Original Lecture #8: 27 October 1992
Topics: Spline Curves
Scribe: Andrew Grenville

1 Degree Raising

An n-ic curve is a curve of degree *at most n*. In this section we consider representing a curve with more control points than required as a degenerate case of a higher order curve.

1.1 Affine Curve as a Degenerate Quadratic

Let $F(t)$ be affine and $G(t)$ quadratic such that

$$G(t) = F(t) \qquad (\forall t). \tag{1}$$

The polar form of $G(t)$ will have two arguments and is given by

$$g(t_1,t_2) = \frac{f(t_1)+f(t_2)}{2} = \frac{F(t_1)+F(t_2)}{2}. \tag{2}$$

This satisfies the three properties of polar forms (see Handout 19 of 1991, p.16):

- symmetric
- biaffine
- $g(t,t) = f(t) = F(t)$

Where is the middle Bézier point? Evaluating $g(0,1)$ we see it is simply the midpoint of the line connecting $f(0)$ and $f(1)$,

$$g(0,1) = \frac{f(0)+f(1)}{2}. \tag{3}$$

1.2 Quadratic Curve as a Degenerate Cubic

We now consider raising a quadratic to a cubic and representing it with four Bézier points instead of three. Let $G(t)$ be quadratic and $H(t)$ be cubic such that

$$H(t) = G(t) \qquad (\forall t). \tag{4}$$

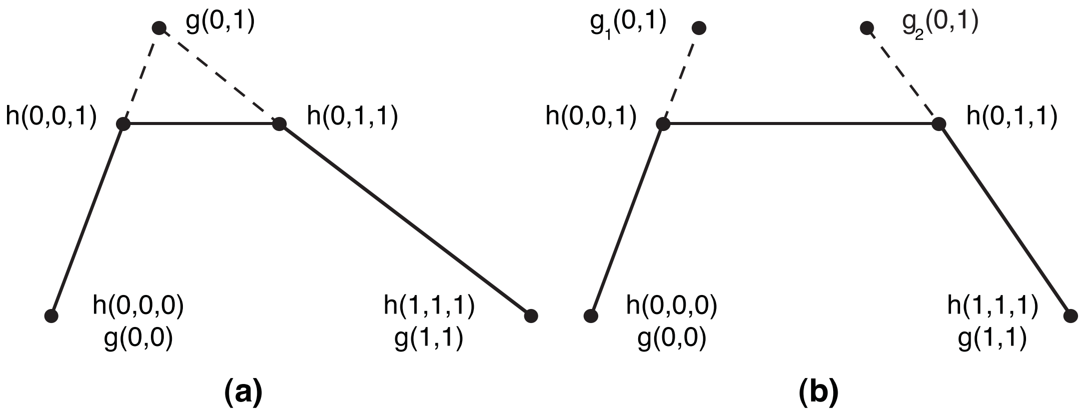

Figure 1: Test to determine whether curve is a quadratic in disguise: a) curve is a quadratic, b) curve is a real cubic.

The polar form of $H(t)$ must be symmetric, triaffine and satisfy $h(t,t,t) = g(t,t) = G(t)$:

$$h(t_1,t_2,t_3) = \frac{g(t_1,t_2)+g(t_1,t_3)+g(t_2,t_3)}{3}. \tag{5}$$

Evaluation of h yields the two new control points

$$h(0,0,1) = \frac{g(0,0)+2g(0,1)}{3}, \tag{6}$$

$$h(0,1,1) = \frac{2g(0,1)+g(1,1)}{3}. \tag{7}$$

Using this relationship between the control points for a quadratic and those for a cubic, we can determine whether a set of four Bézier points describes a real cubic or simply a quadratic in disguise. Solving both Equations 6 and 7 for $g(0,1)$ and noting that $g(t,t) = h(t,t,t)$, we obtain respectively

$$g(0,1) = \frac{3}{2}h(0,0,1) - \frac{1}{2}h(0,0,0), \tag{8}$$

$$g(0,1) = \frac{3}{2}h(0,1,1) - \frac{1}{2}h(1,1,1). \tag{9}$$

If $g(0,1)$ found in both approaches is the same, the curve must actually be of degree 2 at most.

Graphically, this is shown in Figure 1. First, we find the point $g(0,1)$ by extending the line from $h(0,0,0)$ to $h(0,0,1)$ half again as far as the separation of those points. We repeat this on the line from $h(1,1,1)$ to $h(0,1,1)$ to find the position of $g(0,1)$ by the second approach. If both points are the same, the curve is indeed a quadratic in disguise, otherwise it is a true cubic.

The method of degree raising and checking for degeneracy described here may be extended to curves of higher degree.

2 Differentiation and Polar Forms

The material covered during the remainder of this lecture is discussed in detail in Handout 19 of this year by the lecturers. The sections that follow will serve to highlight what was presented in class.

Let F be a parametric, polynomial, n-ic curve and f its polar form. Our goal here is to relate the derivative of F to its polar values. We start by finding the homogeneous polar form of the function as this simplifies the process of finding the derivative.

