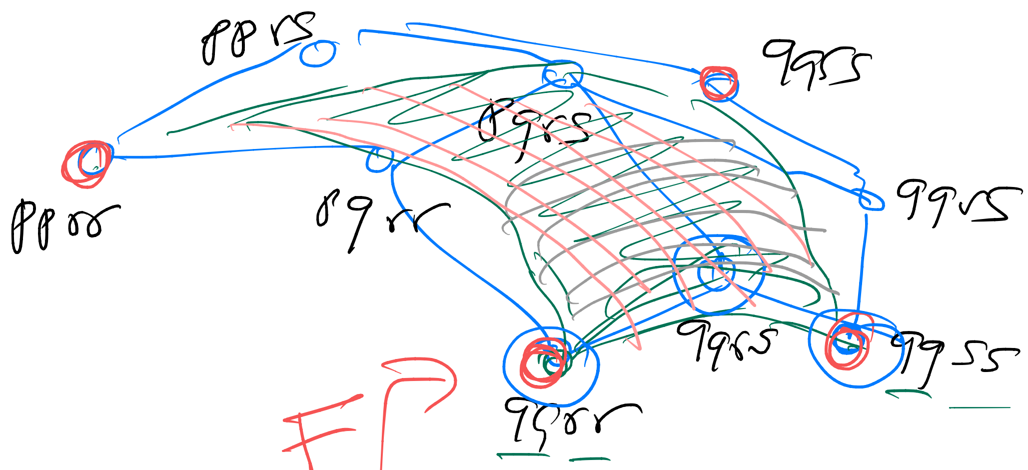
9 Control points

f(pp; rr) f(pq, rr) f(qq, rr)

f(pp; rs) f(pq, rs) f(qq, rs)

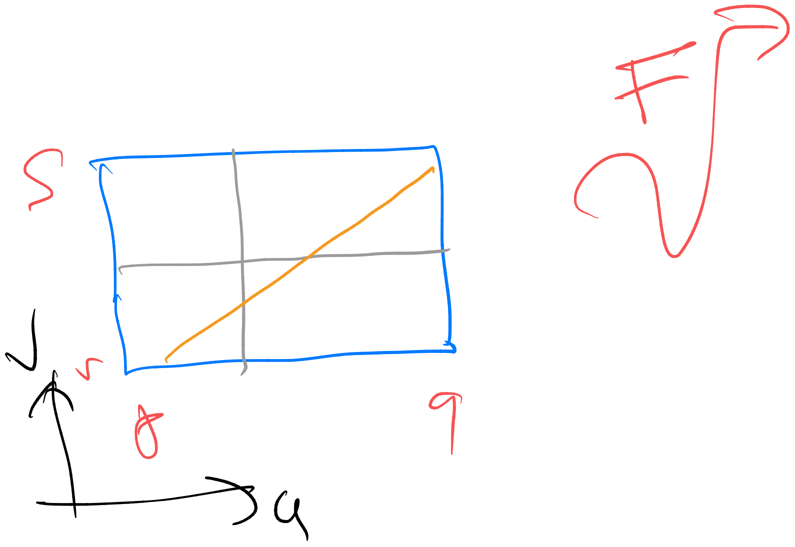
f(pp; rr) f(pq; rs) f(qq, ss)





bi-quadratic control polygon

Bézier control points



de Casteljau F(a,b)?

f(pp,rr) f(pq,rr) f(qq,rr)

f(pa,rr) f(aq,rr)

f(aa,rr)

f(aa,rs) f(aa,ss)

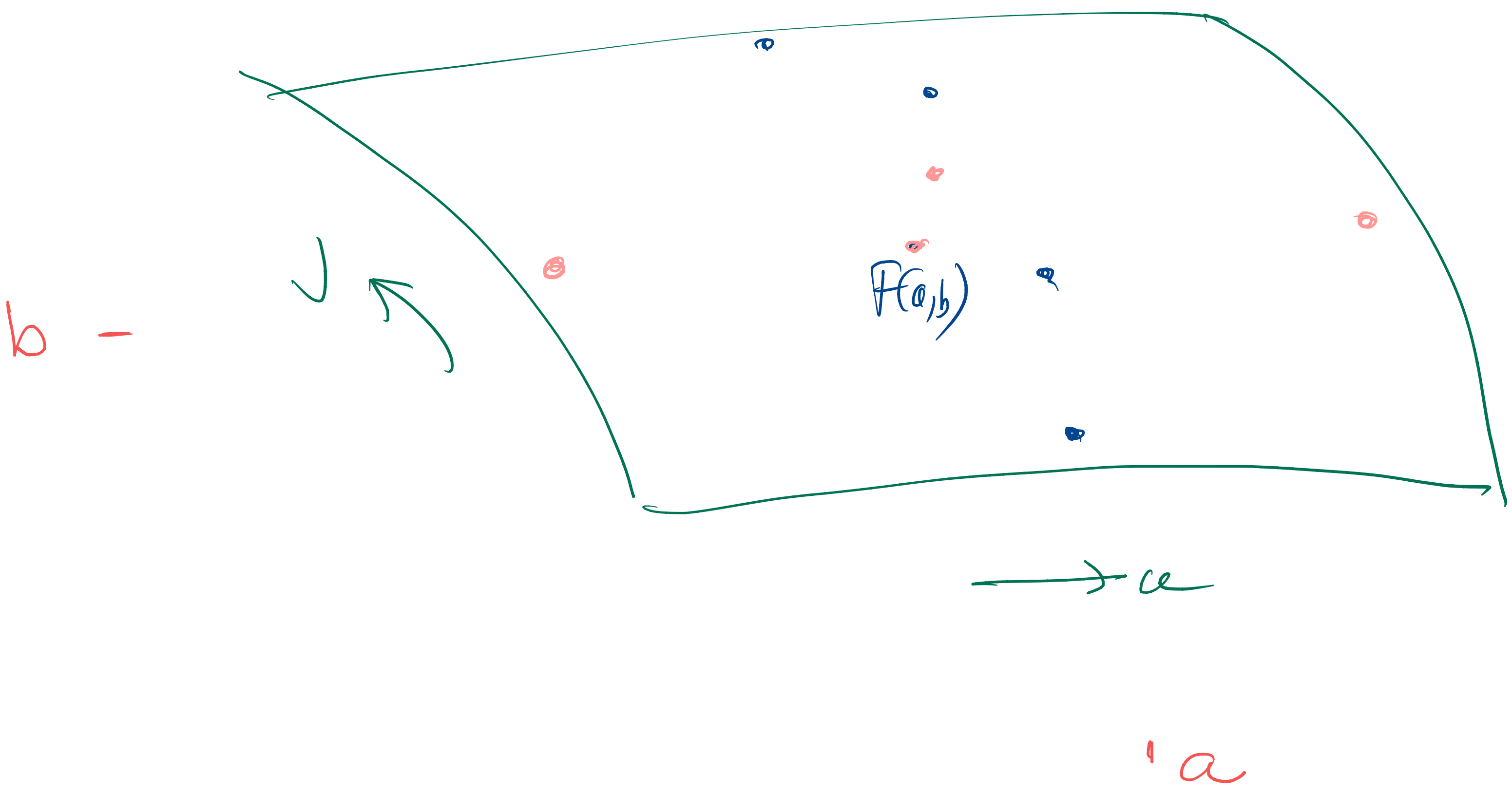
f(aa,bb) = F(a,b)

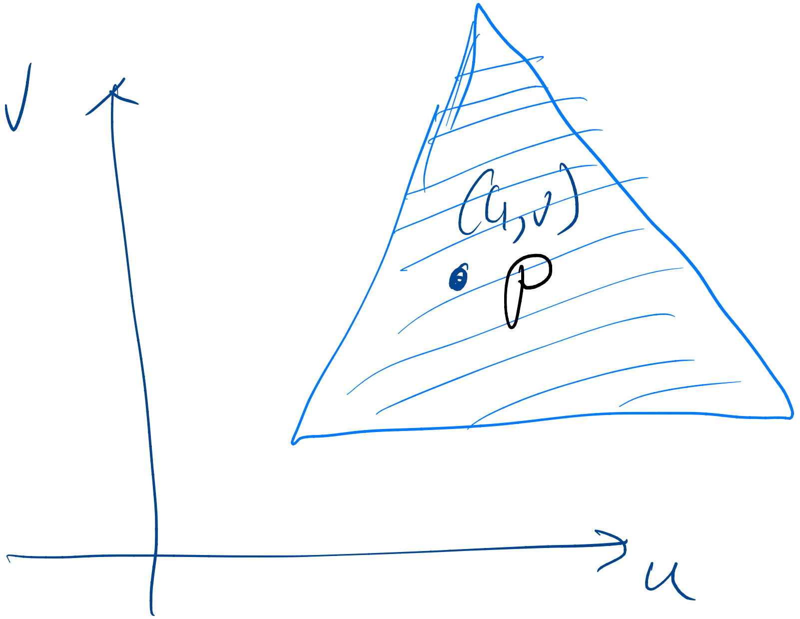
u v

2 3

F(a,b)

f(aa,bbb)





$$X(u,v) = 2uv + \underbrace{3uv^2}_{3} + 7u^2$$

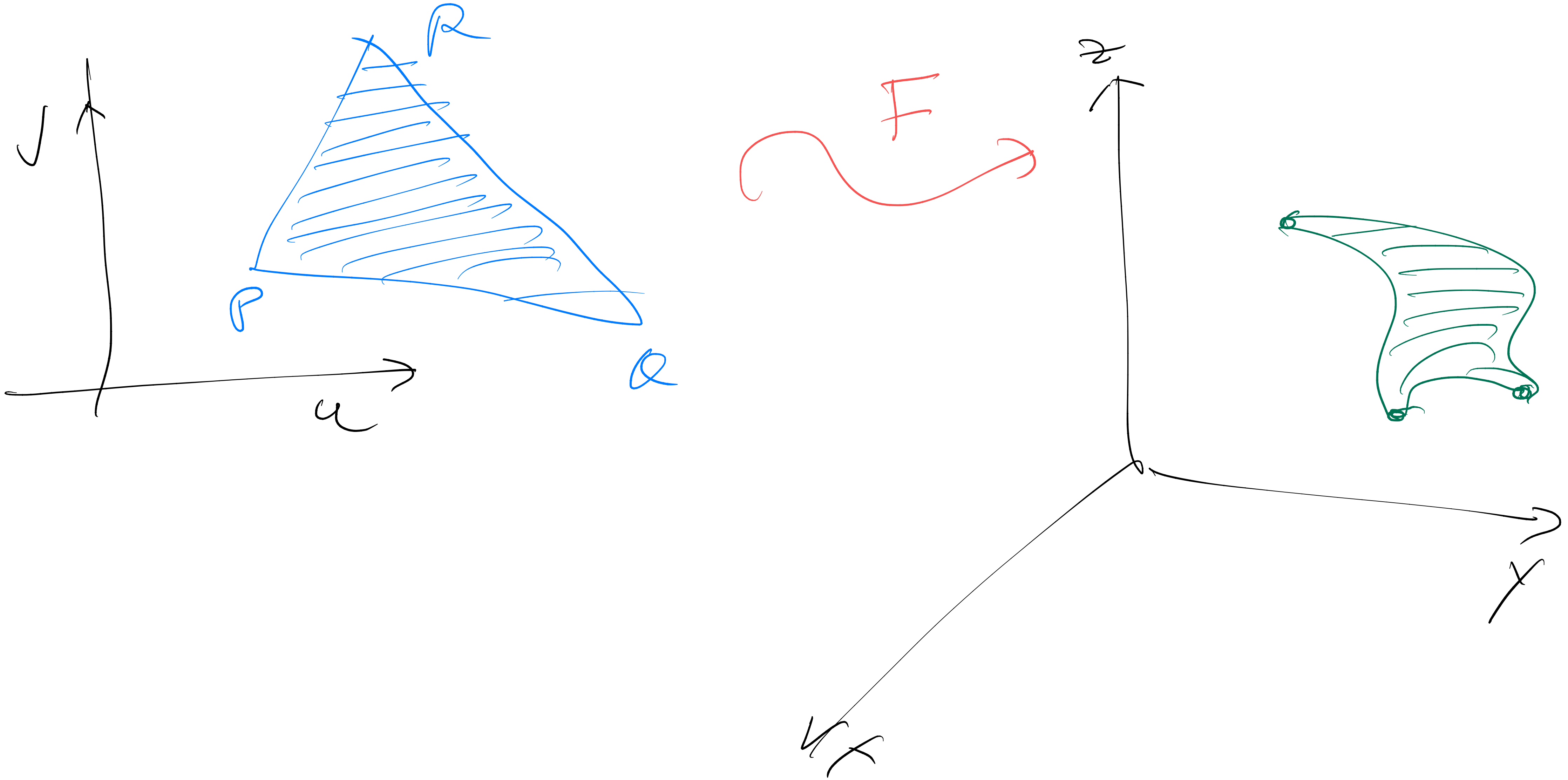
$(u,v) \rightarrow (u_1,v_1),\ (u_2,v_2),\ (u_3,v_3)$

Total degree Surfaces

$$uv = \frac{1}{6}\left(u_1v_2 + u_1v_3 + u_2v_1 + u_2v_3 + u_3v_1 + u_3v_2\right)$$

$$uv^2 = \frac{1}{3}\left(u_1v_2v_3 + u_2v_1v_3 + u_3v_1v_2\right)$$

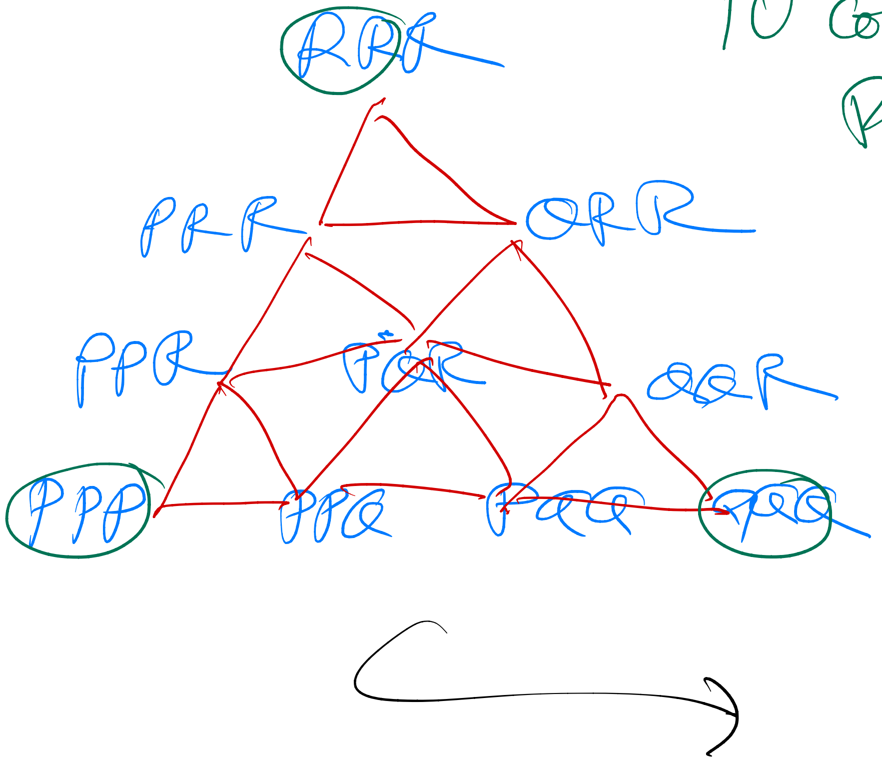
$$u^2 = \frac{1}{3}\left(u_1u_2 + u_2u_3 + u_3u_1\right)$$

