Monte Carlo I

Previous lecture

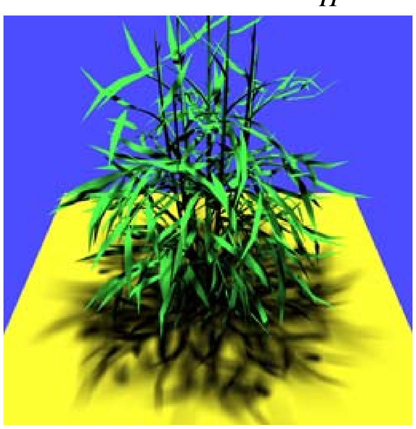
- Analytical illumination formula

This lecture

- Numerical calculation of illumination
- Review random variables and probability
- Monte Carlo integration
- Sampling from distributions
- Sampling from shapes
- Variance and efficiency

Lighting and Soft Shadows

$$E(x) = \int_{H^2} L_i(x,\omega)\cos\theta\, d\omega$$

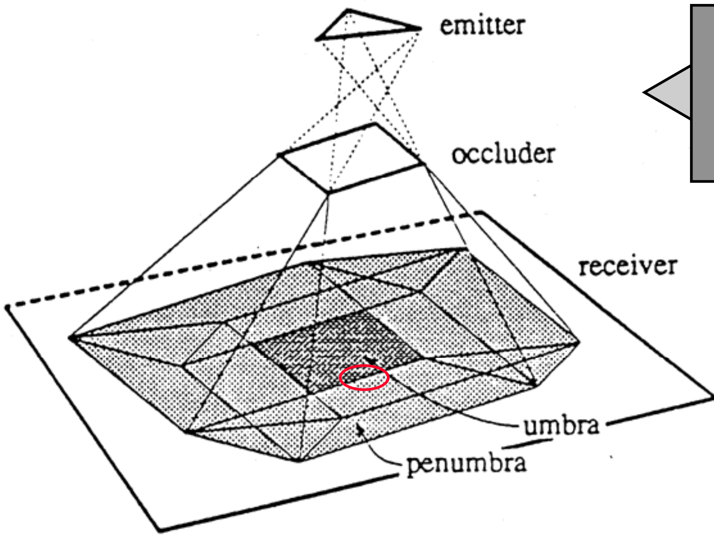


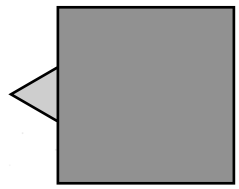
Challenges

- Visibility and blockers
- Varying light distribution
- Complex geometry

Source: Agrawala. Ramamoorthi, Heirich, Moll, 2000

Penumbras and Umbras





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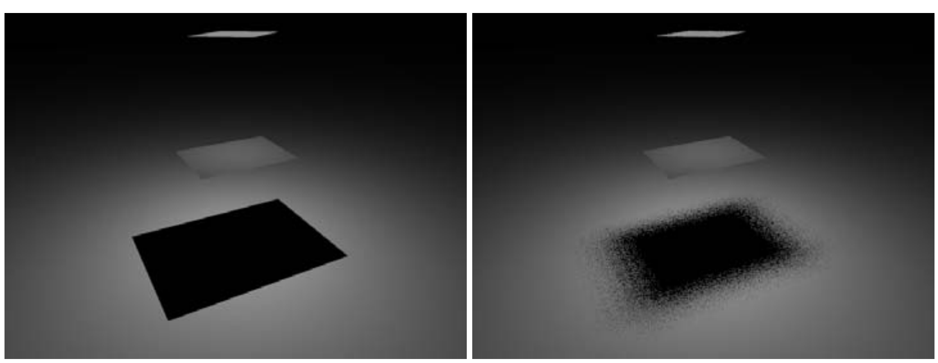
Monte Carlo Lighting



