Overview

Earlier lecture

- Monte Carlo integration

Today

- Variance reduction
- Importance sampling
- Stratified sampling
- Multidimensional sampling patterns
- Discrepancy and Quasi-Monte Carlo

Next lecture

- Signal processing view of sampling

Cameras (5D integral)

$$R = \int_T \iint_\Omega \iint_A L(x,\omega,t)\ P(x)\ S(t) \cos\theta \, dA \, d\omega \, dt$$

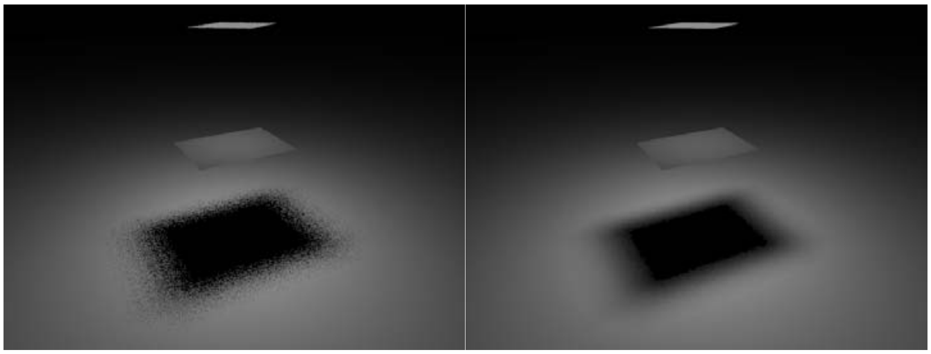
Motion Blur

Cook, Porter, Carpenter, 1984

Depth of Field

Mitchell, 1991

Variance



1 shadow ray per eye ray | 16 shadow rays per eye ray

Variance

Definition

$$V[Y] \equiv E[(Y - E[Y])^2]$$
$$= E[Y^2] - E[Y]^2$$

Variance decreases linearly with sample size

$$V[\frac{1}{N}\sum_{i=1}^{N} Y_i] = \frac{1}{N^2}\sum_{i=1}^{N} V[Y_i] = \frac{1}{N^2} N V[Y] = \frac{1}{N} V[Y]$$

Variance Reduction

Efficiency measure

$$Efficiency \propto \frac{1}{Variance \bullet Cost}$$

If one technique has twice the variance as another technique, then it takes twice as many samples to achieve the same variance

If one technique has twice the cost of another technique with the same variance, then it takes twice as much time to achieve the same variance

Techniques to increase efficiency

- Importance sampling
- Stratified sampling

Biasing

Previously used a uniform probability distribution

Can use another probability distribution

$$X_i \sim p(x)$$

But must change the estimator

$$Y_i = \frac{f(X_i)}{p(X_i)}$$

Unbiased Estimate

Probability $X_i \sim p(x)$

Estimator $Y_i = \frac{f(X_i)}{p(X_i)}$

Proof

$$E[Y_i] = E\left[\frac{f(X_i)}{p(X_i)}\right]$$
$$= \int \left[\frac{f(x)}{p(x)}\right] p(x)\,dx$$
$$= \int f(x)\,dx$$

Importance Sampling

Sample according to *f*

$$\tilde{p}(x) = \frac{f(x)}{E[f]}$$

$$\tilde{f}(x) = \frac{f(x)}{\tilde{p}(x)}$$

Variance

$$V[f] = E[f^2] - E^2[f]$$

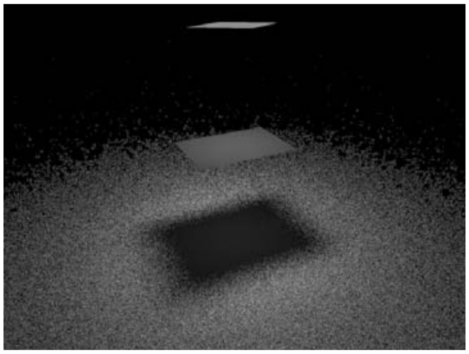
Zero variance!

$$V[\tilde{f}^2] = 0$$

$$E[\tilde{f}^2] = \int \left[\frac{f(x)}{\tilde{p}(x)}\right]^2 \tilde{p}(x)\,dx$$
$$= \int \left[\frac{f(x)}{f(x)/E[f]}\right]^2 \frac{f(x)}{E[f]}\,dx$$
$$= E[f]\int f(x)\,dx$$
$$= E^2[f]$$