2.1 Homogenizing Polar Forms

Until now, we have been treating time, t, as a scalar. However, it has really been a point on the parameter line. Also, to perform differentiation, it is convenient to evaluate F at parameter sites which include vectors as well as points. This requires homogenization of both the function F *and* of time as discussed in Section 1.1 of Handout 19.

In particular, a weight coordinate, s, has been added to time where

$$T := (s;t).$$

For brevity, overbar notation is introduced to refer to points on lines ($s = 1$) by

$$\bar{t} := (1;t).$$

It is also useful to define a vector representing the difference between times $T = 1$ and $T = 0$:

$$\delta := (1;1) - (1;0) = \overline{1} - \overline{0} = (0;1).$$

As shown on p. 4 of Handout 19, there exists a simple relationship between the derivative of a function and its polar form. Starting with the definition of the derivative,

$$F'(\bar{t}) = \lim_{h \to 0} \frac{F(\overline{t+h}) - F(\bar{t})}{h}, \tag{10}$$

we obtain the final answer,

$$F'(\bar{t}) = 3f(\bar{t}, \bar{t}, \delta). \tag{11}$$

As expected, the derivative is a quadratic and has only two variable arguments remaining.

Graphically, the derivative at $\bar{t}$ is three times the vector from $f(\bar{t}, \bar{t}, 0)$ to $f(\bar{t}, \bar{t}, 1)$ as shown in Figure 2.

Note that this result holds only for parametrization along an interval of length 1. More generally, given the Bézier points of the segment $F([p \mathinner{..} q])$, the derivative at $\overline{p}$ is

$$F'(\overline{p}) = \frac{3}{q-p}(f(\overline{p}, \overline{p}, \overline{q}) - f(\overline{p}, \overline{p}, \overline{p})). \tag{12}$$

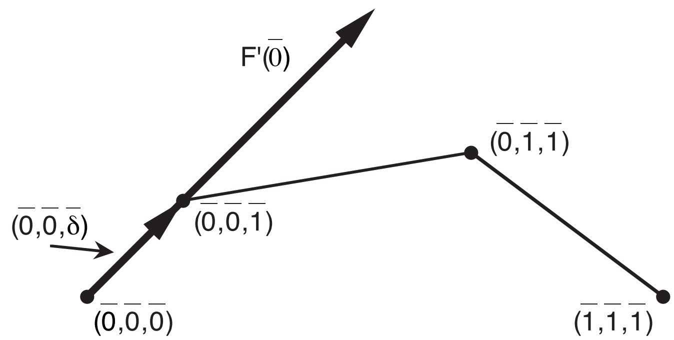

Figure 2: Derivative at $f(\overline{0},\overline{0},\overline{0})$ is three times $f(\overline{0},\overline{0},\overline{\delta})$.

3 Spline Curves

In practice, to form a polynomial curve, it makes most sense to specify a sequence of lower order curves (typically cubic) rather than a single higher order curve. Before these curves can be assembled however, we must formulate a method for controlling the continuity at each joint in the sequence.

3.1 Continuity

There are two types of continuity:

- *Parametric* continuity, denoted by C^k requires smoothness to k^{th} order of both the curve and its parametrization (eg. a movie of particle would look smooth).

- *Geometric* continuity, denoted by G^k, requires only that the curve itself be smooth to k^{th} order (eg. the tracks in the snow left after the particle passed by would look smooth).

Even though geometric continuity is a weaker condition, parametric continuity is simpler to achieve mathematically so we will concentrate on the latter in CS348a.

For example, G^1 means that the slope is continuous, whereas C^1 requires continuity of the velocity vector. Similarly, G^2 dictates continuous slope and curvature, whereas C^2 holds when both the velocity and acceleration vectors are continuous.

Polar forms are particularly well suited to describing parametric continuity. The following key result is proved in Handout 19. Two n-ic curve segments F and G join with C^k continuity at joint q if and only if

$$f(\underbrace{q,q,\ldots,q}_{n-k},t_1,t_2,\ldots,t_k) = g(\underbrace{q,q,\ldots,q}_{n-k},t_1,t_2,\ldots,t_k) \tag{13}$$

for all t_1 through t_k.

To illustrate parametric continuity, we consider the degrees of continuity for a cubic:

C^{-1}	$f(q,q,q)$	$\neq$	$g(q,q,q)$	no continuity
C^0	$f(q,q,q)$	$=$	$g(q,q,q)$	position continuous
C^1	$f(q,q,t_1)$	$=$	$g(q,q,t_1)$	velocity continuous
C^2	$f(q,t_1,t_2)$	$=$	$g(q,t_1,t_2)$	acceleration continuous
C^3	$f(t_1,t_2,t_3)$	$=$	$g(t_1,t_2,t_3)$	identical functions

3.2 Knots

We can now assemble a spline curve out of a sequence of curves and specify a degree of continuity C^k at each joint. If the spline is constructed from segments of degree at most n, it makes sense to allow continuity from C^{-1} to C^{n-1}. At one extreme, C^{-1} allows the curve to make a discontinuous jump and at the other extreme, continuity greater than C^{n-1} would make the joint redundant.

