Monte Carlo II: Direct Illumination

cs348b
Matt Pharr
Review: Basic MC Estimator

\[
E \left[ \frac{1}{N} \sum_{i=1}^{N} f(x_i) \right] = \frac{1}{N} \sum_{i=1}^{N} E[f(x_i)] \\
= \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{1} f(x)p(x) \, dx \\
= \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{1} f(x) \, dx \\
= \int_{0}^{1} f(x) \, dx
\]

\[\text{if } p(x) \text{ is uniform}\]

*Increasing number of samples } N \text{ reduces variance (noise)*
Overview

• The direct lighting problem
• Improving efficiency
  • Splitting and Russian roulette
  • Importance sampling
  • Stratified sampling
• Building blocks
  • Sampling BRDFs
  • Sampling light sources
• Multiple importance sampling
Applications: Imaging

\[ \int \int \int \int L(x, y, t, u, v) dx \, dy \, dt \, du \, dv \]

Motion Blur  
Cook, Porter, Carpenter, 1984

Depth of Field  
Mitchell, 1991

AntiAliasing  

Matt Pharr, Spring 2003
Applications: Direct Lighting

$$L(x, \omega_o) = \int_{\Omega} f(\omega \to \omega_o) L_i(x, \omega) \cos \theta d\omega$$

$$= \int_A f(\omega \to \omega_o) L_e(x_l, \omega_l) \frac{\cos \theta \cos \theta_l}{|x - x_l|^2} V(x, x_l) dA(x_l)$$

Sample light uniformly by area

$$L(x, \omega_o) \approx$$

$$\frac{1}{N} \sum_{i=1}^{N} f(\omega \to \omega_o) L_e(x_l[i], \omega_l[i]) \frac{\cos \theta \cos \theta_l[i]}{|x - x_l[i]|^2} V(x, x_l[i])$$
Improving Efficiency: Splitting

• Take additional samples in some dimensions

\[
\int_0^1 \int_0^1 f(x)g(y) \, dx \, dy \approx \frac{1}{N} \sum_{i=1}^{N_f} f(x_i)g(y_i)
\]

\[
\approx \frac{1}{N_f} \sum_{i=1}^{N_f} f(x_i) \left( \frac{1}{N_g} \sum_{j=1}^{N_g} g(y_j) \right)
\]

• e.g. multiple shadow rays
Improving Efficiency: Russian Roulette

- Randomly skip computations with low expected value...in an unbiased manner

- Algorithm:
  - Determine probability of skipping $p_s$
  - Choose uniform random number $\xi$
  - Skip if $\xi < p_s$
  - Otherwise estimate is

$$\frac{f(x_i)}{1 - p_s}$$

- Expected value

$$p_s \cdot 0 + (1 - p_s) \cdot E[f/(1 - p_s)] = E[f]$$
Importance Sampling

\[ E \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)} \right] = \frac{1}{N} \sum_{i=1}^{N} E \left[ \frac{f(x_i)}{p(x_i)} \right] \]

\[ = \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{1} \frac{f(x)}{p(x)} p(x) \, dx \]

\[ = \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{1} f(x) \, dx \]

\[ = \int_{0}^{1} f(x) \, dx \]
Importance Sampling

• Very useful for reducing variance in rendering problems
• Must have $p(x) > 0$ for all $x$ where $f(x) > 0$
• Application to generalizing domains:
  \[
  \int_{a}^{b} f(x) \, dx = (b - a) \int_{0}^{1} f(a + x(b - a)) \, dx
  \]

• or
  \[
  p(x) = \frac{1}{b - a}, \quad a \leq x \leq b \quad (0 \text{ otherwise})
  \]
  \[
  \frac{1}{N} \sum_{i=1}^{N} f(x) / p(x) = (b - a) \frac{1}{N} \sum_{i=1}^{N} f(x)
  \]
Direct Lighting
Sampling the Hemisphere

• Recall that

\[
\int_{\Omega} f(\omega) d\omega = \int_0^{\pi/2} \int_0^{2\pi} f(\theta, \phi) \sin \theta d\theta d\phi
\]

• Sample

\[
\theta \sim [0, \pi/2], \phi \sim [0, 2\pi]
\]

• Estimate by

\[
\pi^2 \frac{1}{N} \sum_{i=1}^{N} f(\theta_i, \phi_i) \sin \theta
\]

• Inefficient but correct
Sampling the Hemisphere (Better)

- Find pdf for uniform samples:

\[ 1 = \int_{\Omega} p(\omega) \, d\omega = \int_{\Omega} c \, d\omega = \int_{0}^{2\pi} \int_{0}^{\pi/2} c \sin \theta \, d\theta \, d\phi = 2\pi c \quad \rightarrow \quad p(x) = 1/2\pi \]

- Sample phi uniformly \( \phi = 2\pi \xi \)
- Sample theta by \( \theta = \arccos \xi \)
Stratified Sampling

- Why not just use rejection sampling?
- Stratification: divide domain into regions, allocate samples to the regions.
- Variance is sum of region variances
- If some are simple, variance is reduced
Sampling a Cone

- Given cone about normal direction of spread angle $\theta_{\text{max}}$
- Find normalization constant for pdf

$$1 = c \int_{0}^{\theta_{\text{max}}} 1 \sin \theta \, d\theta$$

- Invert to sample using $\xi$
- Change of basis for arbitrary center direction
Sampling a Phong Lobe

- Sample an offset from the reflected direction

\[
\int_{\Omega} (\cos \theta)^n \, d\omega \rightarrow \int_0^{2\pi} \int_0^{\pi/2} (\cos \theta)^n \sin \theta \, d\theta \, d\phi
\]
Direct Lighting
Sampling a Circle

- Shirley’s Mapping
Sampling Strategies

Sample Light

Sample BRDF
Multiple Importance Sampling

• Have a family of sampling distributions $p_i(x)$
• Take $N_i$ samples from each one, $N = \sum N_i$ total samples
• Balance heuristic: compute MC estimate

$$\int f(x) \, dx \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{\sum_{j}(N_j/N)p_j(x_i)}$$

• Reduces spiky noise due to surprisingly large value of $f(x)$
Multiple Importance Sampling

Result: better than either of the two strategies alone