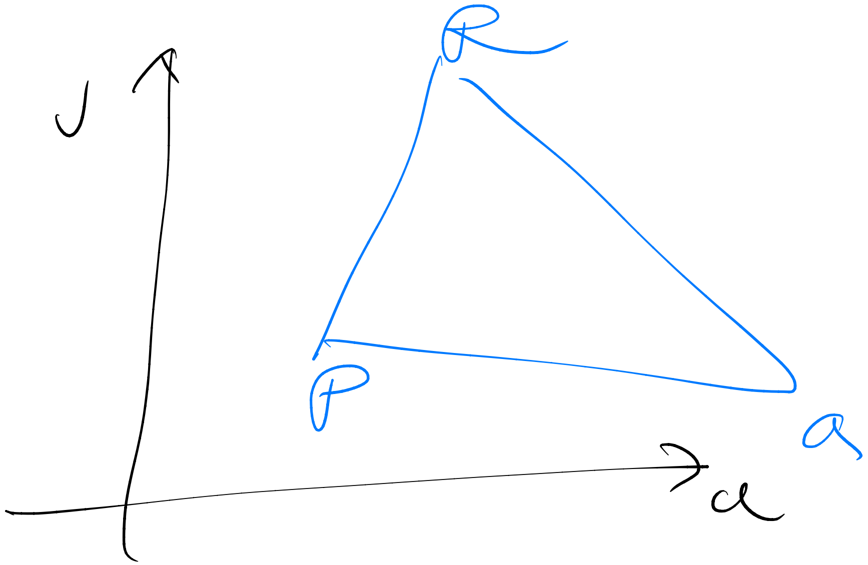

v
u
R
P
Q
F
z
y
x

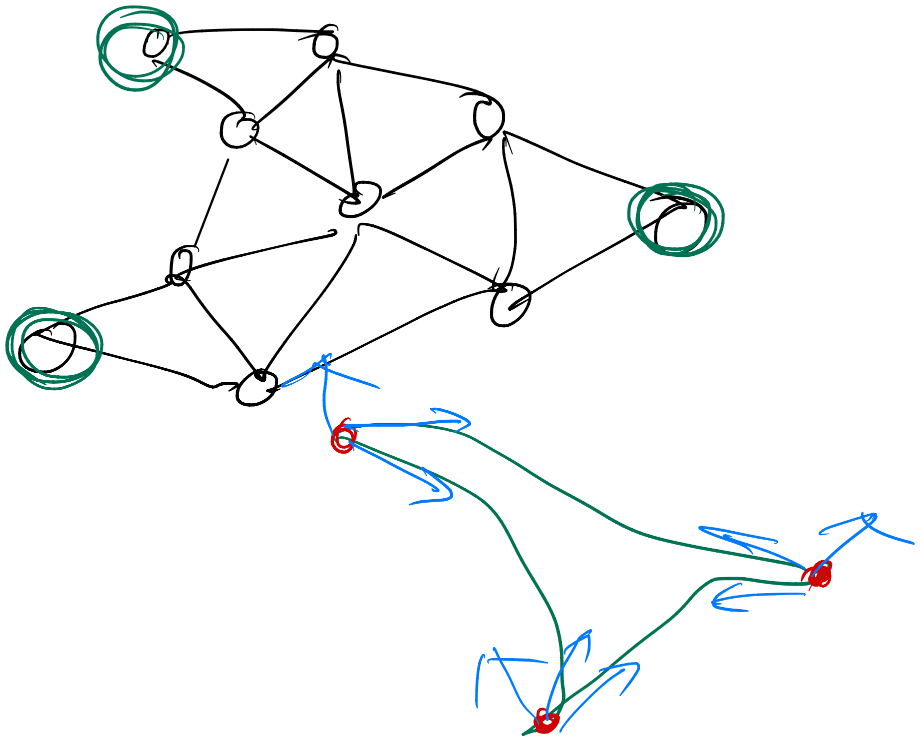
Bézier Control Points?

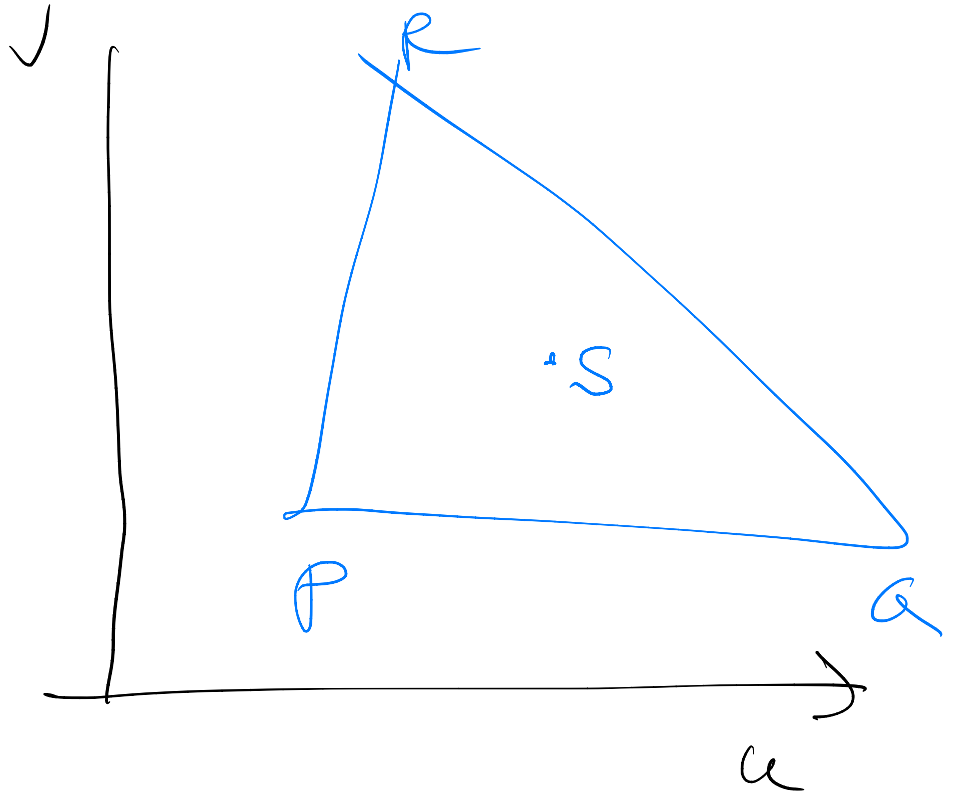
$d = 3$

10 control Points





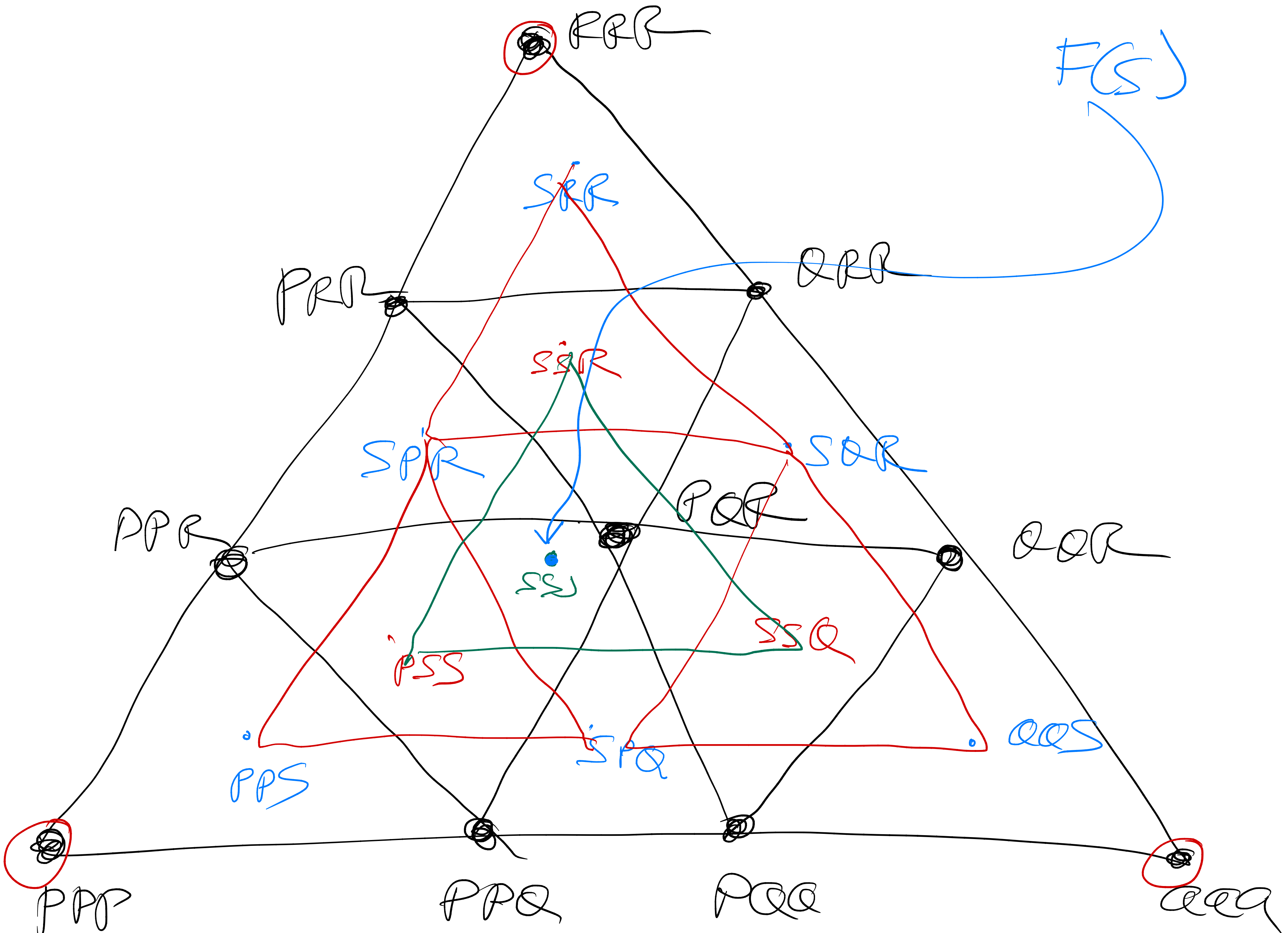


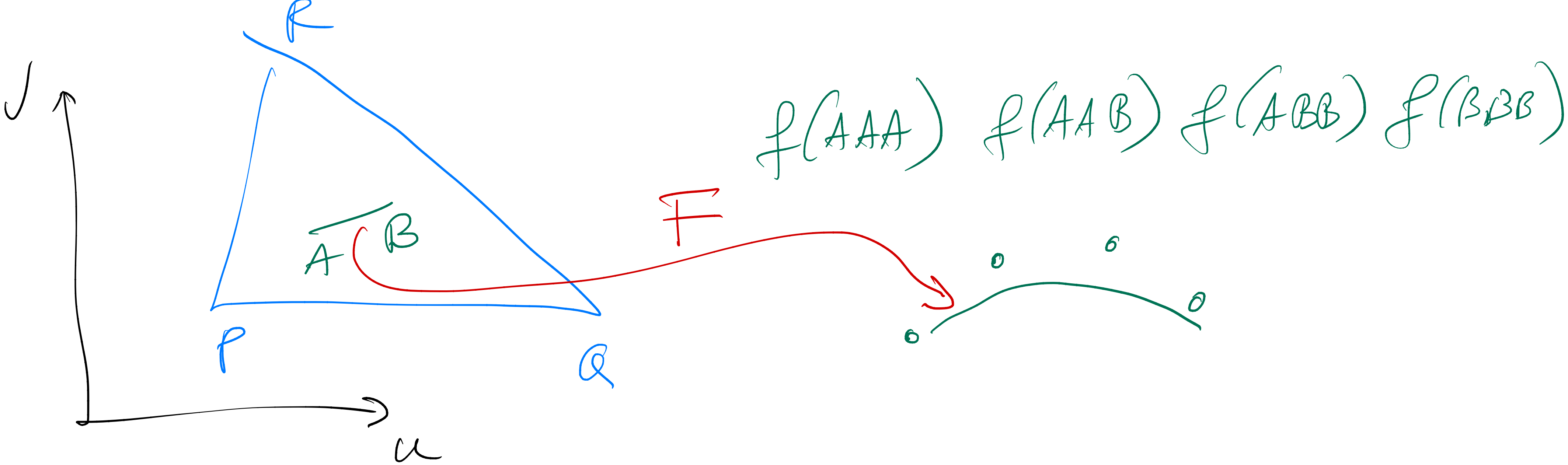


$$F(S) = f(S,S,S)$$

$$S = \tau_P P + \tau_Q Q + \tau_R R$$

$$\tau_P + \tau_Q + \tau_R = 1$$

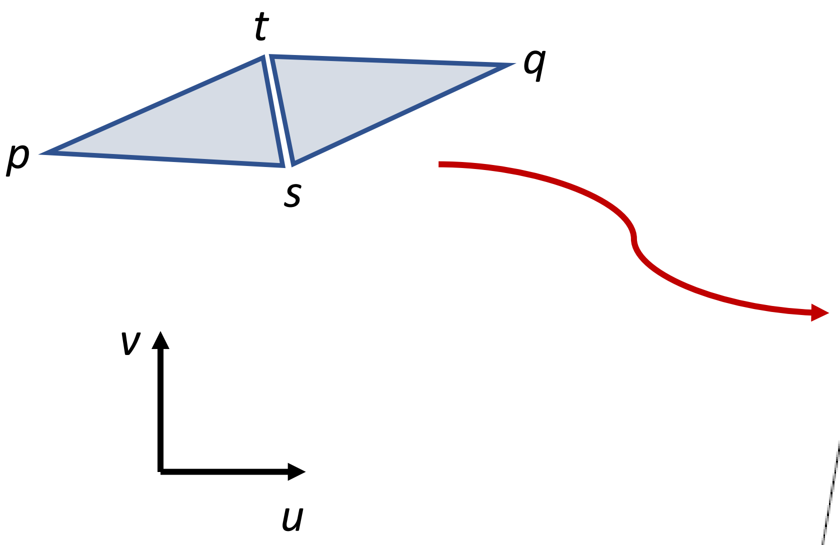

RRR
F(S)
SRR
PRR
QRR
SSR
SPR
SQR
PPR
PQR
QQR
S(S)
PSS
SSQ
PPS
SPQ
QQS
PPP
PPQ
PQQ
QQQ



v
u
P
Q
R
A
B
F
f(AAA) f(AAB) f(ABB) f(BBB)