1 shadow ray per eye ray

Center Random

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Monte Carlo Algorithms

Advantages

- Easy to implement
- Easy to think about (but be careful of subtleties)
- Robust when used with complex integrands (lights, ...) and domains (shapes)
- Efficient for high dimensional integrals
- Efficient solution method for a few selected points

Disadvantages

- Noisy
- Slow (many samples needed for convergence)

Random Variables

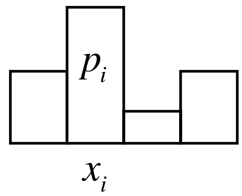
X is chosen by some random process

$X \sim p(x)$ probability distribution function (PDF)

Discrete Probability Distributions

Discrete events X_i

with probability p_i



$$p_i \geq 0$$

$$\sum_{i=1}^{n} p_i = 1$$

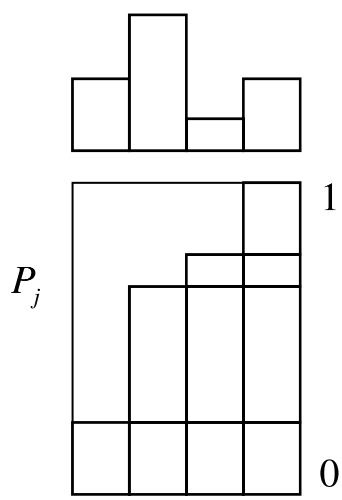
Discrete Probability Distributions

Cumulative PDF

$$P_j = \sum_{i=1}^{j} p_i$$

$$0 \leq P_i \leq 1$$

$$P_n = 1$$



Discrete Probability Distributions

Construction of samples

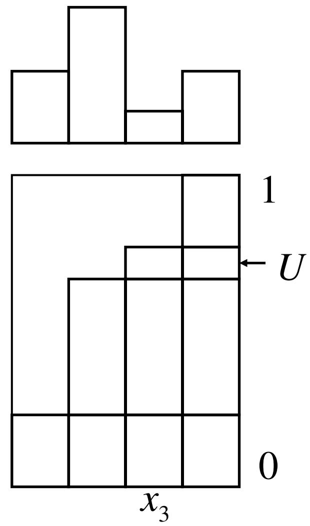
To randomly select an event,

Select x_i if

$$P_{i-1} < U \le P_i$$

$\uparrow$

Uniform random variable



Continuous Probability Distributions

PDF $p(x)$

$$p(x) \ge 0$$

CDF $P(x)$

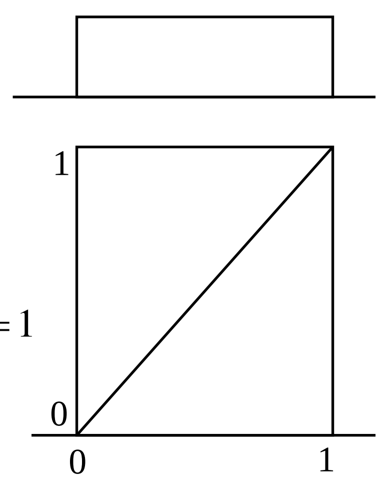
$$P(x) = \int_0^x p(x)dx$$

$$P(x) = \Pr(X < x) \qquad P(1) = 1$$

$$\Pr(\alpha \le X \le \beta) = \int_\alpha^\beta p(x)\,dx$$

$$= P(\beta) - P(\alpha)$$

Uniform



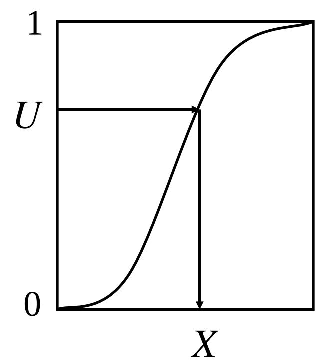
Sampling Continuous Distributions

Cumulative probability distribution function

$$P(x) = \Pr(X < x)$$

Construction of samples

Solve for $X = P^{-1}(U)$



Must know:

1. **The integral of** $p(x)$
2. **The inverse function** $P^{-1}(x)$

Example: Power Function

Assume

$$p(x) = (n+1)x^n$$

$$P(x) = x^{n+1}$$

$$X \sim p(x) \Rightarrow X = P^{-1}(U) = \sqrt[n+1]{U}$$

$$\int_0^1 x^n dx = \left.\frac{x^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1}$$

Sampling a Circle

$$A = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \int_0^1 r\,dr \int_0^{2\pi} d\theta = \left(\frac{r^2}{2}\right)\Bigg|_0^1 \theta\Big|_0^{2\pi} = \pi$$

$$p(r,\theta)\,dr\,d\theta = \frac{1}{\pi} r\,dr\;d\theta \Rightarrow p(r,\theta) = \frac{r}{\pi}$$

$$p(r,\theta) = p(r)\,p(\theta)$$

$$p(\theta) = \frac{1}{2\pi}$$

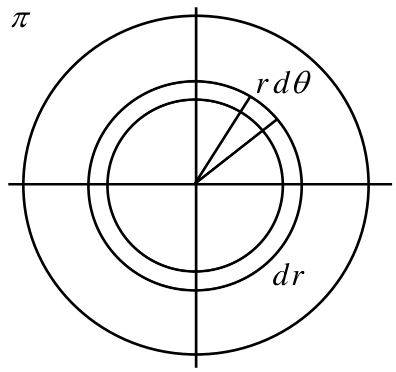
$$P(\theta) = \frac{1}{2\pi}\theta$$

$$\theta = 2\pi U_1$$

$$p(r) = 2r$$

$$P(r) = r^2$$

$$r = \sqrt{U_2}$$



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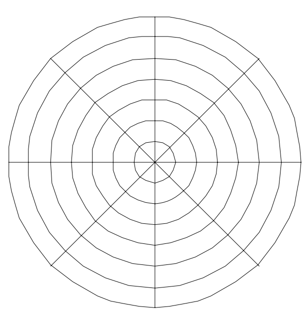
Sampling a Circle

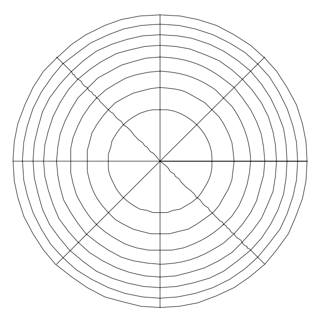
WRONG ≠ Equi-Areal



$$\theta = 2\pi U_1$$

$$r = U_2$$

RIGHT = Equi-Areal



$$\theta = 2\pi U_1$$

$$r = \sqrt{U_2}$$

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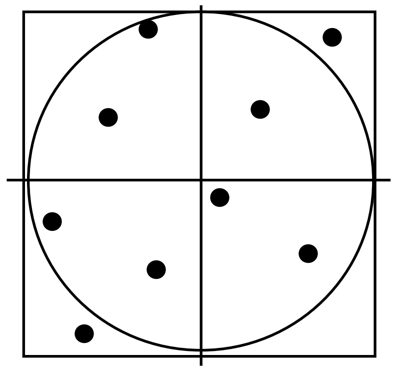
Computing Area of a Circle

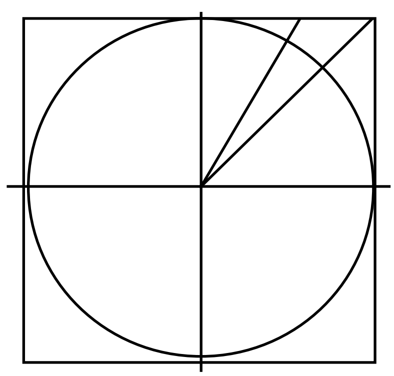


```
A = 0
for i=0 to N
  X=1-2*U1
  Y=1-2*U2
  if(X*X+Y*Y < 1)
    A += 1
A *= 4/N
```

Efficiency?

