Sampling and Reconstruction

The sampling and reconstruction process

- Real world: continuous
- Digital world: discrete

Basic signal processing

- Fourier transforms
- The convolution theorem
- The sampling theorem

Aliasing and antialiasing

- Uniform supersampling
- Nonuniform supersampling

Camera Simulation

\[ R = \iiint_{\Lambda} \left( P(x', \lambda) S(x', \omega', t) L(T(x', \omega', \lambda), t, \lambda) \right) \, d\lambda \, d\omega' \, dt \, dx' \]

Sensor response \( P(x', \lambda) \)
Lens \((x, \omega) = T(x', \omega', \lambda)\)
Shutter \(S(x', \omega', t)\)
Scene radiance \(L(x, \omega, t, \lambda)\)
Imagers = Signal Sampling

All imagers convert a continuous image to a discrete sampled image by integrating over the active "area" of a sensor.

\[ R = \int \int \int L(x,\omega,t)P(x)S(t)\cos\theta dA d\omega dt \]

Examples:
- Retina: photoreceptors
- CCD array

Virtual CG cameras do not integrate, they simply sample radiance along rays ... 

Displays = Signal Reconstruction

All physical displays recreate a continuous image from a discrete sampled image by using a finite sized source of light for each pixel.

Examples:
- DACs: sample and hold
- Cathode ray tube: phosphor spot and grid
Sampling in Computer Graphics

Artifacts due to sampling - Aliasing

- Jaggies
- Moire
- Flickering small objects
- Sparkling highlights
- Temporal strobing

Preventing these artifacts - Antialiasing

Jaggies

Retort sequence by Don Mitchell

Staircase pattern or jaggies
Fourier Transforms

Spectral representation treats the function as a weighted sum of sines and cosines.

Each function has two representations:
- Spatial domain - normal representation
- Frequency domain - spectral representation

The Fourier transform converts between the spatial and frequency domain:

\[ \mathcal{F}(f) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx \]

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(\omega) e^{i\omega x} \, d\omega \]
### Spatial and Frequency Domain

<table>
<thead>
<tr>
<th>Spatial Domain</th>
<th>Frequency Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Spatial Domain Image" /></td>
<td><img src="image2" alt="Frequency Domain Image" /></td>
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</table>

### More Examples

<table>
<thead>
<tr>
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<tr>
<td><img src="image3" alt="Spatial Domain Image" /></td>
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More Examples

Spatial Domain | Frequency Domain

More Examples

Spatial Domain | Frequency Domain
Pat’s Frequencies

Marc’s Frequencies
Convolution

**Definition**

\[ h(x) = f \otimes g = \int f(x')g(x-x')dx' \]

*Convolution Theorem*: Multiplication in the frequency domain is equivalent to convolution in the space domain.

\[ f \otimes g \leftrightarrow F \times G \]

*Symmetric Theorem*: Multiplication in the space domain is equivalent to convolution in the frequency domain.

\[ f \times g \leftrightarrow F \otimes G \]

---

**The Sampling Theorem**
Sampling: Spatial Domain

\[
\begin{align*}
 f(x) & \times \int_{-\infty}^{\infty} \delta(x - nT) f(nT) \\
\text{III}(x) & = \sum_{n=-\infty}^{\infty} \delta(x - nT)
\end{align*}
\]

Sampling: Frequency Domain

\[
\begin{align*}
 F(\omega) & \times \left( \sum_{n=-\infty}^{\infty} \delta(\omega - n / T) \right) \\
\text{III}_{1/T}(\omega) & = \sum_{n=-\infty}^{\infty} \delta(\omega - n / T)
\end{align*}
\]
Reconstruction: Frequency Domain

Rect function of width $T$:

$$
\Pi_{1/T}(x) = \begin{cases} 
1 & |x| \leq \frac{T}{2} \\
0 & |x| > \frac{T}{2} 
\end{cases}
$$

Sinc function:

$$
sinc(x) = \frac{\sin \pi x}{\pi x}
$$
Sampling and Reconstruction

Sampling Theorem

This result is known as the Sampling Theorem and is due to Claude Shannon who first discovered it in 1949.

A signal can be reconstructed from its samples without loss of information, if the original signal has no frequencies above 1/2 the Sampling frequency.