Continuity Constraints

C^0 Continuity



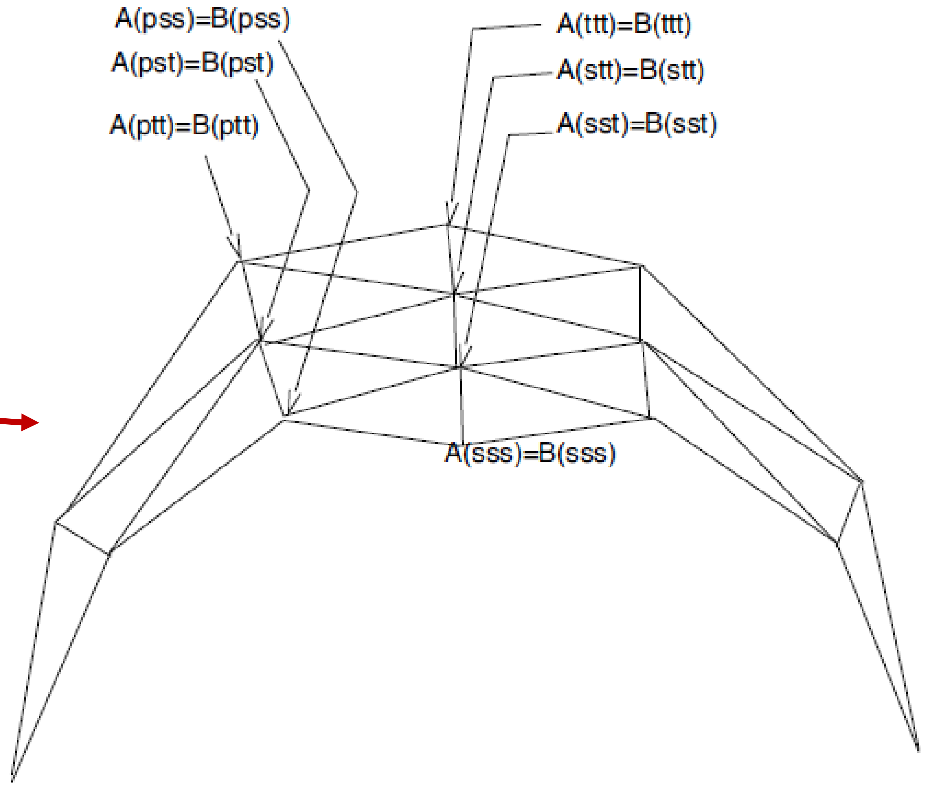


Figure 4: The Bezier points of two cubic surfaces that join with C^0 continuity

C^1 Continuity

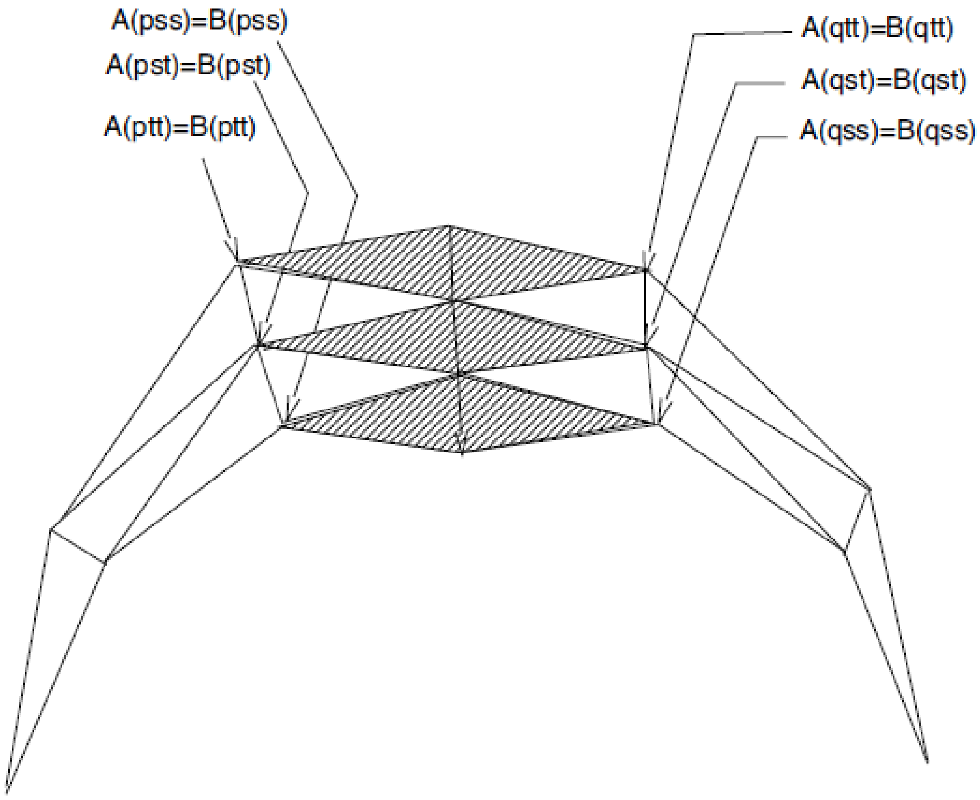


Figure 5: The Bezier points of two cubic surfaces that join with C^1 continuity

C^2 Continuity

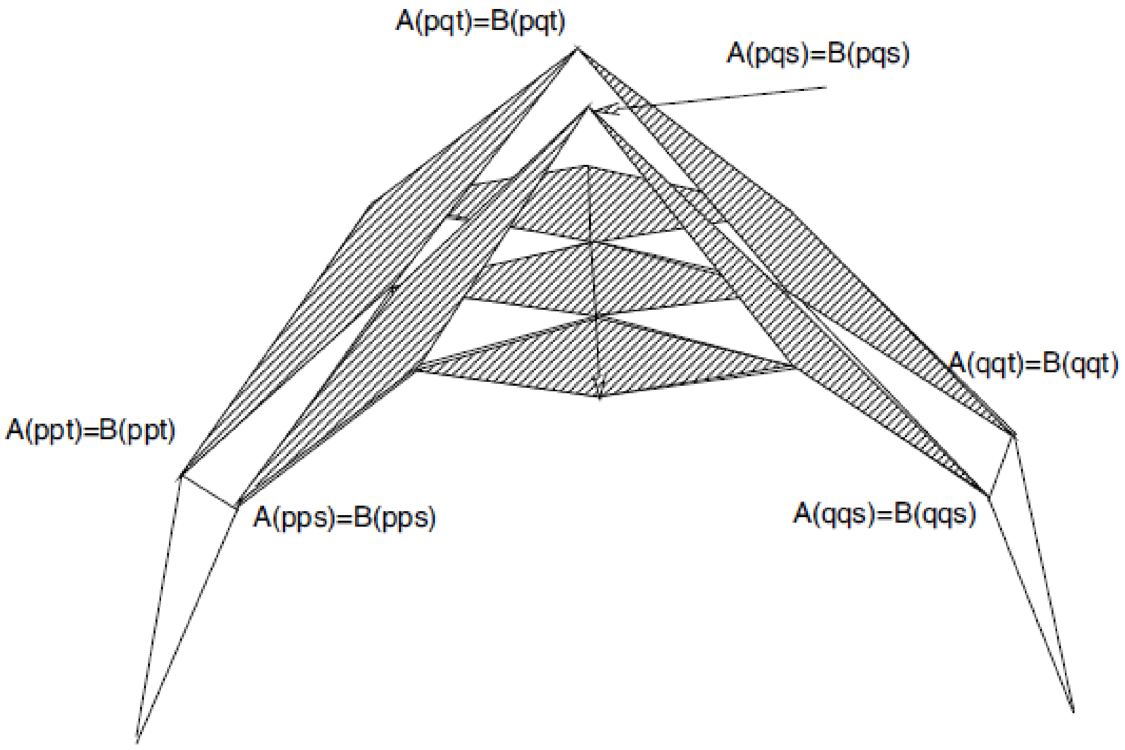


Figure 6: The Bezier points of two cubic surfaces that join with C^2 continuity

C^3 Continuity

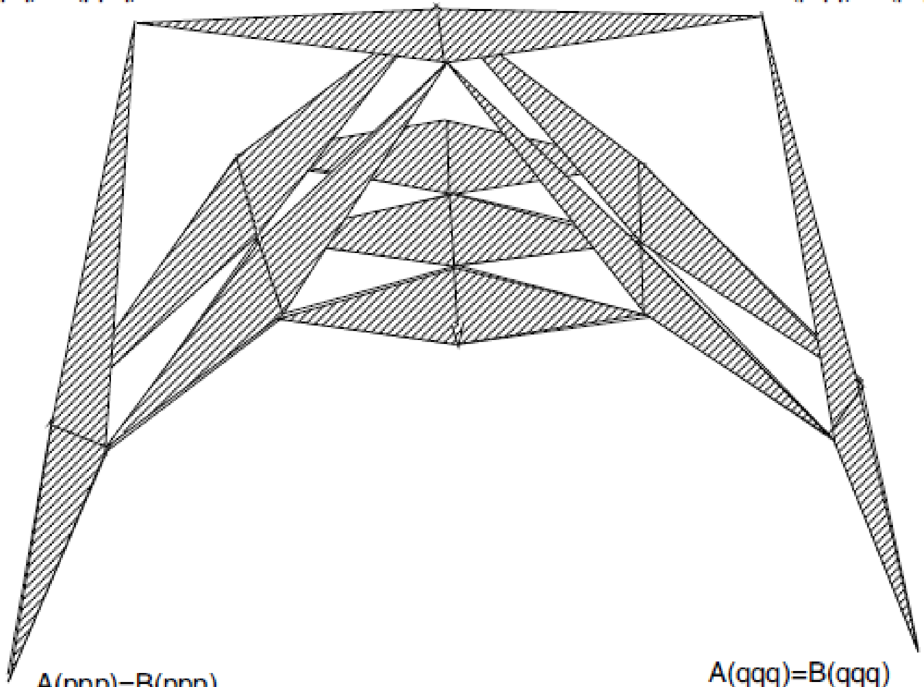


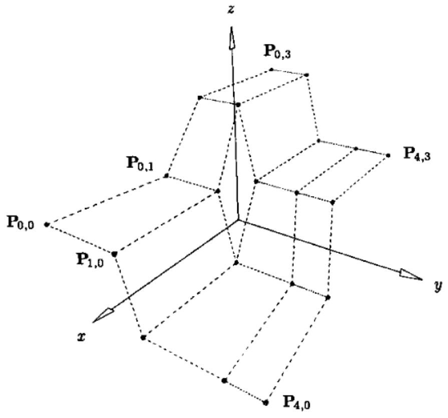
Figure 7: The Bezier points of two cubic surfaces that join with C^3 continuity

Tensor Product B-Spline Surfaces

B-Spline Surfaces (Tensor Product)

B-Spline surface - tensor product surface of B-Spline curves

$$S(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} N_{i,p}(u) N_{j,q}(v) \boldsymbol{P}_{ij}$$

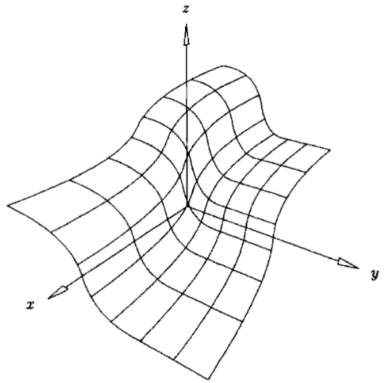


Building blocks:

Control net, $m + 1$ rows, $n + 1$ columns: $\boldsymbol{P}_{ij}$

Knot vectors $U = \{ u_0 , u_1 , \dots , u_h \}$, $V = \{ v_0 , v_1 , \dots , v_k \}$

The degrees p and q for the u and v directions



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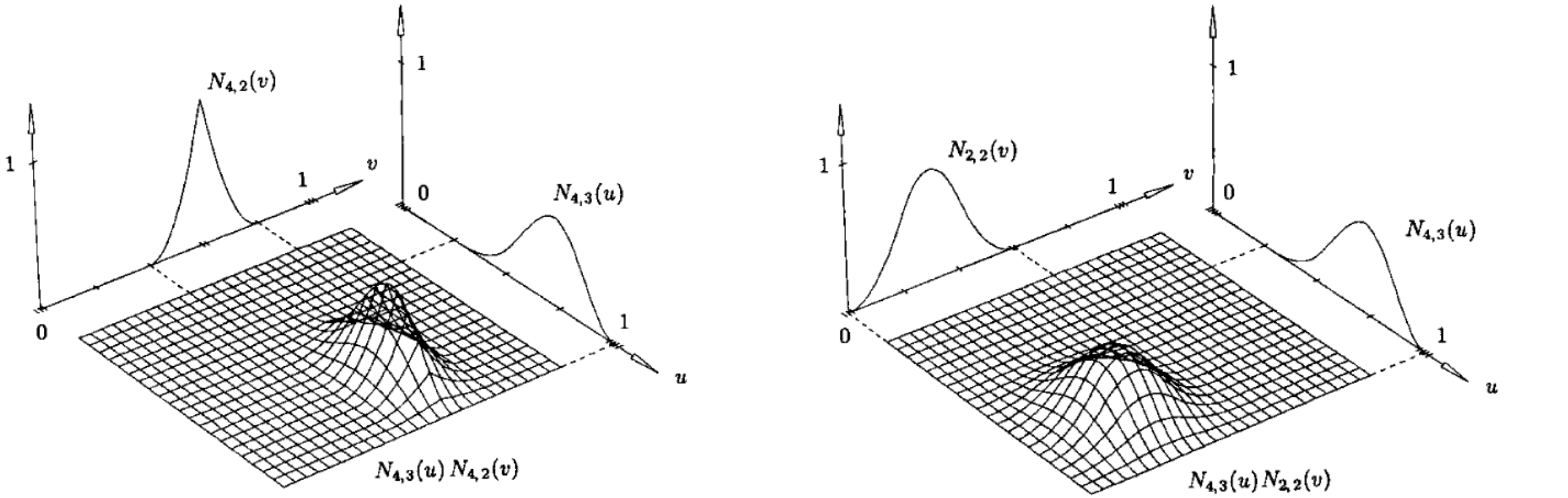
Polynomial Basis Functions

Cubic × Quadratic basis functions:



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Properties

- Non negativity

$$N_{i,p}(u)N_{j,q}(v) \geq 0, \qquad \text{for all } i, j, p, q, u, v$$

- Partition of unity

$$\sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,p}(u)N_{j,q}(v) = 1, \qquad \text{for all } (u,v) \in [0,1]\times[0,1]$$

- Affine invariance

$$A\boldsymbol{S}_{\{\boldsymbol{P}_{ij}\}}(u,v) + B = \boldsymbol{S}_{\{A\boldsymbol{P}_{ij}+B\}}(u,v)$$

$$\text{for all } A \in R^{3x3}, B \in R^{3}, (u,v) \in [0,1]\times[0,1]$$

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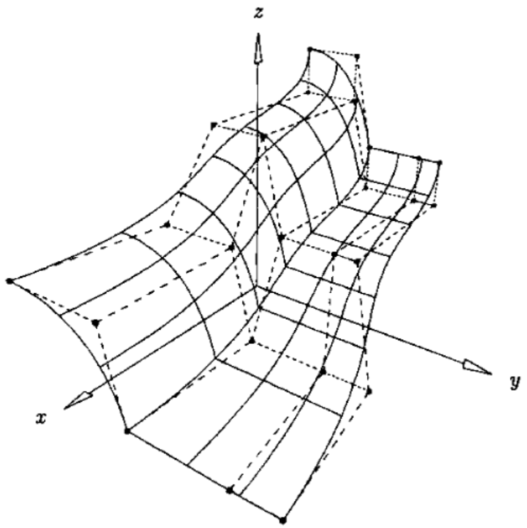
Properties

- If $n = p$, $m = q$, $U = \{ 0,\ldots,0,1,\ldots,1\}$ and $V = \{0,\ldots,0,1,\ldots,1\}$ then

$$N_{i,p}(u)N_{j,q}(v) = B_i^n(u)B_j^m(v)$$

  and $\boldsymbol{S}(u,v)$ is a Bézier surface

- $\boldsymbol{S}(u,v)$ is C^{p-k} continuous in the u direction at a u knot of multiplicity k, and similar for v direction



$U = \{0,0,0,0.5,0.5,1,1,1\}$
$V = \{0,0,0,0,0.5,1,1,1,1\}$

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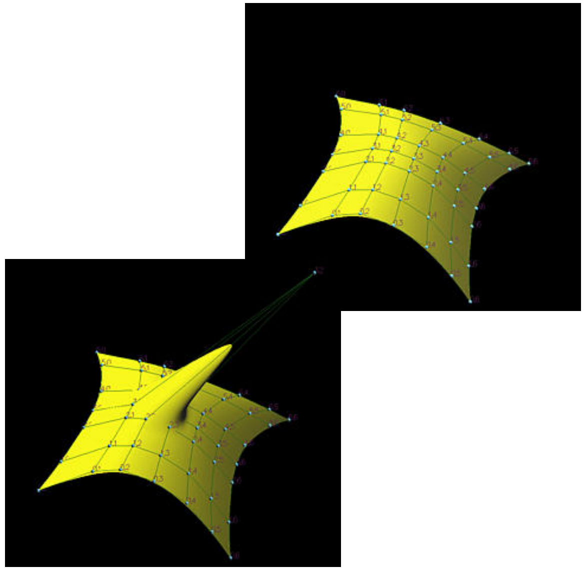
Properties

- Compact support

$$N_{i,p}(u)N_{j,q}(v) = 0, \qquad \text{for all } (u,v) \notin \left[u_i, u_{i+p+1}\right] \times \left[v_j, v_{j+q+1}\right]$$

- Local modification scheme
  - Moving $\boldsymbol{P}_{ij}$ affects the surface only in the rectangle

$$\left[u_i, u_{i+p+1}\right] \times \left[v_j, v_{j+q+1}\right]$$



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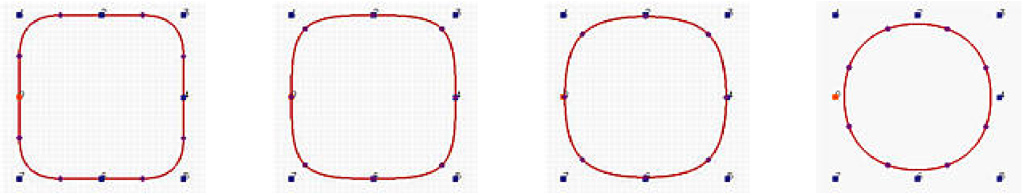
Reminder: NURBS Curves

- B-spline curves cannot represent exactly circles and ellipses



$p = 2$ $p = 3$ $p = 5$ $p = 10$

- Generalize to *rationals (ratios of polynomials)*

$$C(u) = \frac{\sum_{i=0}^{n} N_{i,p}(u) w_i \boldsymbol{P}_i}{\sum_{i=0}^{n} N_{i,p}(u) w_i}$$