Area of circle / Area of square

Sampling 2D Directions

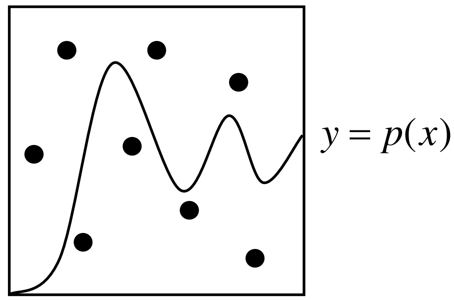


```
do {
  X=1-2*U1
  Y=1-2*U2
while(X*X+Y*Y>1)

R = sqrt(X*X+Y*Y)
dx = X/R
dy = Y/R
```

Rejection Methods

$$\int_0^1 p(x)\,dx = \iint_{y<p(x)} dx\,dy$$



Algorithm

Pick U_1 **and** U_2

Accept U_1 **if** $U_2 < f(U_1)$

Monte Carlo Integration

Definite integral $$I(f) \equiv \int_0^1 f(x)\,dx$$

Expectation of f $$E[f] \equiv \int_0^1 f(x)\,p(x)\,dx$$

Random variables $$X_i \sim p(x)$$
$$Y_i = f(X_i)$$

Estimator $$F_N = \frac{1}{N}\sum_{i=1}^{N} Y_i$$

Unbiased Estimator

$$E[F_N] = I(f)$$

$$E[F_N] = E[\frac{1}{N}\sum_{i=1}^{N} Y_i]$$
$$= \frac{1}{N}\sum_{i=1}^{N} E[Y_i] = \frac{1}{N}\sum_{i=1}^{N} E[f(X_i)]$$
$$= \frac{1}{N}\sum_{i=1}^{N} \int_0^1 f(x)p(x)\,dx$$
$$= \frac{1}{N}\sum_{i=1}^{N} \int_0^1 f(x)dx$$
$$= \int_0^1 f(x)\,dx$$

Properties

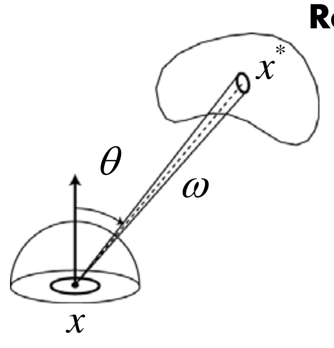
$$E[\sum_i Y_i] = \sum_i E[Y_i]$$
$$E[aY] = aE[Y]$$

Assume uniform probability distribution for now

Direct Lighting – Directional Sampling

$$E(x) = \int_{H^2} L_i(x,\omega)\cos\theta\, d\omega$$

Ray intersection $x^*(x,\omega)$



Sample ω **uniformly by** Ω

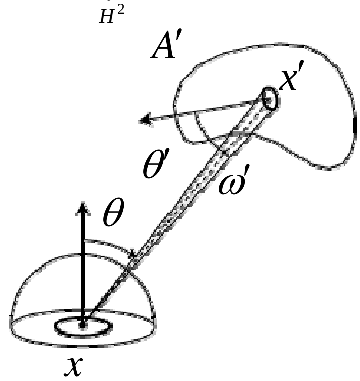
$$\int_{H^2} p(\omega)d\omega = 1 \Rightarrow p(\omega) = \frac{1}{2\pi}$$

Estimator

$$Y_i = L_o(x^*(x,\omega_i),-\omega_i)\cos\theta_i\ 2\pi$$

Direct Lighting – Area Sampling

$$E(x) = \int_{H^2} L(x,\omega)\cos\theta \, d\omega = \int_{A'} L_o(x',\omega')V(x,x') \frac{\cos\theta\cos\theta'}{|x-x'|^2} dA'$$