Each joint is consequently characterized by the number of derivatives that are broken. This number is defined as the multiplicity of the *knot* corresponding to that joint. A simple knot occurs when a single derivative is broken (C^{n-1}), a double knot when two derivatives are broken (C^{n-2}), and so on. At an $(n+1)$-fold knot, we break all derivatives including the position, resulting in C^{-1} continuity. We can therefore think of a knot as "the right to break one derivative".

3.3 Knot Sequences

The knot sequence is a list of the time of each joint with the time repeated according to the multiplicity of that particular knot. As an example, we consider a cubic spline F with the knot sequence:

$$(0,0,0,0,1,2,2,2,4,5,5,6,7,8,8,8,8)$$

C^{-1} continuity exists at the endpoints to ensure isolation from other splines. There is C^0 continuity at time 2 between $F([1..2])$ and $F([2..4])$, C^1 continuity at time 5 between $F([4..5])$ and $F([5..6])$ and C^2 continuity everywhere else. Construction of this spline is illustrated in Figure 3.

The points labeled 222, 224, 244, and 444 in Figure 3 are the Bézier points of the single cubic segment $F([2..4])$ that the spline follows in the interval $[2..4]$ between adjacent knots. The figure also includes various points with 3's in their polar labels. Note that these points are carrying out the de Casteljau Algorithm to compute the point $F(3)$ from the Bézier points of the segment $F([2..4])$.

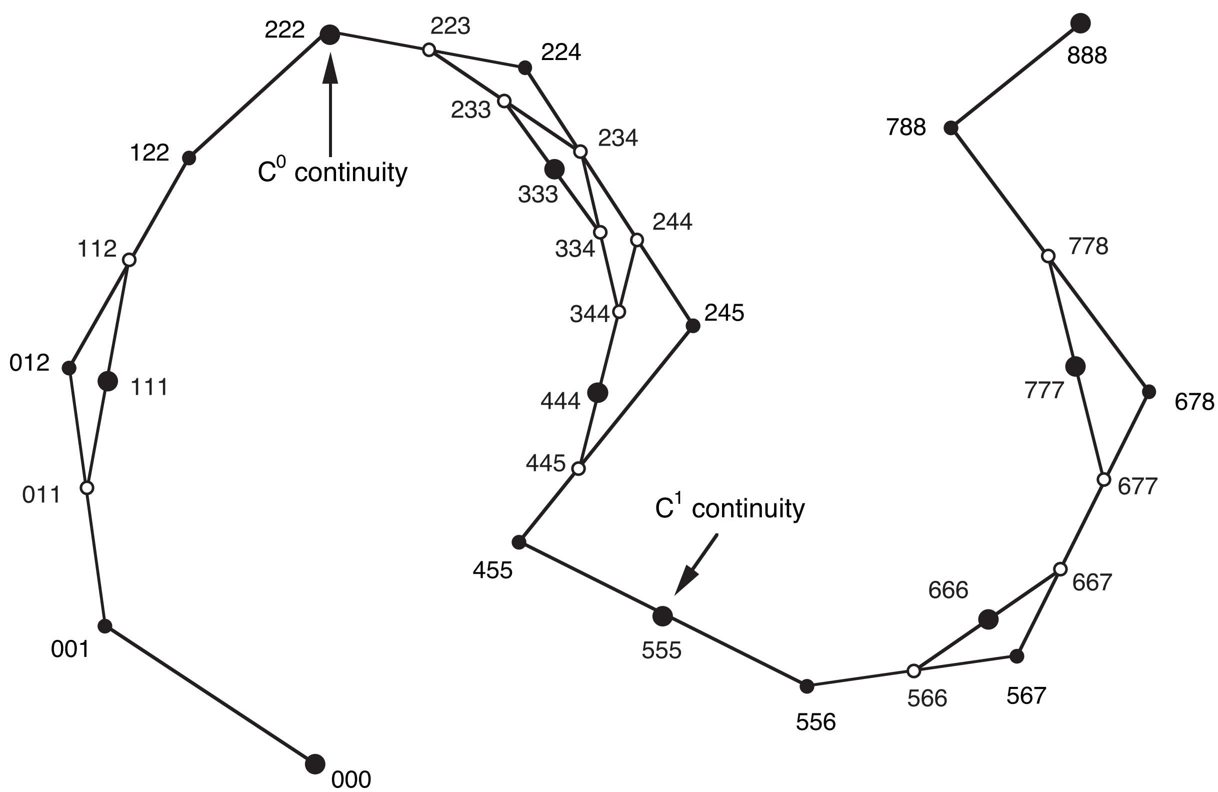

Figure 3: Construction of the spline from the knot sequence. The de Boor points are the vertices of the outer polygon. There is C^2 continuity except where noted — and at 333, where there is C^3 continuity.