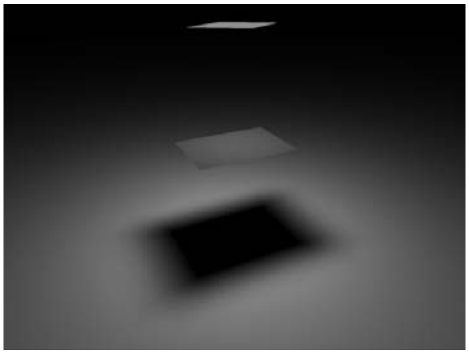
Gotcha?

Examples



Projected solid angle

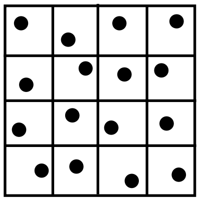
4 eye rays per pixel
100 shadow rays



Area

4 eye rays per pixel
100 shadow rays

Stratified Sampling



Allocate samples per region

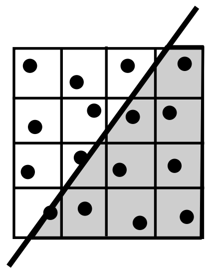
Estimate each region separately

$$F_N = \frac{1}{N}\sum_{i=1}^{N} F_i$$

New variance

$$V[F_N] = \frac{1}{N^2}\sum_{i=1}^{N} V[F_i]$$

Stratified Sampling



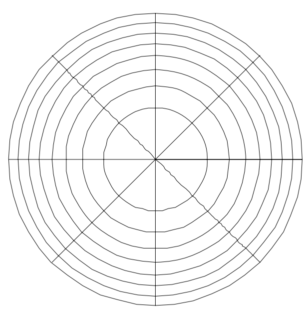
Sample a polygon

$$V[F_N] = \frac{1}{N^2} \sum_{i=1}^{\sqrt{N}} V[F_j] = \frac{V[F_E]}{N^{1.5}}$$

If the variance in each region is less than the overall variance, there will be a reduction in overall variance

Sampling a Circle

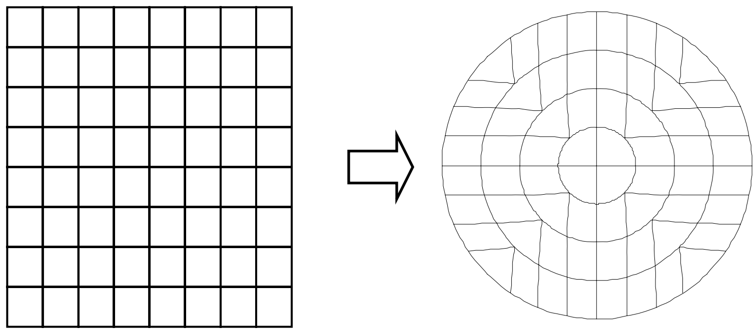
Equi-Areal

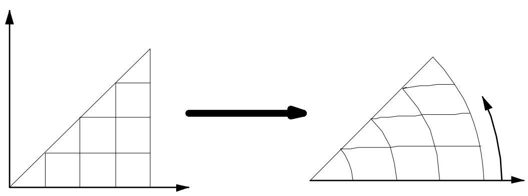


$$\theta = 2\pi U_1$$

$$r = \sqrt{U_2}$$

Shirley's Mapping





$$r = U_1$$

$$\theta = \frac{\pi}{4}\frac{U_2}{U_1}$$

High-dimensional Sampling

Complete set of samples $N = \underbrace{n \times n \times \ldots \times n}_{d} = n^d$