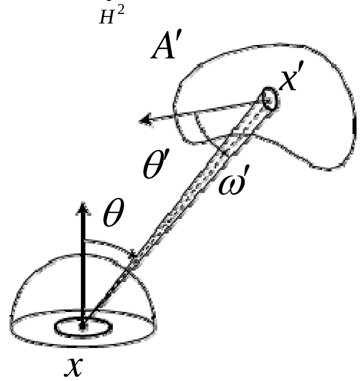


Ray direction $\omega' = x - x'$

$$V(x,x') = \begin{cases} 0 & \text{blocked} \\ 1 & \text{visible} \end{cases}$$

Direct Lighting – Area Sampling

$$E(x) = \int_{H^2} L(x,\omega)\cos\theta \, d\omega = \int_{A'} L_o(x',\omega')V(x,x') \frac{\cos\theta\cos\theta'}{|x-x'|^2} dA'$$



Ray direction $\omega' = x - x'$

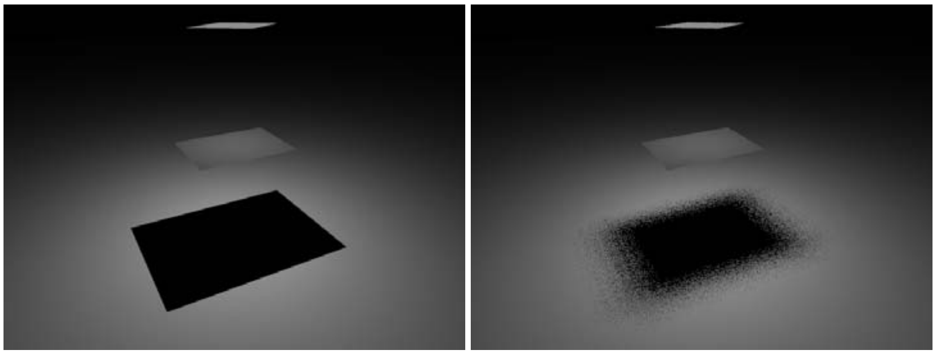
Sample shape uniformly by area *A*

$$\int_A p(u,v) dA = 1$$

Estimator

$$Y_i = L_o(x_i', \omega_i') \, V(x, x_i') \frac{\cos\theta_i \, \cos\theta_i'}{|x - x_i'|^2} A$$

Example – Area Sampling



1 shadow ray per eye ray

Center | Random

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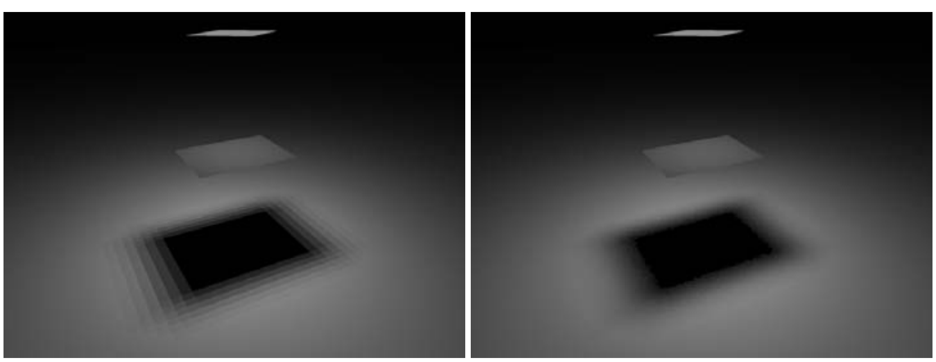
Example – Area Sampling