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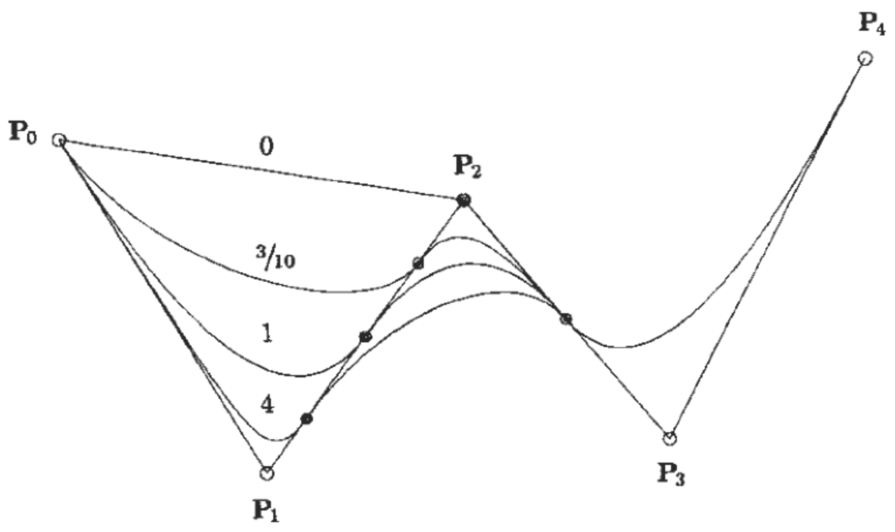
Reminder: NURBS Curves

A weight per control point allows to change the influence of a point on the curve, without moving the point



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NURBS Surfaces

Add a weight for every control point of a B-spline surface, and normalize

$$\boldsymbol{S}(u,v)=\frac{\sum_{i=0}^{n}\sum_{j=0}^{m}N_{i,p}(u)N_{j,q}(v)w_{ij}\boldsymbol{P}_{ij}}{\sum_{i=0}^{n}\sum_{j=0}^{m}N_{i,p}(u)N_{j,q}(v)w_{ij}}$$

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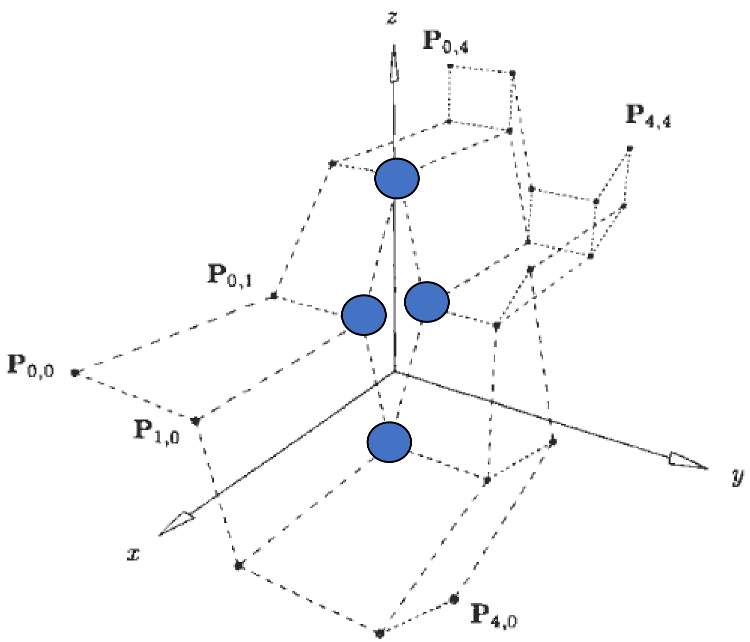
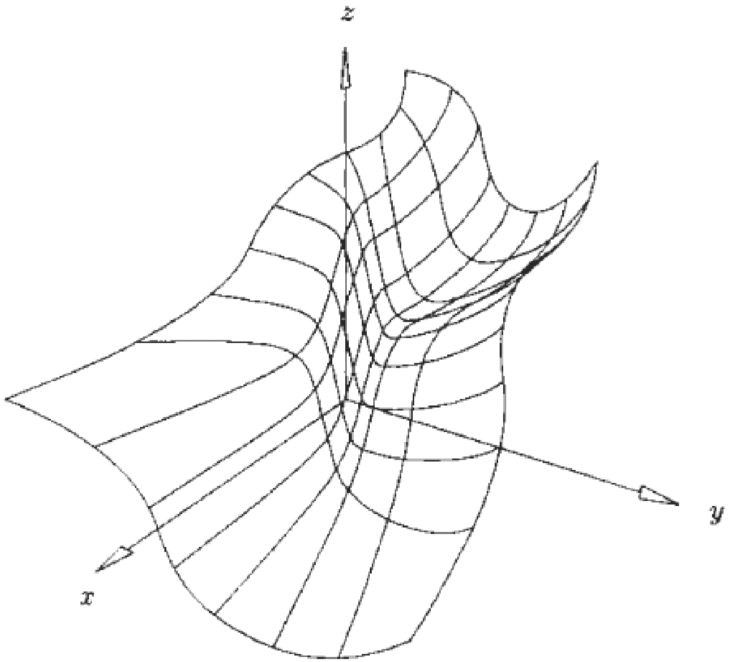
NURBS Surface Example



Control net

$$U = V = \{0,0,0,1/3,2/3,1,1,1\}$$

$$w_{ij}(\ \bullet\ ) = 10,\ w_{ij}(\ \cdot\ ) = 1$$



NURBS Surface

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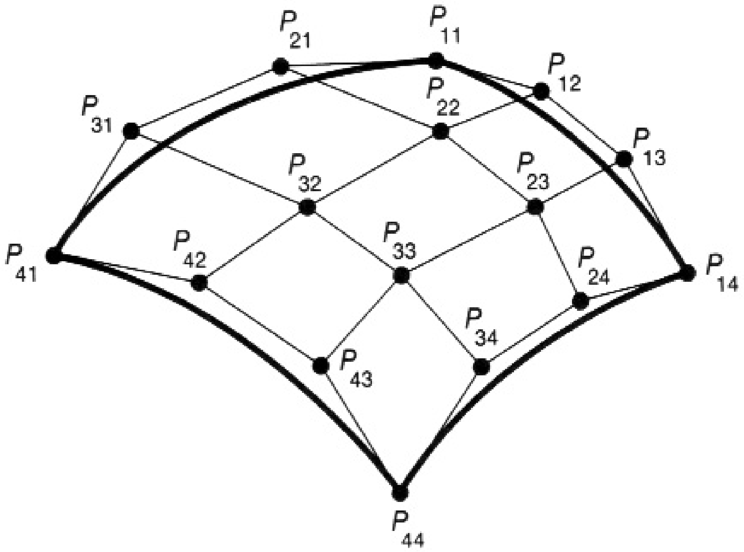
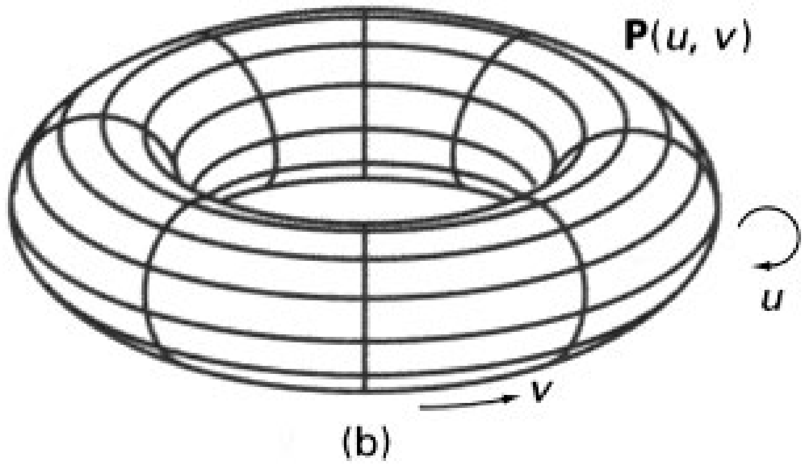
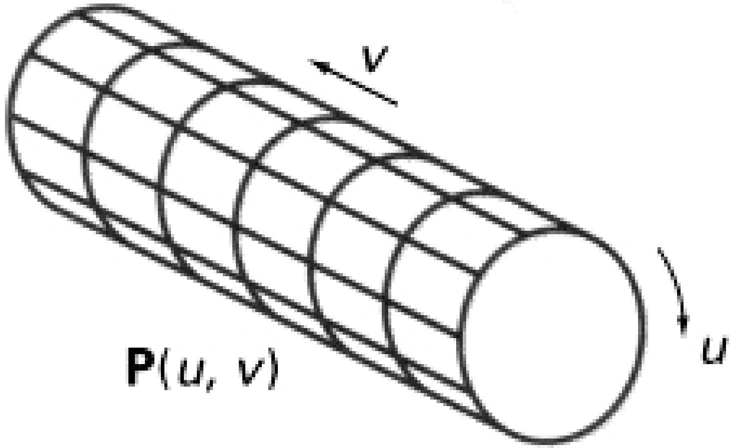
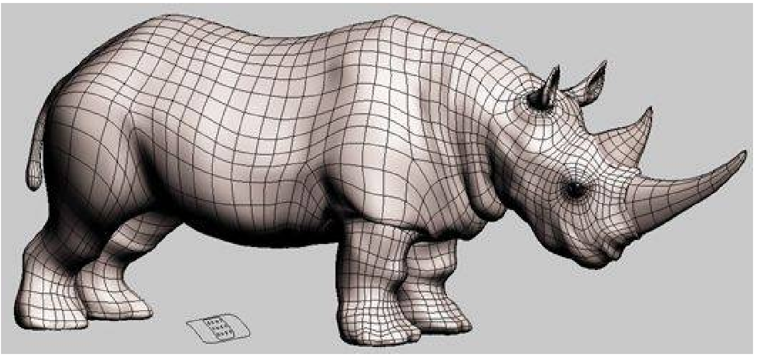
Tensor Product B-Spline Topology Limitations

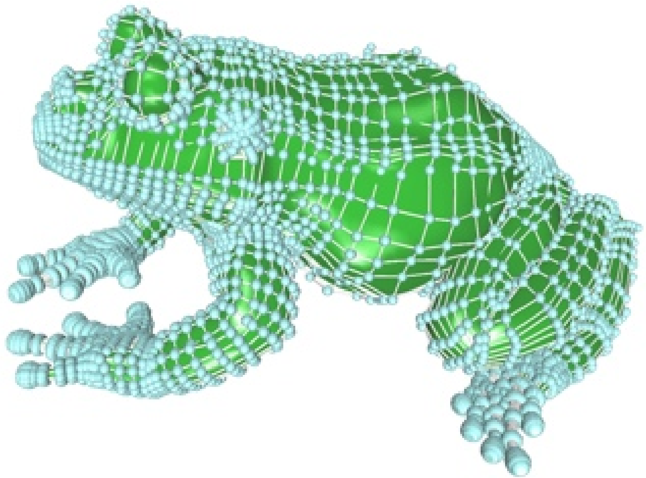


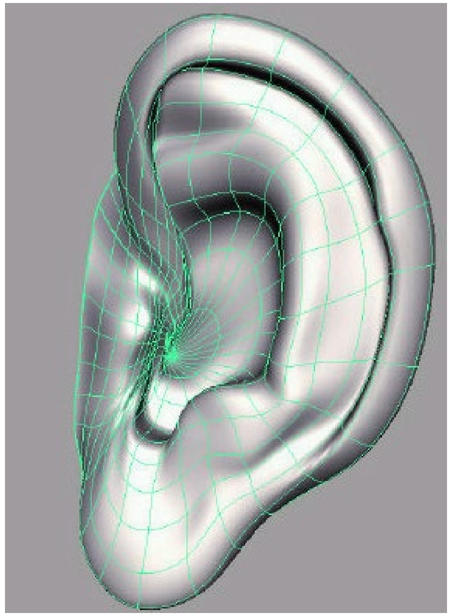


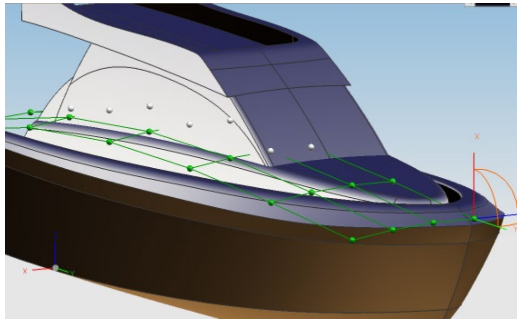


NURBS Surfaces









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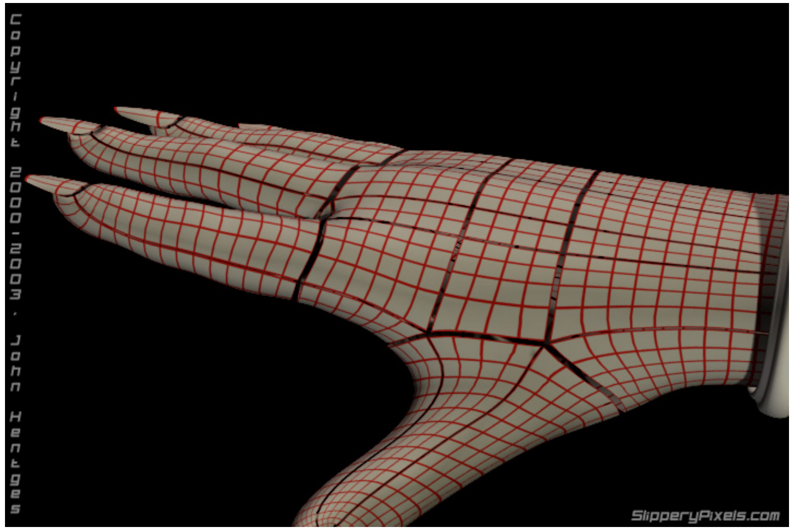
Problems with NURBS

- A single NURBS patch is either a topological disk, a tube or a torus

- Must use many NURBS patches to model complex geometry

- When deforming a surface made of NURBS patches, cracks arise at the seams



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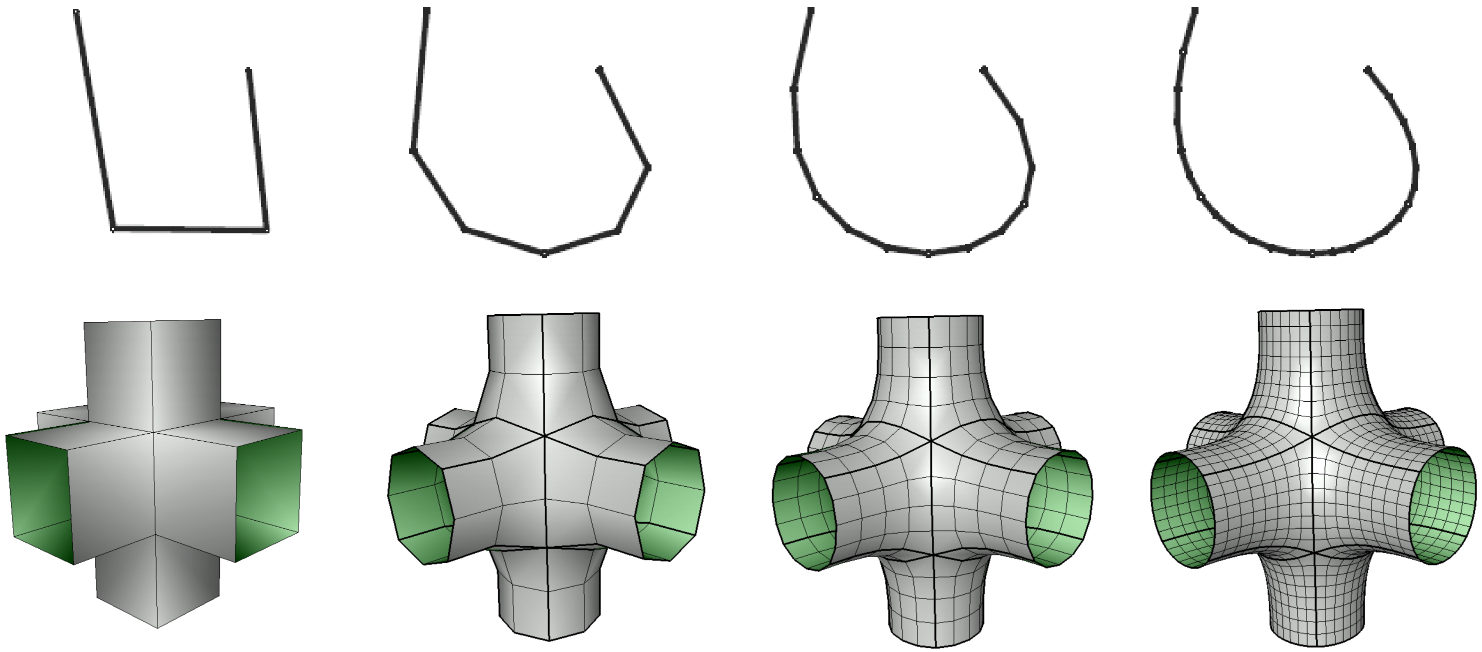
Subdivision