For a given bandlimited function, the rate at which it must be sampled is called the Nyquist Frequency.
Aliasing

Undersampling: Aliasing

\[ \times \]

\[ = \]

\[ = \]
Sampling a “Zone Plate”

Zone plate: \( \sin(x^2 + y^2) \)

Sampled at 128x128
Reconstructed to 512x512
Using a 30-wide
Kaiser windowed sinc

Left rings: part of signal
Right rings: prealiasing

---

Ideal Reconstruction

Ideally, use a perfect low-pass filter - the sinc function - to bandlimit the sampled signal and thus remove all copies of the spectra introduced by sampling.

Unfortunately,

- The sinc has infinite extent and we must use simpler filters with finite extents. Physical processes in particular do not reconstruct with sincs.
- The sinc may introduce ringing which are perceptually objectionable.
Sampling a “Zone Plate”

Zone plate: \[ \sin x^2 + y^2 \]

Sampled at 128x128
Reconstructed to 512x512
Using optimal cubic

Left rings: part of signal
Right rings: prealiasing
Middle rings: postaliasing

Mitchell Cubic Filter

\[
h(x) = \frac{1}{6} \begin{cases} 
(12 - 9B - 6C)x^3 + (-18 + 12B + 6C)x^2 + (6 - 2B) & |x| < 1 \\
(-B - 6C)x^3 + (6B + 30C)x^2 + (-12B - 48C)x + (8B + 24C) & 1 < |x| < 2 \\
0 & \text{otherwise}
\end{cases}
\]

Good: \((1/3, 1/3)\)

Properties:
\[ \sum_{n=-\infty}^{\infty} h(x) = 1 \]

B-spline: \((1, 0)\)
Catmull-Rom: \((0, 1/2)\)

From Mitchell and Netravali
Aliasing

- Prealiasing: due to sampling under Nyquist rate
- Postaliasing: due to use of imperfect reconstruction filter

Antialiasing
Antialiasing

Antialiasing = Preventing aliasing

1. Analytically prefilter the signal
   - Solvable for points, lines and polygons
   - Not solvable in general
     e.g. procedurally defined images
2. Uniform supersampling and resample
3. Nonuniform or stochastic sampling

Antialiasing by Prefiltering

Frequency Space
Uniform Supersampling

Increasing the sampling rate moves each copy of the spectra further apart, potentially reducing the overlap and thus aliasing.

Resulting samples must be resampled (filtered) to image sampling rate.

\[ \text{Pixel} = \sum_s w_s \cdot \text{Sample}_s \]

Point vs. Supersampled

Checkerboard sequence by Tom Duff
Analytic vs. Supersampled

Exact Area 4x4 Supersampled

Distribution of Extrafoveal Cones

Monkey eye cone distribution

Fourier transform

Yellot theory
- Aliases replaced by noise
- Visual system less sensitive to high freq noise
Non-uniform Sampling

Intuition

Uniform sampling
- The spectrum of uniformly spaced samples is also a set of uniformly spaced spikes
- Multiplying the signal by the sampling pattern corresponds to placing a copy of the spectrum at each spike (in freq. space)
- Aliases are coherent, and very noticeable

Non-uniform sampling
- Samples at non-uniform locations have a different spectrum; a single spike plus noise
- Sampling a signal in this way converts aliases into broadband noise
- Noise is incoherent, and much less objectionable

Jittered Sampling

Add uniform random jitter to each sample
Jittered vs. Uniform Supersampling

4x4 Jittered Sampling

4x4 Uniform

Analysis of Jitter

Non-uniform sampling

\[ s(x) = \sum_{n=-\infty}^{\infty} \delta(x - x_n) \]
\[ x_n = nT + j_n \]

Jittered sampling

\[ j_n \sim j(x) \]
\[ j(x) = \begin{cases} 1 & |x| \leq 1/2 \\ 0 & |x| > 1/2 \end{cases} \]
\[ J(\omega) = \text{sinc} \ \omega \]

\[ S(\omega) = \frac{1}{T} \left[ 1 - |J(\omega)|^2 \right] + \frac{2\pi}{T^2} |J(\omega)|^2 \sum_{n=-\infty}^{\infty} \delta(\omega - \frac{2\pi n}{T}) \]
\[ = \frac{1}{T} \left[ 1 - \text{sinc}^2 \ \omega \right] + \delta(\omega) \]
Poisson Disk Sampling

Dart throwing algorithm