16 shadow rays per eye ray

Uniform grid | Stratified random

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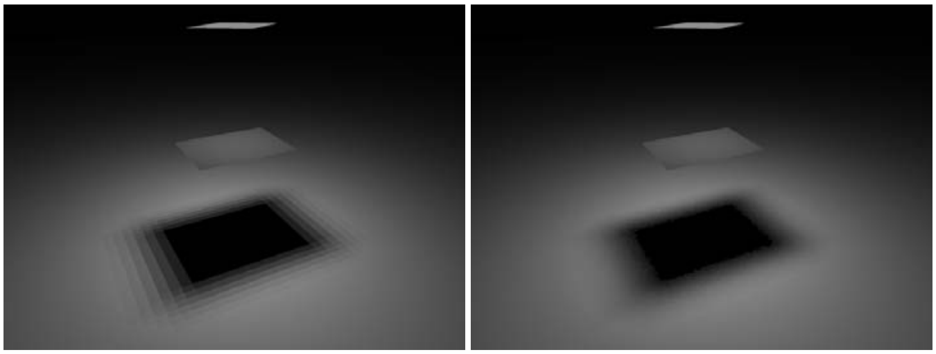
Example – Area Sampling

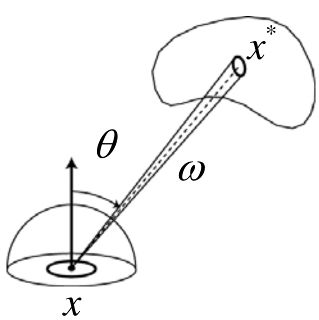


64 shadow rays per eye ray

Uniform grid | **Stratified random**

Direct Lighting – Directional Sampling

$$E(x) = \int_{H^2} L(x,\omega)\cos\theta d\omega$$



Ray intersection $x^*(x,\omega)$

Sample ω **uniformly by** Ω

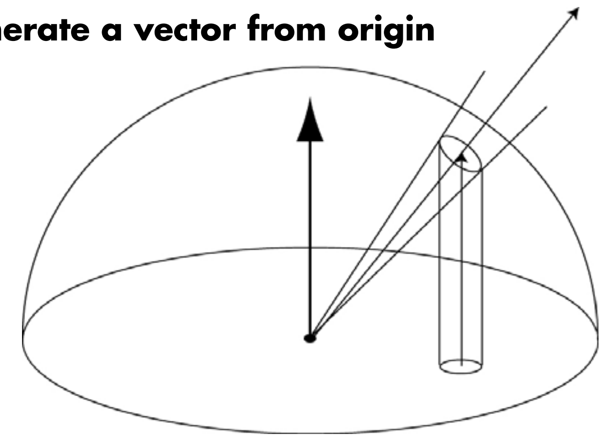
$$Y_i = L(x^*(x,\omega_i),-\omega_i)\cos\theta_i \; 2\pi$$

Sample ω **uniformly by** $\tilde{\Omega}$

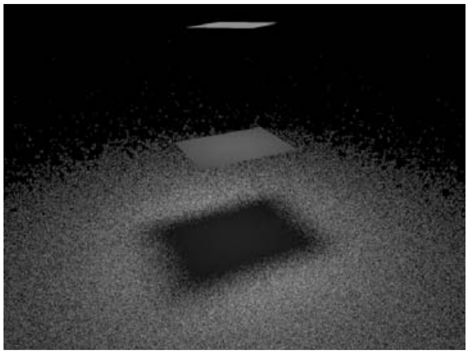
$$Y_i = L(x^*(x,\omega_i),-\omega_i)\;\pi$$

