Monte Carlo I: Foundations

cs348b
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Integration Challenges in Rendering

- Complex light geometry
- Complex visibility
- Discontinuous integrands
- Many times per pixel

\[ E(x) = \int_{H^2} L_i(x, \omega) |\cos \theta| d\omega \]
Advantages of MC Integration

• Few restrictions on the integrand
  • Doesn’t need to be continuous, smooth, ...
  • Only need to be able to evaluate at a point
• Extends to high-dimensional problems
• Conceptually straightforward
• Efficient for solving at just a few points
Disadvantages of MC

• Noisy
• Slow convergence
• Good implementation is hard
  • Debugging
  • Choosing appropriate techniques
Overview

• Random variables
  • Sampling from distributions
• Basic MC estimator
• Variance
  • Definition and causes
  • Variance reduction: importance sampling
• Efficiency
  • Splitting & Russian Roulette
Random Variables

• Definition
  • $\bar{x}$ is chosen by some random process
  • $\bar{x} \sim p(x)$ probability distribution function (pdf)
  • pdf is non-negative, integrates to one
  • *not* necessarily $\leq 1!$

• Pdf examples
  • $p(x) = 1, 0 \leq x < 1$
  • $p(x) = 1/2, 5 \leq x < 7$
  • $p(x) = \cos(x), 0 \leq x < \pi/2$
  • $p(x) = \delta(x)$
Probability Distributions

- **Discrete**
  - Events $x_i$ with probability $p_i \geq 0$, \( \sum_{i=1}^{N} p_i = 1 \)
  - Cumulative distribution $P_j = \sum_{i=1}^{j} p_i$
  - Can choose event with uniform random num. $\xi$

- **Continuous**
  
  $\bar{x} \sim p(x)$
  
  $p(x) \geq 0$

  $P(x) = \int_{0}^{x} p(x)dx$

  $P(x) = \Pr(\bar{x} < x)$

  $\Pr(\alpha \leq \bar{x} \leq \beta) = P(\beta) - P(\alpha)$
Sampling the Power Function

• Given \( f(x) = x^n \)
• Normalize to find pdf \( p(x) \)
• Find cumulative distribution function \( P(x) \)
• Solve to find \( \bar{x} = P^{-1}(\xi) \)

• Trick: \( \bar{x} = \max_n(\xi_1, \ldots, \xi_n) \)
Sampling a Circle

Wrong

\[ \theta = 2\pi \xi_1 \]
\[ r = \xi_2 \]

Right

\[ \theta = 2\pi \xi_1 \]
\[ r = \sqrt{\xi_2} \]
Basic Rejection Sampling

- Pick $\xi_1$ and $\xi_2$
- Accept $\xi_1$ if $\xi_2 < c \cdot f(\xi_1)$

- Applications
  - Sampling points in a circle
  - Sampling directions on the sphere
Expected Value

• Definition

\[ E[f] \equiv \int_0^1 f(x)p(x) \, dx \]

• Key property: linearity

• Unbiased MC Estimator:

\[ E\left[ \frac{1}{N} \sum_{i=1}^{N} f(x_i) \right] = \int_0^1 f(x) \, dx \]
Unbiased MC Estimator

\[
E \left[ \frac{1}{N} \sum_{i=1}^{N} f(x_i) \right] = \frac{1}{N} \sum_{i=1}^{N} E[f(x_i)] \quad x_i \sim p(x)
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{1} f(x)p(x)dx
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{1} f(x)dx \quad \text{if } p(x) \text{ is uniform}
\]

\[
= \int_{0}^{1} f(x)dx
\]

Natural extension to multiple dimensions...

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Matt Pharr, Spring 2003
Direct Lighting: Area Sampling

\[ E(x) = \int_{\Omega} L(x, \omega) \cos \theta d\omega \]

\[ = \int_{A} L(x_o, \omega_o) \frac{\cos \theta \cos \theta_o}{|x - x_o|^2} V(x, x_o) dA(x_o) \]

Sample light uniformly by area

\[ E(x) \approx \frac{1}{N} \sum_{i=1}^{N} L(x_o[i], \omega_o[i]) \frac{\cos \theta \cos \theta_o[i]}{|x - x_o[i]|^2} V(x, x_o[i]) \]
Direct Lighting

4 eye rays, 1 shadow ray
Direct Lighting

4 eye rays, 16 shadow rays
Direct Lighting

4 eye rays, 64 shadow rays
Variance

- Definition


- Properties

\[ V[\sum Y_i] = \sum V[Y_i] \quad V[aY] = a^2 \sum V[Y] \]

- Variance decreases with sample size

\[ V \left[ \frac{1}{N} \sum Y_i \right] = \frac{1}{N^2} \sum V[Y_i] = \frac{1}{N} V[Y] \]
Direct Lighting

- Directional sampling wastes samples where the integrand is zero

Basic MC strategy: focus samples where integrand is high (easier said than done)
Improving Efficiency: Splitting

- Name from neutron transport applications
- Take additional samples in some dimensions

\[
\int_0^1 \int_0^1 f(x)g(y) \, dx \, dy \approx \frac{1}{N} \sum f(x_i)g(y_i)
\]

\[
\approx \frac{1}{N_f} \sum_{i=1}^{N_f} f(x_i) \left( \frac{1}{N_g} \sum_{j=1}^{N_g} g(y_j) \right)
\]

- e.g. multiple shadow rays
Improving Efficiency: Russian Roulette

• Randomly skip computations with low expected value...in an unbiased manner

• Algorithm:
  • Determine probability of skipping $p_s$
  • Choose uniform random number $\xi$
  • Skip if $\xi < p_s$
  • Otherwise estimate is $\frac{f(x_i)}{1 - p_s}$

• Expected value

$$p_s \cdot 0 + (1 - p_s) \cdot E[f/(1 - p_s)] = E[f]$$
Importance Sampling

• Try to sample where integrand is large

\[
E \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)} \right] = \frac{1}{N} \sum_{i=1}^{N} E \left[ \frac{f(x_i)}{p(x_i)} \right] = \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{1} \frac{f(x)}{p(x)} p(x) dx = \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{1} f(x) dx = \int_{0}^{1} f(x) dx
\]

\[x_i \sim p(x)\]