"Subdivision defines a smooth curve or surface as the limit of a sequence of successive refinements"



Subdivision Surfaces

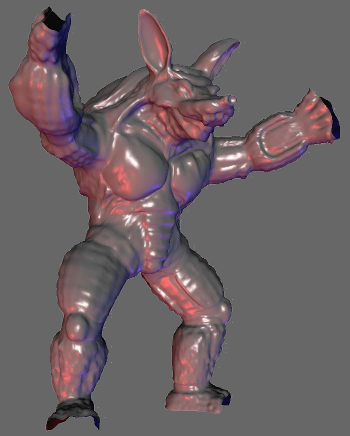
Subdivision Surfaces

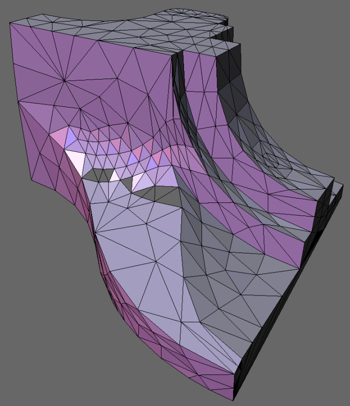
Peter Schröder, Caltech
Denis Zorin, NYU

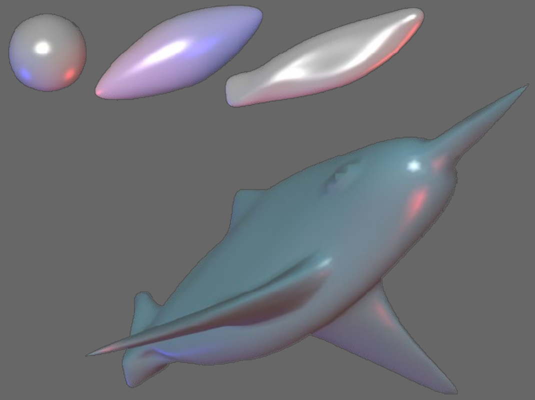
Geometric Modeling

Surface representations

- large class of surfaces
- interactive manipulation/display
- numerical modeling tasks







Representations

General philosophy

- same core representation for multiple purposes
  - transmission
  - rendering
  - simulation
  - editing

Splined Surfaces

Advantages

- high level control (control points)
- compact representation
- multiresolution structure

Disadvantages

- difficult to maintain and manage
- topology limitations

Polygonal Meshes

Advantages

- very general
- direct hardware implementation

Disadvantages

- heavy weight representation
- good editing semantics difficult
- limited multiresolution structure

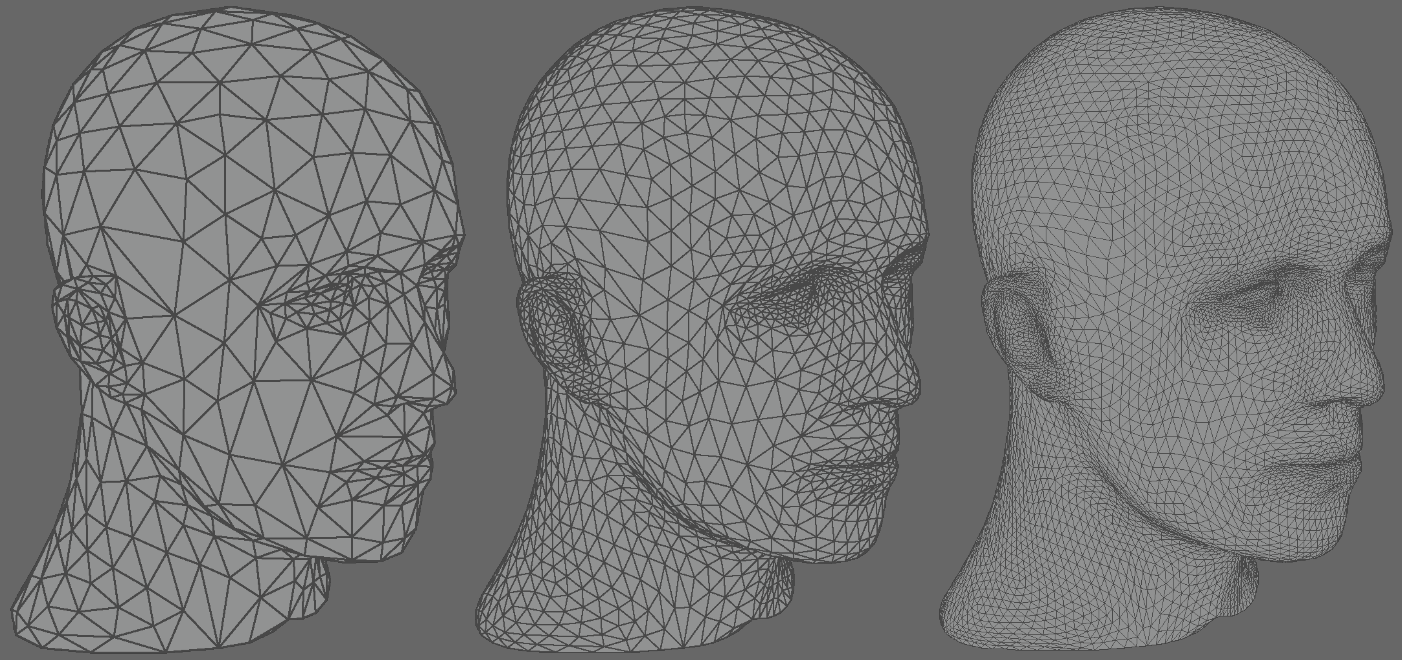
Subdivision Surfaces

Important modeling primitive

- smooth, arbitrary topology surface modeling
- generalizes spline patches
- covers range of representations from “pure” spline patches to “pure” meshes
  - BUT: special connectivity (more on that later)

Subdivision

Smooth surfaces as the limit of a sequence of successive refinements



Why Subdivision?

Advantages

- arbitrary topology, smooth surfaces
- support for compression and LOD
- suitable for wavelet-based numerical solvers

Scalability

- large datasets on small machines

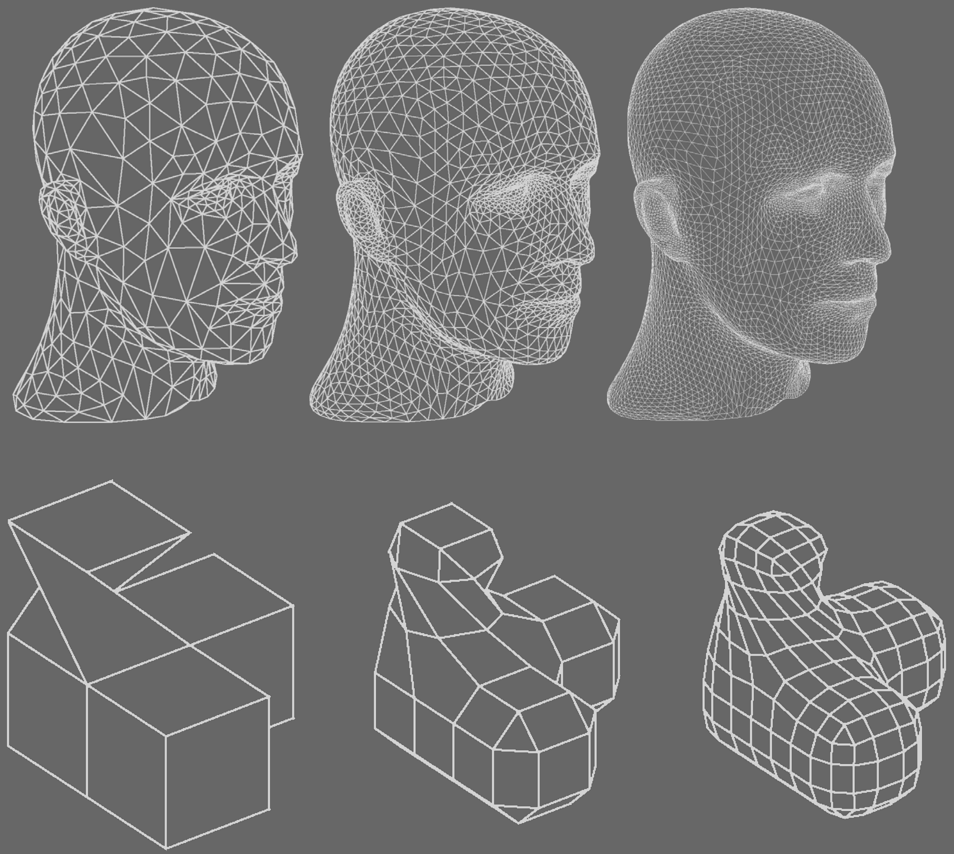
Algorithms

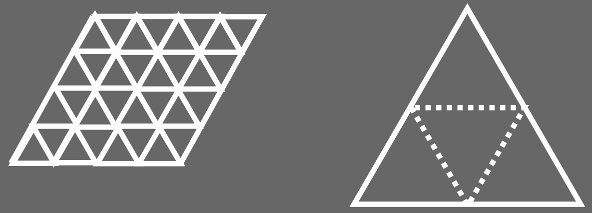
Properties

- exact evaluation
  - value, tangent planes, derivatives
  - moments
- efficient computation
- simple data structures
- integrates well with spline methods

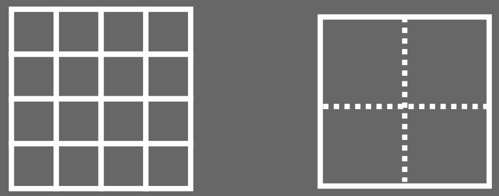
SUBDIVISION ZOO

Triangular and quadrilateral subdivision





Triangular



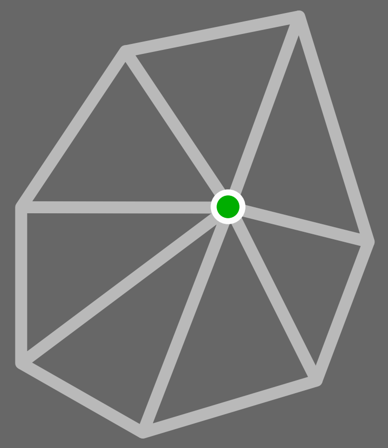
Quadrangular

Subdivision

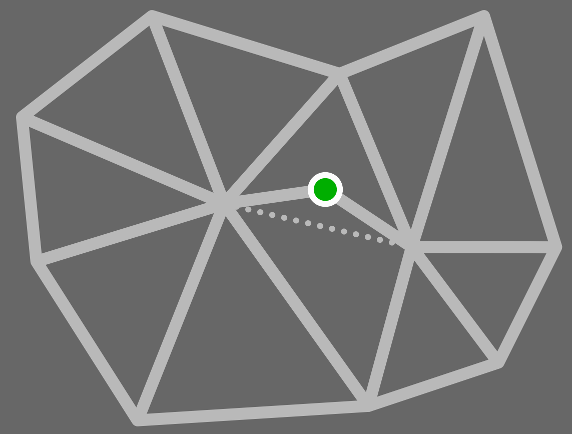
Geometric rules for adding points

- geometry is a map defined on graph
- extension rule

even = “old”



even at level i



odd = “new”

odd at level i

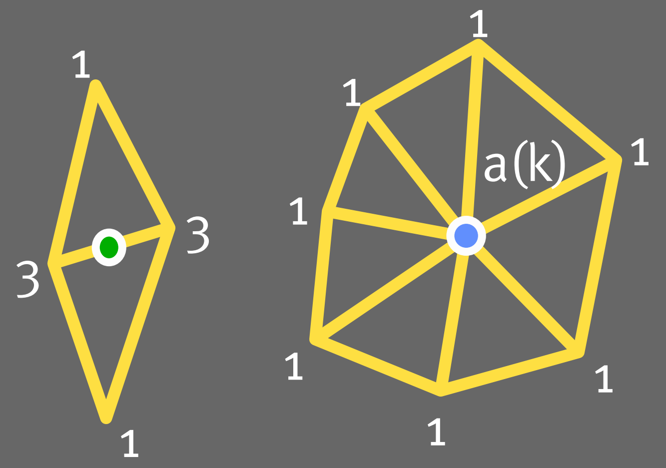
Advantages

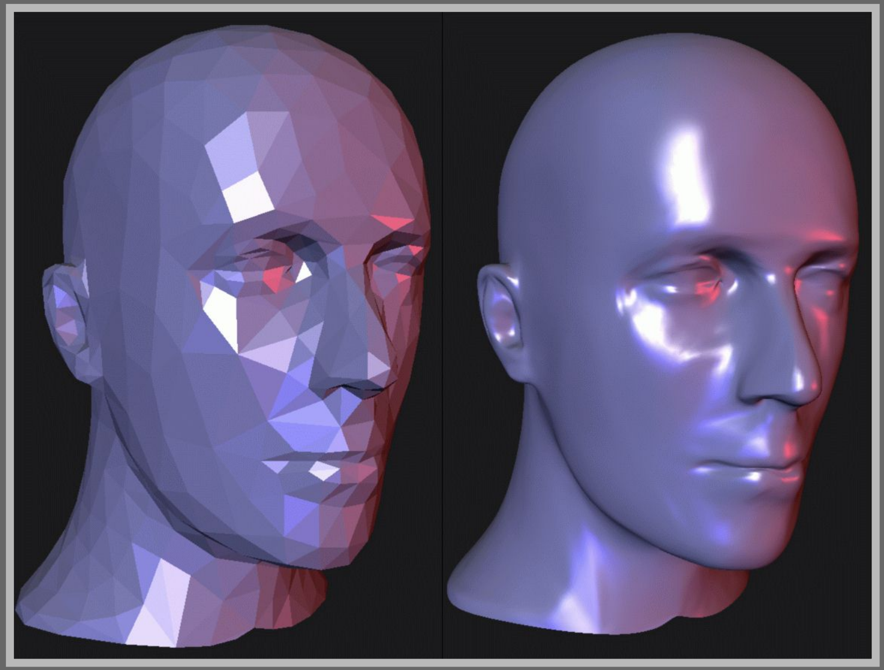
- approximating schemes
  - based on splines, small support
- interpolating schemes
  - control points on surface
  - in-place implementation