Random sampling

Error ... $E \sim V^{1/2} \sim \frac{1}{N^{1/2}}$

Numerical integration

Error ... $E \sim \frac{1}{n} = \frac{1}{N^{1/d}}$

Monte Carlo requires fewer samples for the same error in high dimensional space

Space-time Patterns

6	10	2	13
3	14	12	8
15	0	7	11
5	9	4	1

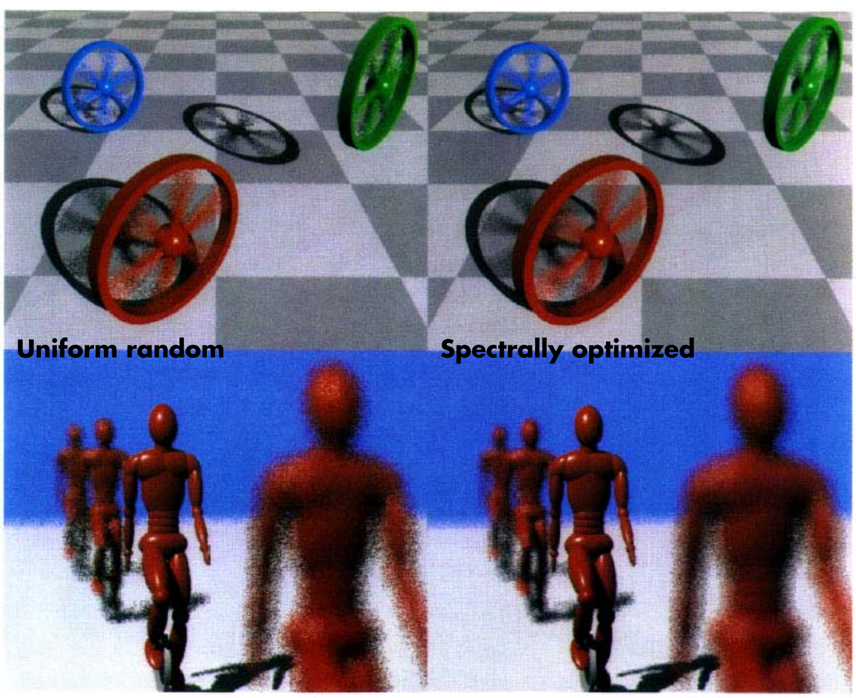
Cook Pattern

15	8	5	2
4	3	14	9
10	13	0	7
1	6	11	12

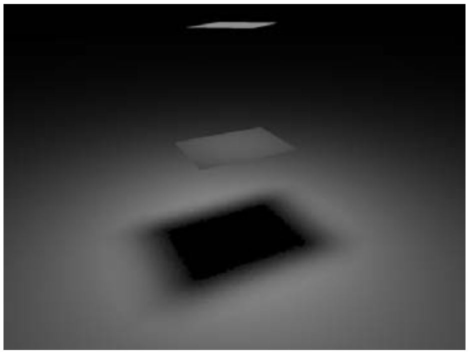
Pan-diagonal Magic Square

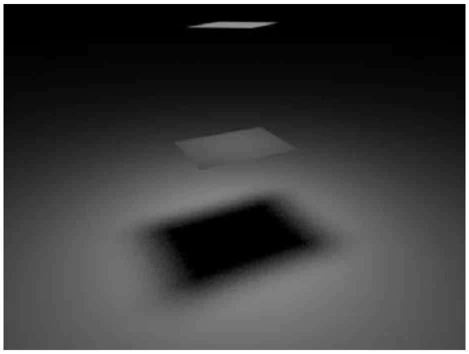
Distribute samples in time

- Complete in space
- Incomplete in time
- Decorrelate space and time
- Nearby samples in space should differ greatly in time



Path Tracing





4 eye rays per pixel
16 shadow rays per eye ray

Complete

64 eye rays per pixel
1 shadow ray per eye ray

Incomplete

Block Design

Latin Square

a	*d*	*c*	*b*
b	*a*	*d*	*c*
c	*b*	*a*	*d*
d	*c*	*b*	*a*

Alphabet of size *n*

Each symbol appears exactly once in each row and column

Rows and columns are stratified

Block Design

N-Rook Pattern

a			
		a	
	a		
			a

Incomplete block design

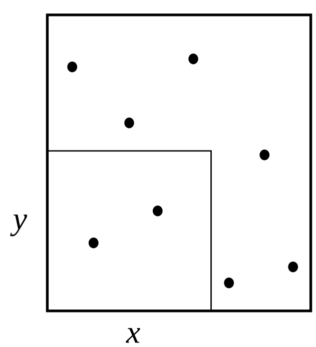
Replaced n^2 samples with n samples

Permutations: $(\pi_1(i), \pi_2(i), \cdots \pi_d(i))$

Generalizations: N-queens, 2D projection

$$(\pi_x = \{1,2,3,4\}, \pi_y = \{4,2,3,1\})$$

Discrepancy



$$\Delta(x,y) = \frac{n(x,y)}{N} - xy$$

$A = xy$

$n(x,y)$ number of samples in A

$$D_N = \max_{x,y} |\Delta(x,y)|$$

Theorem on Total Variation

Theorem: $$\left|\frac{1}{N}\sum_{i=1}^{N} f(X_i) - \int f(x)\,dx\right| \le V(f) D_N$$