Sampling Projected Solid Angle

1. Generate a random point inside the circle
2. Project from circle to sphere
3. Generate a vector from origin

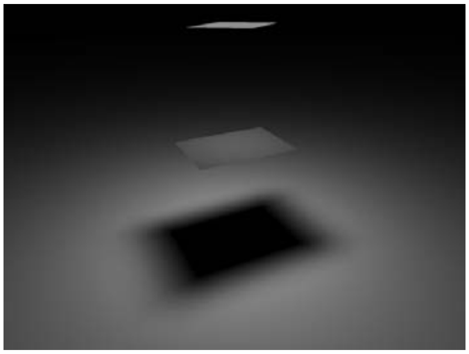


Examples



Projected solid angle

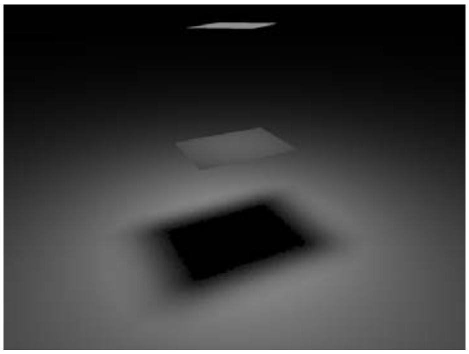
100 shadow rays

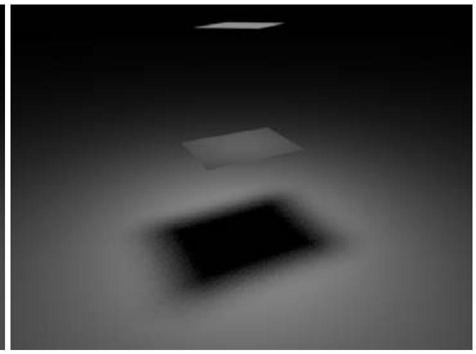


Area

100 shadow rays

Examples





4 eye rays per pixel
16 shadow rays per eye ray

64 eye rays per pixel
1 shadow ray per eye ray

Variance

Definition

$$V[Y] \equiv E[(Y - E[Y])^2]$$
$$= E[Y^2 - 2YE[Y] + E[Y]^2]$$
$$= E[Y^2] - E[Y]^2$$

Properties

$$V[\sum_i Y_i] = \sum_i V[Y_i]$$
$$V[aY] = a^2 V[Y]$$

Variance decreases linearly with sample size

$$V[\frac{1}{N}\sum_{i=1}^{N} Y_i] = \frac{1}{N^2}\sum_{i=1}^{N} V[Y_i] = \frac{1}{N} V[Y]$$

Efficiency

$$Quality \propto \frac{1}{Variance}$$

$$Efficiency = \frac{Quality}{Cost} \propto \frac{1}{Variance \bullet Cost}$$

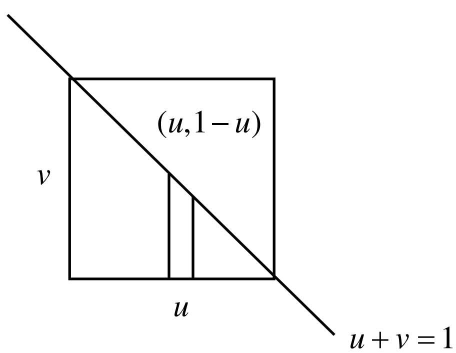
Efficiency constant as you increase sample size *N*

Comparing sampling strategies

- **Low variance, low cost -> good**
- **High variance, high cost -> bad**
- **High variance, low cost vs low variance, high cost ??**

Sampling a Triangle

$$u \ge 0$$
$$v \ge 0$$
$$u + v \le 1$$



$$A = \int_0^1 \int_0^{1-u} dv\, du = \int_0^1 (1-u)\, du = -\frac{(1-u)^2}{2}\Bigg|_0^1 = \frac{1}{2}$$

$$p(u,v) = 2$$

Sampling a Triangle

Here u and v are not independent! $p(u,v)=2$

Conditional probability

$$p(u) \equiv \int p(u,v)\,dv \qquad p(u \mid v) \equiv \frac{p(u,v)}{p(u)}$$

$$p(u) = 2\int_0^{1-u} dv = 2(1-u)$$

$$u_0 = 1-\sqrt{U_1}$$

$$P(u_0) = \int_0^{u_0} 2(1-u)\,du \quad = (1-u_0)^2$$

$$p(v \mid u) = \frac{1}{(1-u)}$$

$$v_0 = \sqrt{U_1}U_2$$

$$P(v_0 \mid u_0) = \int_0^{v_o} p(v \mid u_0)\,dv \quad = \int_0^{v_o} \frac{1}{(1-u_0)}\,dv \quad = \frac{v_0}{(1-u_0)}$$