Approximating

Insert new, smooth new and old

- generalizes spline patches

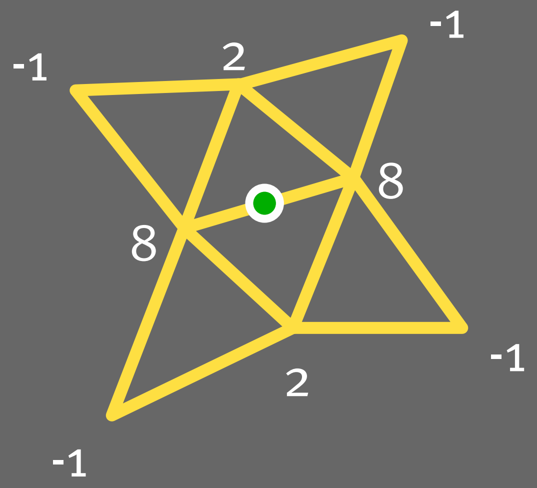


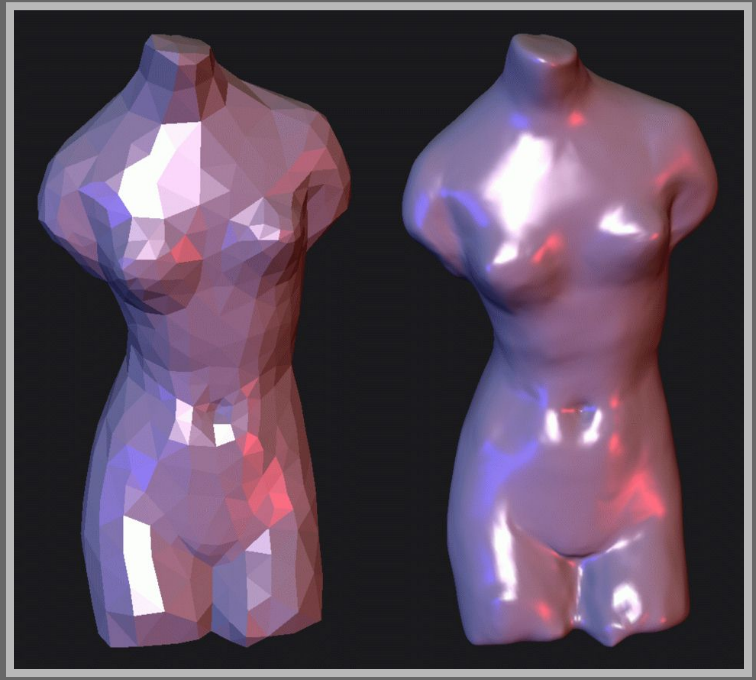


Interpolating

Keep old points, insert new ones

- affine combination of nearby points

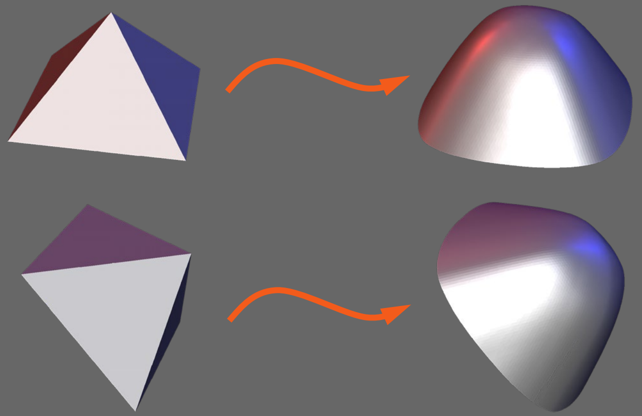


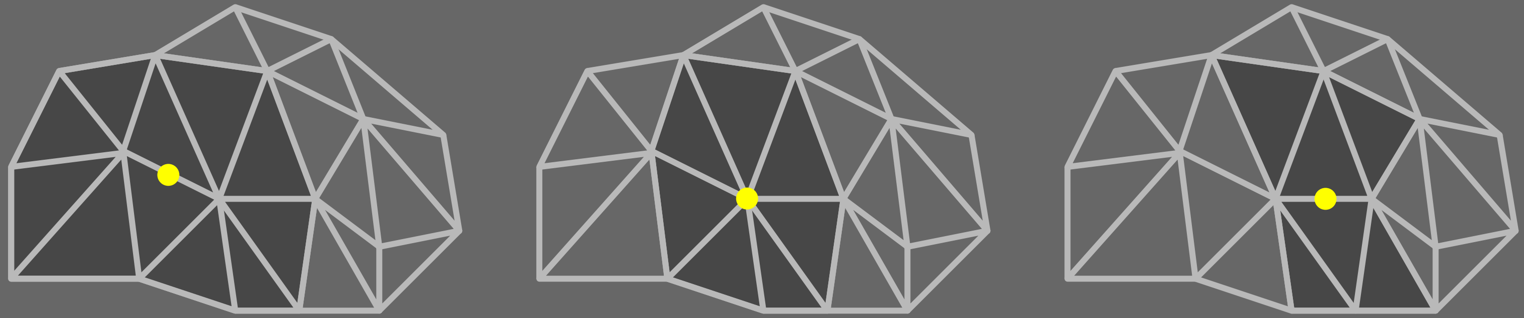


Subdivision

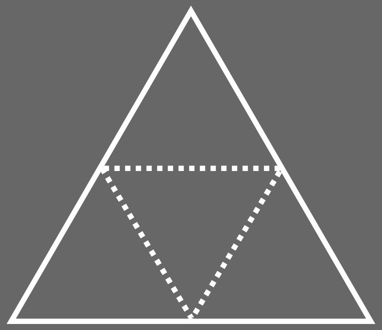
Properties

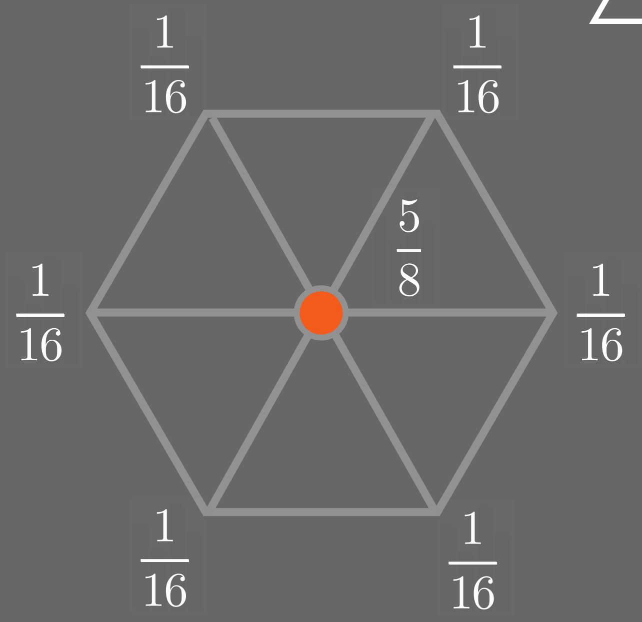
- affine invariance
- local definition
- compact support

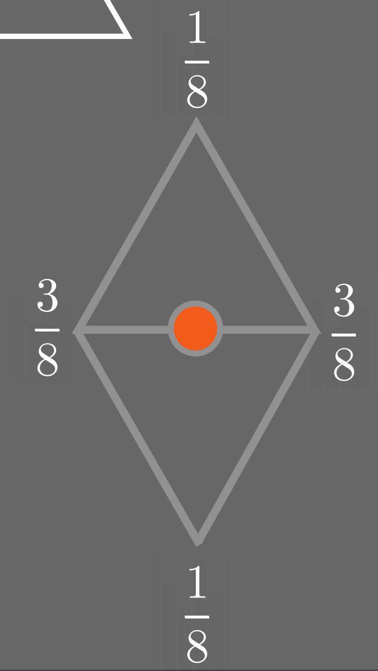


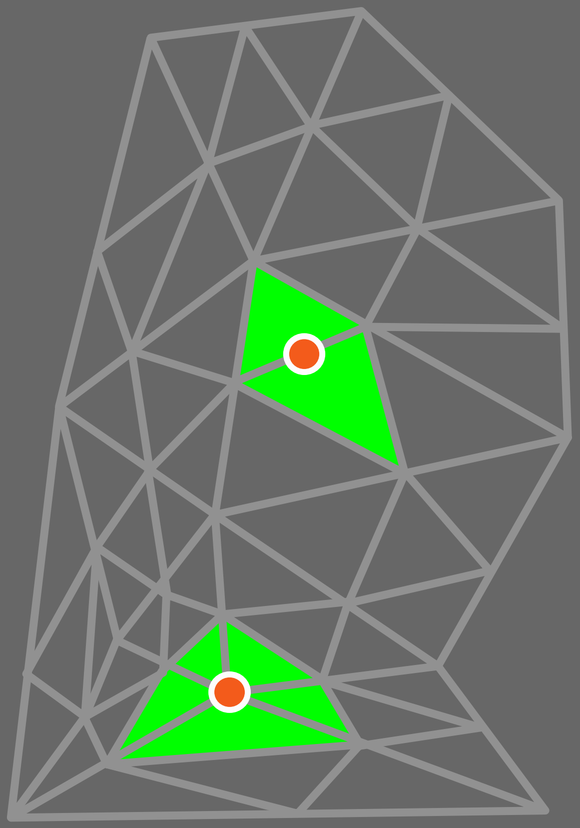


Example: Loop scheme

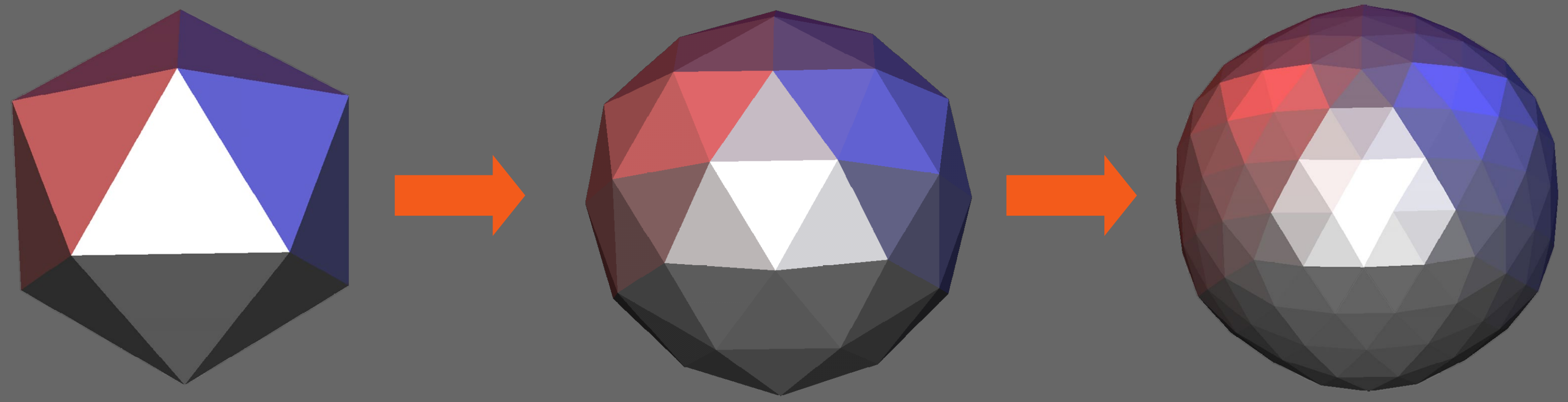








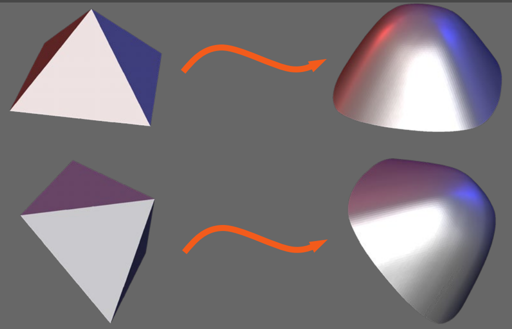
Example: Loop scheme



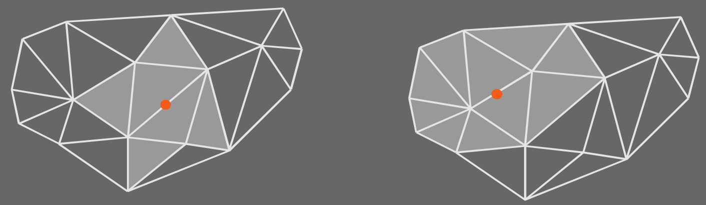
- For a “good” scheme, recursive application approximates a smooth surface

Constructing the Rules

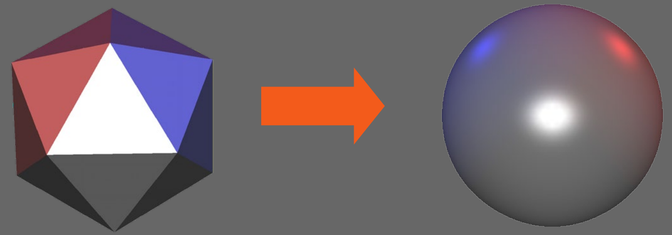
- Invariance under rotations and translations



- Small support

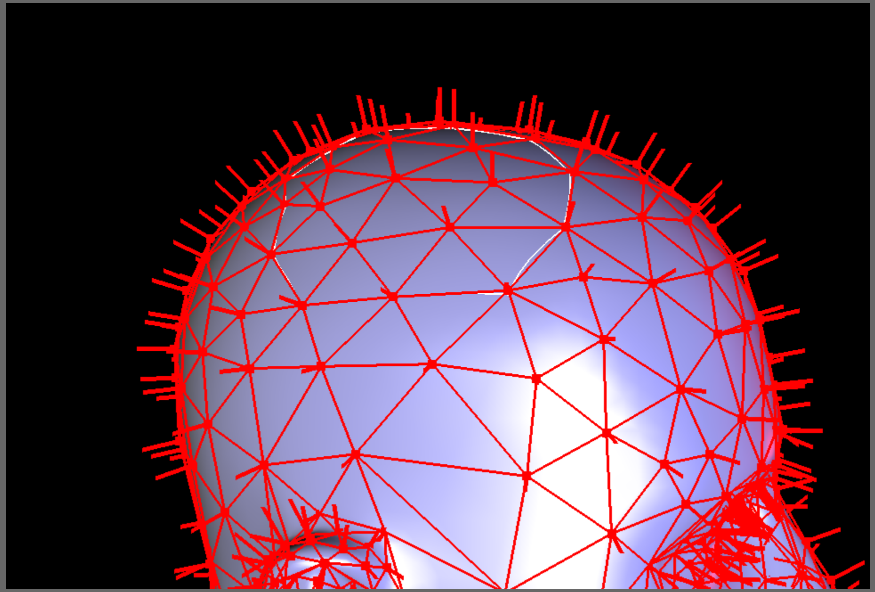


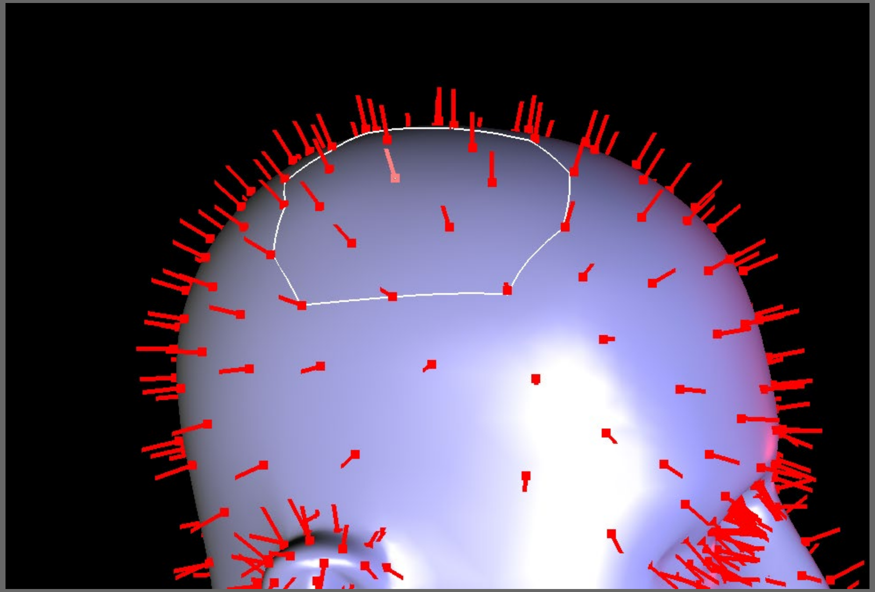
- Smoothness and Fairness



Control points

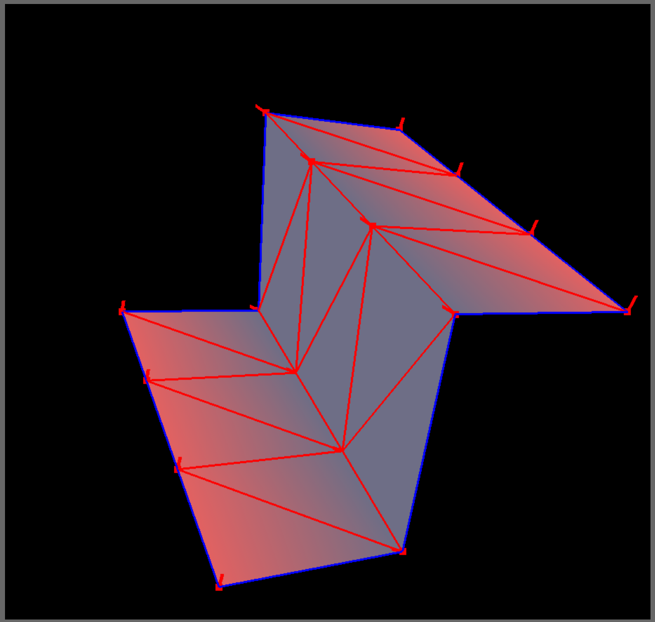
- vertices of the initial mesh define the surface
- each vertex influences a finite part of the surface