Proof: Integrate by parts

$$\int f(x) \ \ [\frac{\delta(x-x_i)}{N} - 1]\,dx \qquad\qquad \frac{\partial\Delta(x)}{\partial x} = \frac{\delta(x-x_i)}{N} - 1$$

$$= \int f(x) \frac{\partial \Delta(x)}{\partial x}\,dx$$

$$= f\Delta\Big|_0^1 - \int \frac{\partial f(x)}{\partial x}\Delta(x)\,dx = -\int \frac{\partial f(x)}{\partial x}\Delta(x)\,dx$$

$$\le D_N \int \left|\frac{\partial f(x)}{\partial x}\right| dx = V(f) D_N$$

Quasi-Monte Carlo Patterns

Radical inverse (digit reverse)

of integer i in integer base b

$$i = d_i \cdots d_2 d_1 d_0$$
$$\phi_b(i) \equiv 0.d_0 d_1 d_2 \cdots d_i$$

			$\phi_2(i)$
1	1	.1	1/2
2	10	.01	1/4
3	11	.11	3/4
4	100	.001	3/8
5	101	.101	5/8

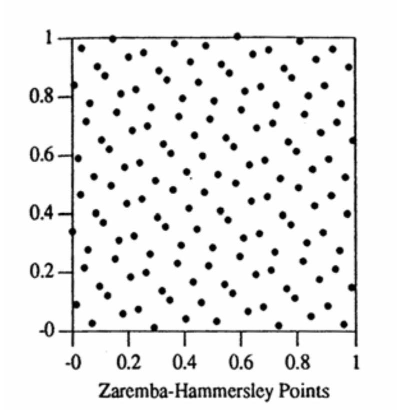
Hammersley points

$$(i/N, \phi_2(i), \phi_3(i), \phi_5(i), \cdots) \qquad D_N = O(\frac{\log^{d-1} N}{N})$$

Halton points (sequential)

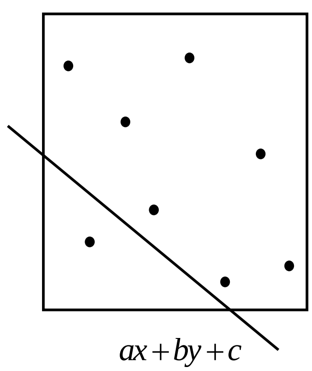
$$(\phi_2(i), \phi_3(i), \phi_5(i), \cdots) \qquad D_N = O(\frac{\log^{d} N}{N})$$

Hammersly Points



$$(i/N, \phi_2(i), \phi_3(i), \phi_5(i), \cdots)$$

Edge Discrepancy



Note: SGI IR Multisampling extension:
8x8 subpixel grid; 1,2,4,8 samples

Low-Discrepancy Patterns

Process	16 points	256 points	1600 points
Zaremba	0.0504	0.00478	0.00111
Jittered	0.0538	0.00595	0.00146
Poisson-Disk	0.0613	0.00767	0.00241
N-Rooks	0.0637	0.0123	0.00488
Random	0.0924	0.0224	0.00866

Discrepancy of random edges, From Mitchell (1992)

Random sampling converges as $N^{-1/2}$
Zaremba converges faster and has lower discrepancy
Zaremba has a relatively poor blue noise spectra
Jittered and Poisson-Disk recommended

Views of Integration

1. Numerical
   - Quadrature/Integration rules
   - Smooth functions
2. Statistical sampling (Monte Carlo)
   - Sampling like polling
   - Variance reduction techniques
   - High dimensional sampling: $1/N^{1/2}$
3. Quasi Monte Carlo
   - Discrepancy
   - Asymptotic efficiency in high dimensions
4. Signal processing
   - Sampling and reconstruction
   - Aliasing and antialiasing
   - Blue noise good