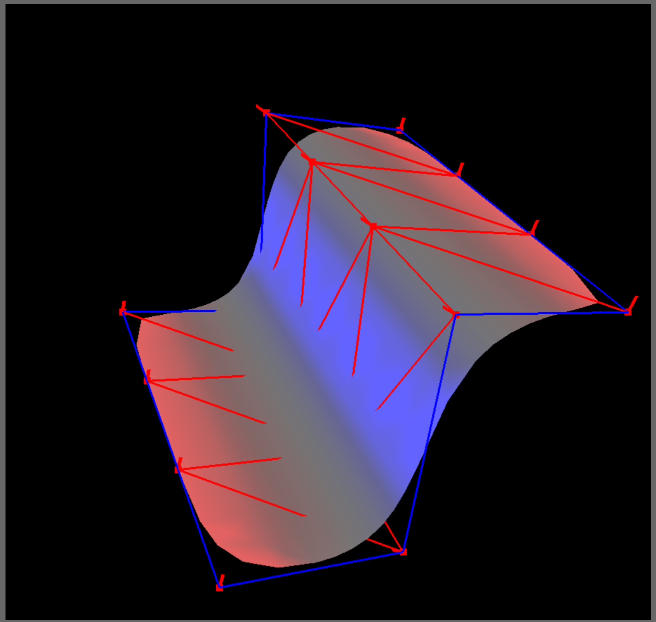




Subdivision and splines

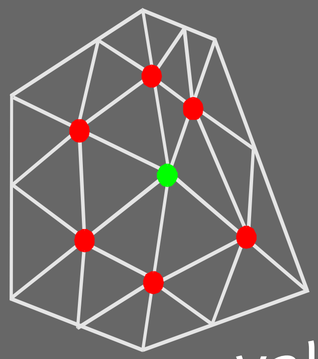
- For splines, the control mesh is regular



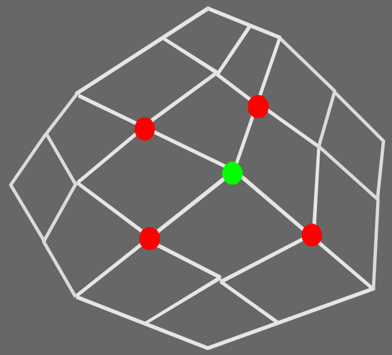


Extraordinary Vertices

Triangular meshes | Quad meshes

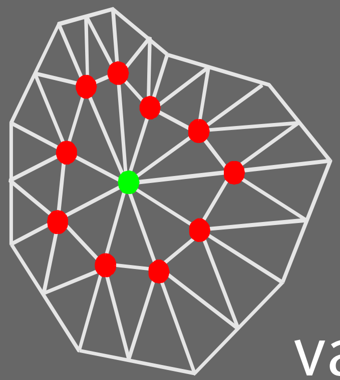


regular

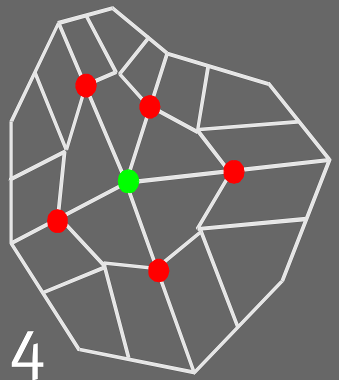


valence 6

valence 4



extraordinary



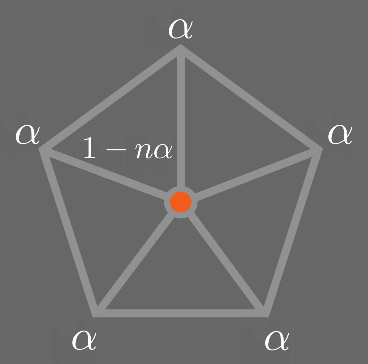
valence $\neq 6$

valence $\neq 4$

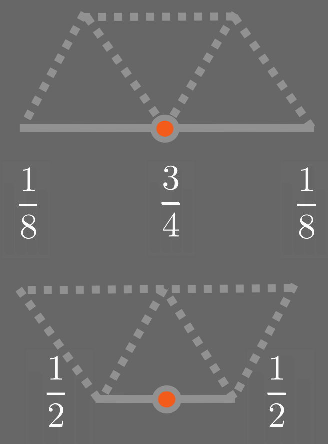
Constructing the rules

- Start with spline rules (or other smooth rules)
- Define rules for:

Extraordinary vertices



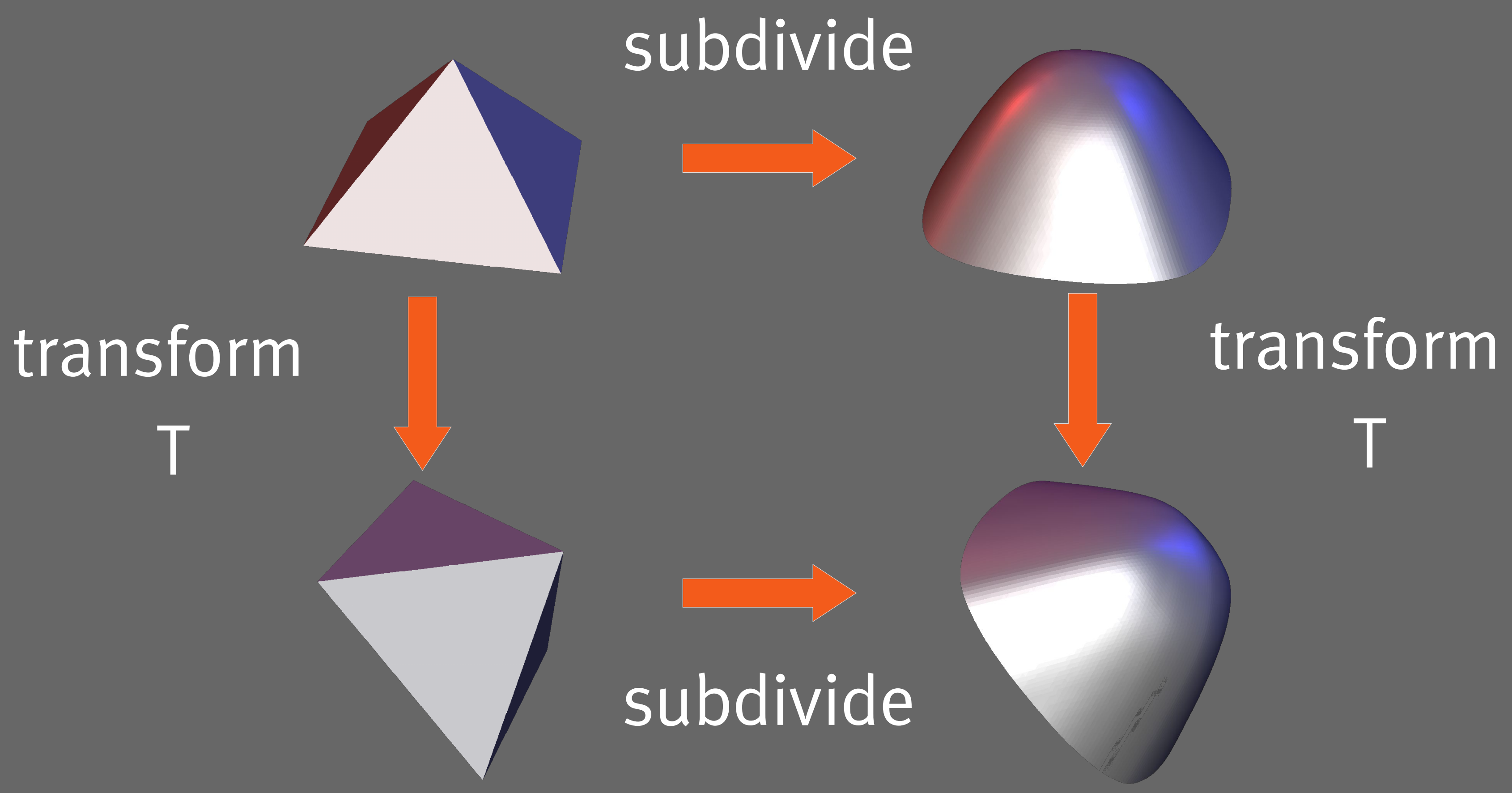
Boundaries



Creases etc.

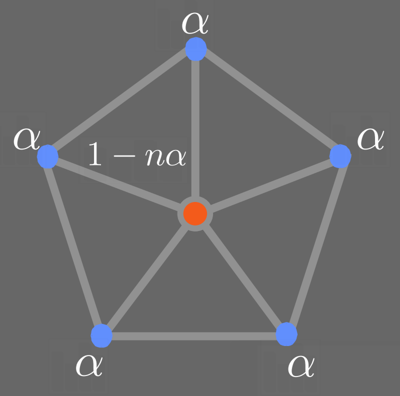
• • •

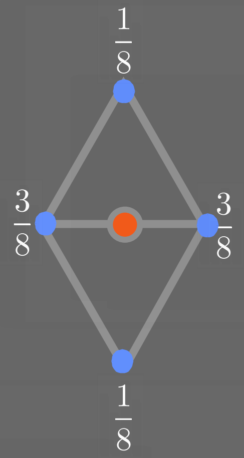
Affine invariance



Affine invariance

- the coeficients of any mask should sum up to 1





$$p = \sum a_i p_i$$

displacement

$$\sum a_i (p_i + t) = \underbrace{\left(\sum a_i\right)}_{1} t + p$$

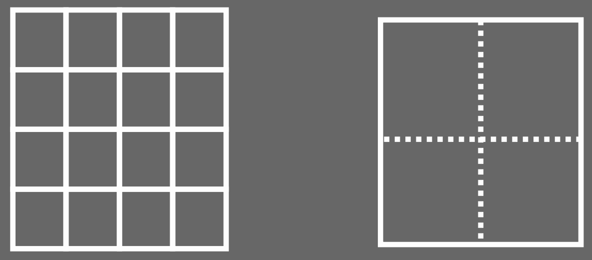
Classification of schemes

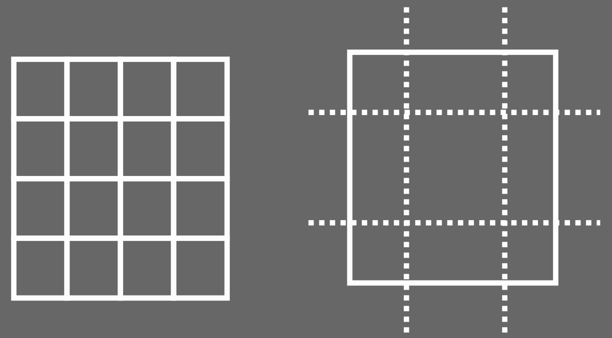
Classification criteria

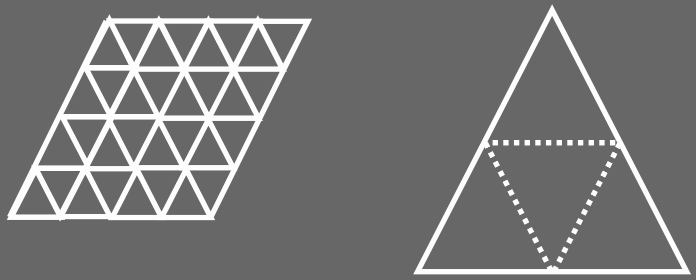
- type of refinement rule (primal or dual)
- type of mesh (triangular or quad or...)
- approximating or interpolating

Refinement rules

- Primal (vertex insertion)
- Dual (corner cutting)

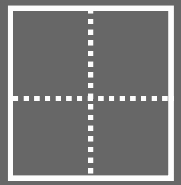
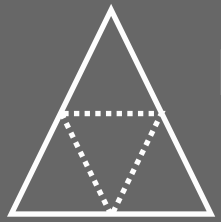




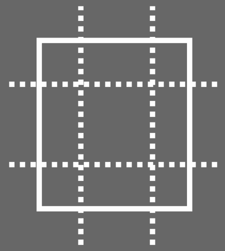


Subdivision schemes

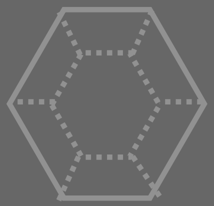
Primal (vertex insertion)

	Approximating	Interpolating
	Catmull-Clark	Kobbelt
	Loop	Butterfly

Dual (corner cutting)



Doo-Sabin,
Midedge

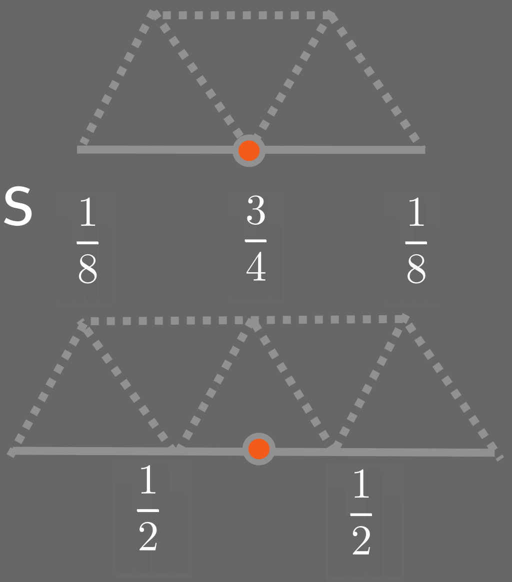


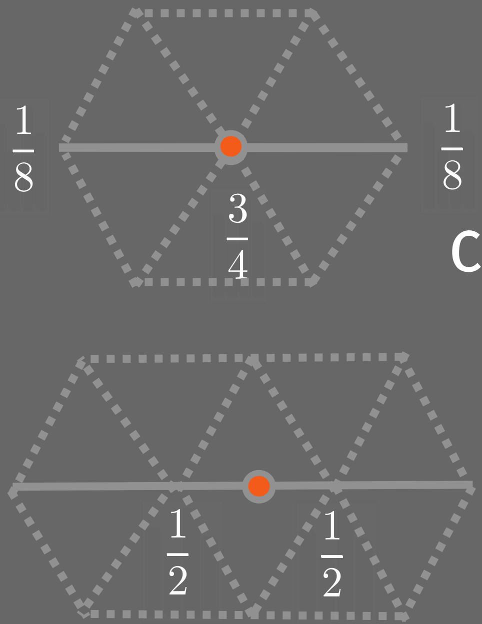
?

Boundaries and creases

- special rules on and near the boundary
- boundary independent of the interior

boundaries



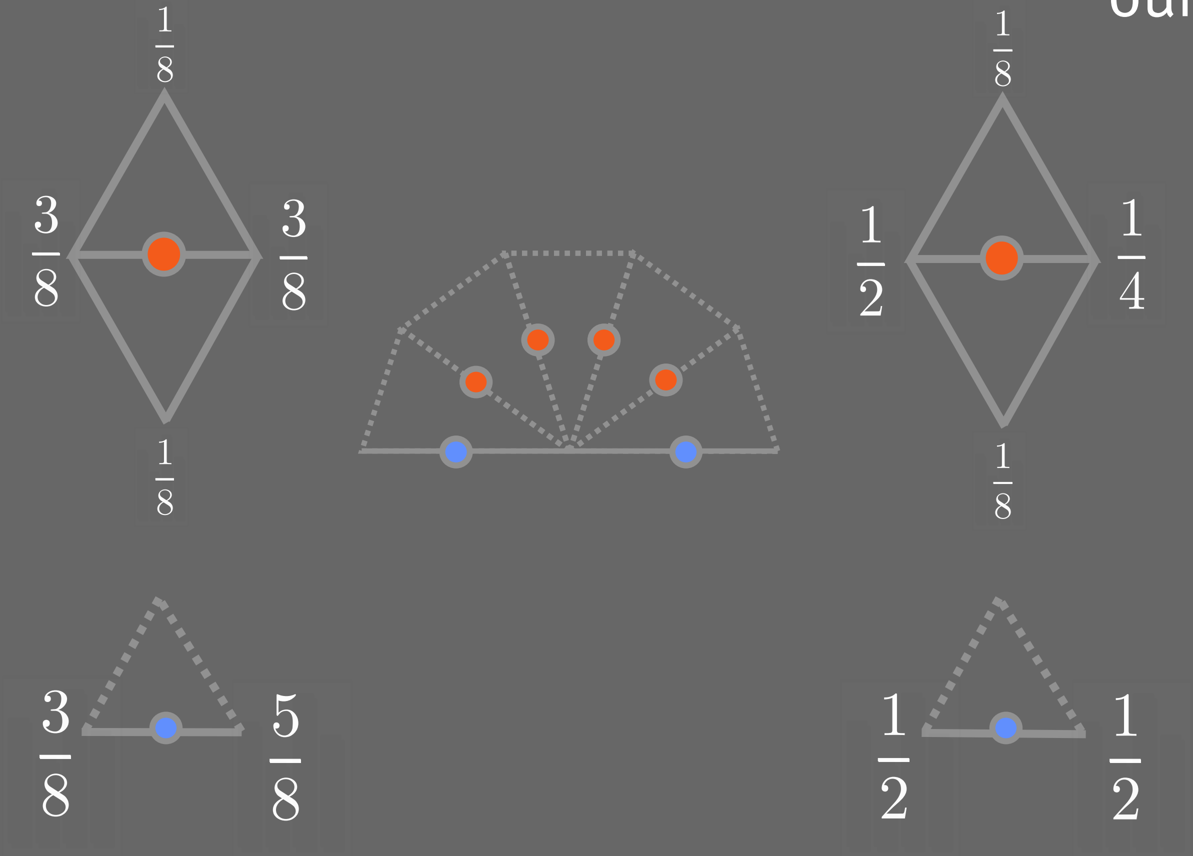


creases

Loop scheme, boundaries and creases

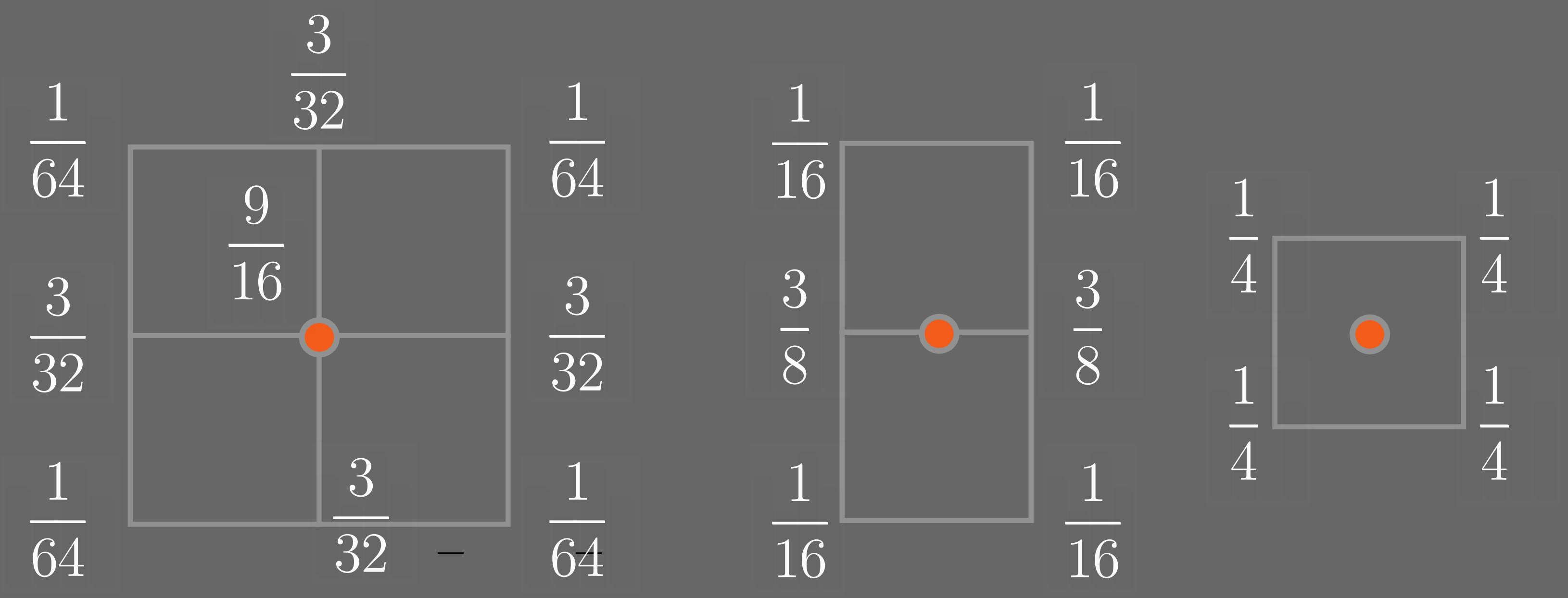
Hoppe et al

our rules



Catmull-Clark scheme

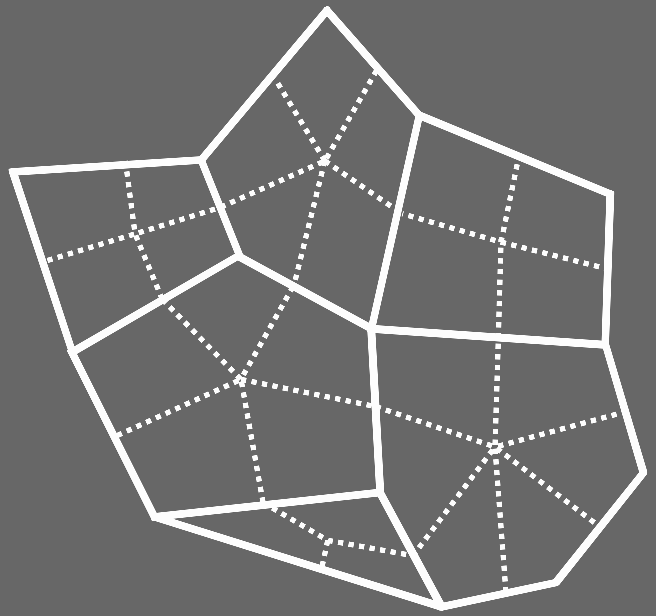
- quadrilateral, approximating
- tensor-product bicubic splines



Catmull-Clark scheme

Reduction to a quadrilateral mesh

- do one step of subdivision with special rules; all polygons become quads

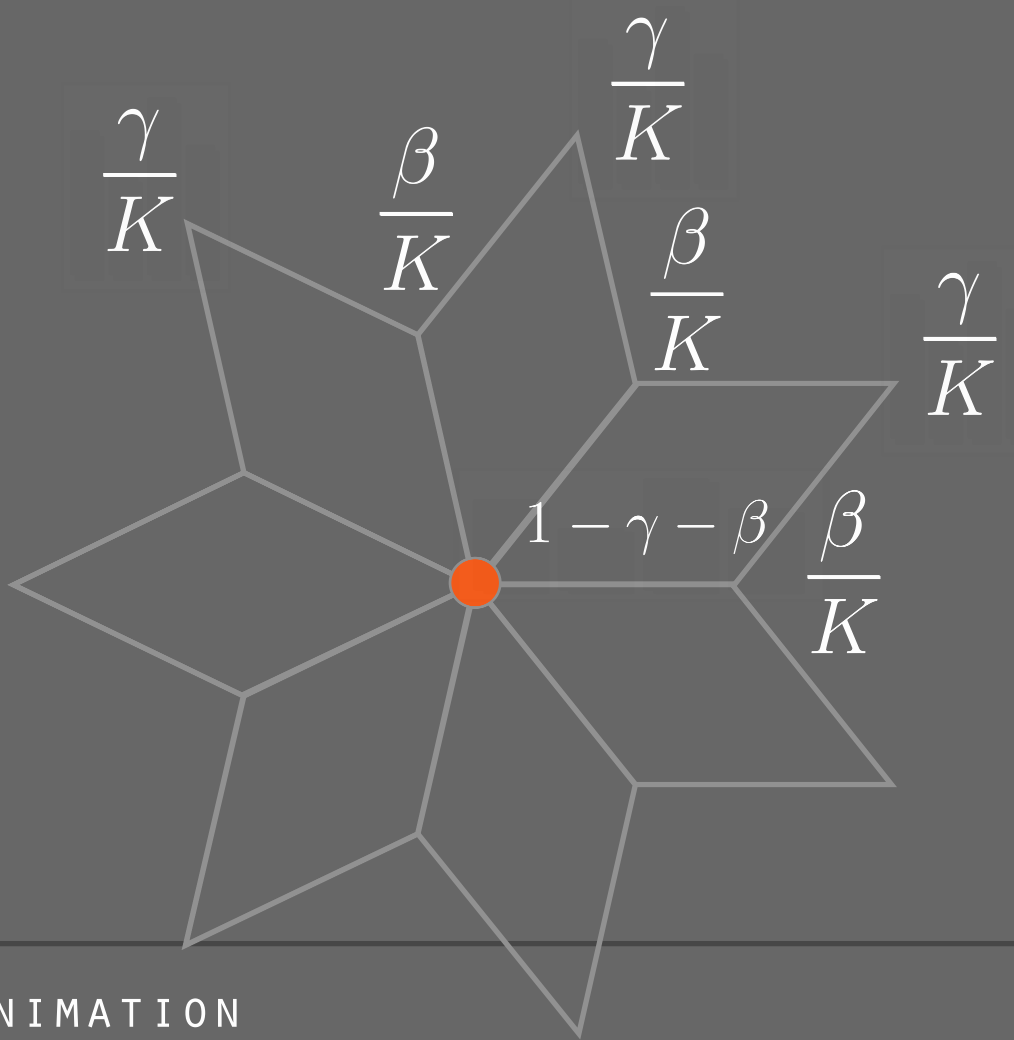


Catmull-Clark scheme

Extraordinary vertices

$$\gamma = \frac{1}{4K}$$

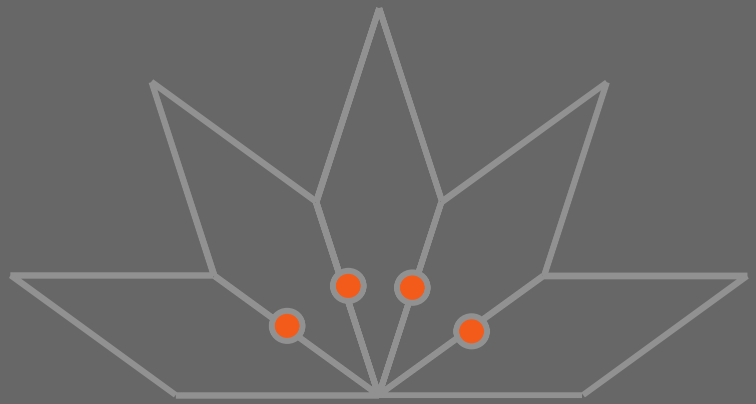
$$\beta = \frac{3}{2K}$$

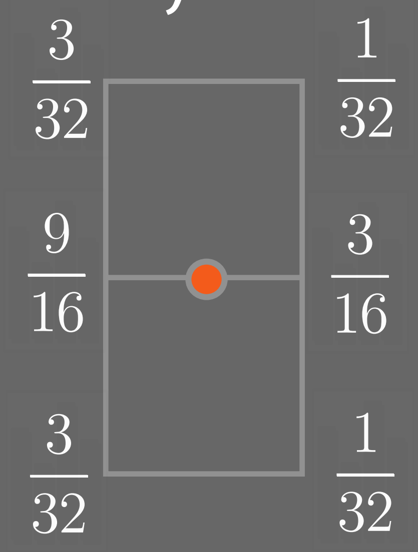


Catmull-Clark scheme

boundaries and creases

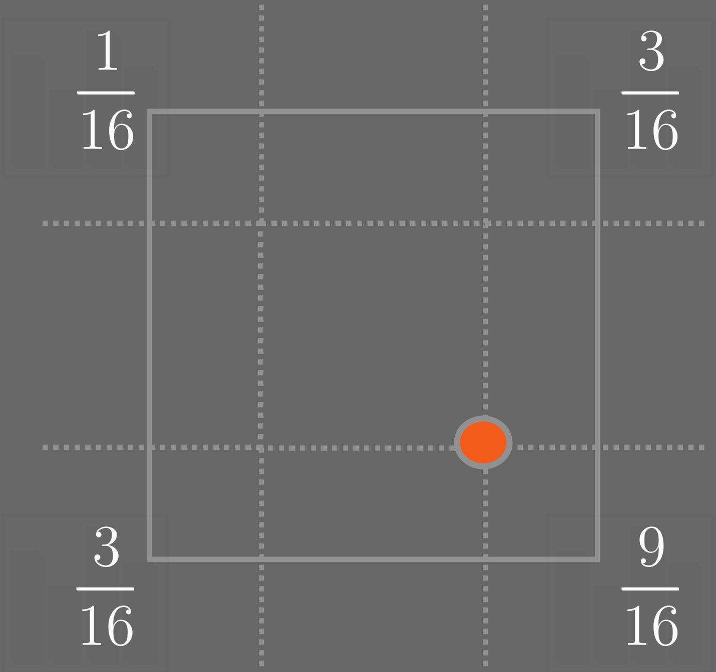
- cubic spline (same as Loop!)
- need to fix rules for C1-continuity





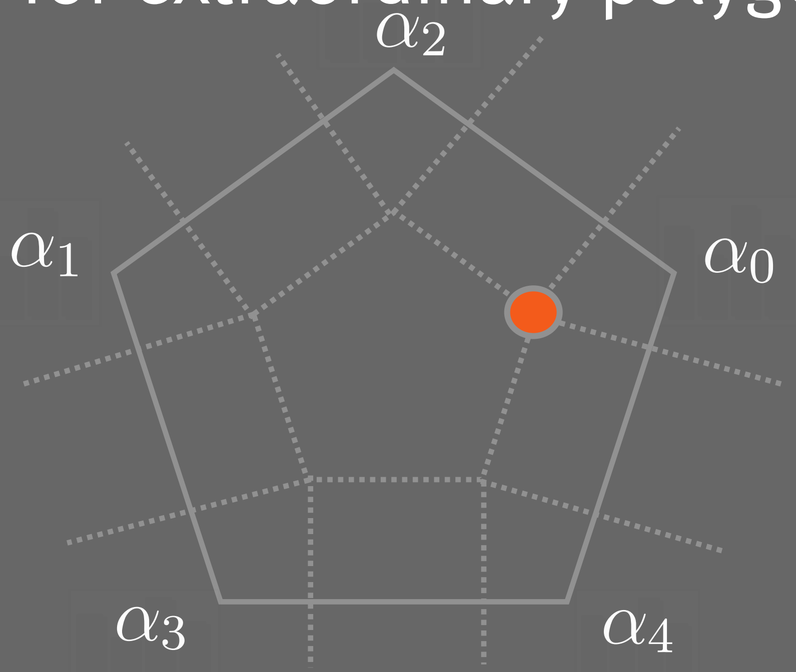
Doo-Sabin scheme

- dual scheme, quadrilateral
- extends tensor-product biquadratic splines



Doo-Sabin scheme

- after one step, all valences = 4
- rule for extraordinary polygons:



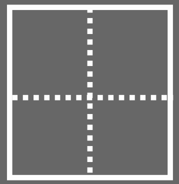
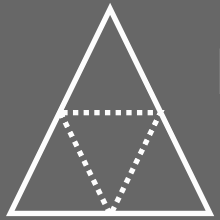
$$\alpha_0 = \frac{1+5K}{4}$$

for $i = 1 \ldots K-1$

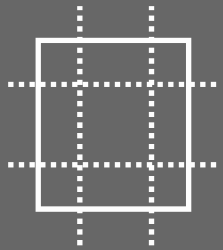
$$\alpha_i = \frac{1}{K}\left(3 + 2\cos\frac{2i\pi}{K}\right)$$

Summary

- Primal (vertex insertion)

	Approximating	Interpolating
	Catmull-Clark	Kobbelt
	Loop	Butterfly

- Dual (corner cutting)



Doo-Sabin,
Midedge

Limitations of subdivision

- no C2 with small support
- decrease of smoothness with valence
- ripples
- no direct control of fairness

That’s All