

# Monte Carlo I

## Previous lecture

- Analytical illumination formula

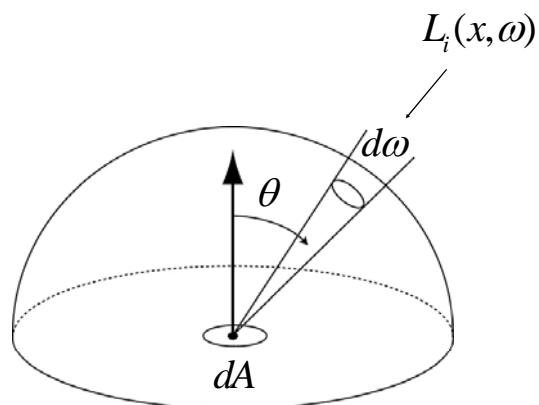
## This lecture

- Review random variables and probability
- Sampling from distributions
- Sampling from shapes
- Monte Carlo integration
- Numerical calculation of illumination

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# Irradiance from the Environment



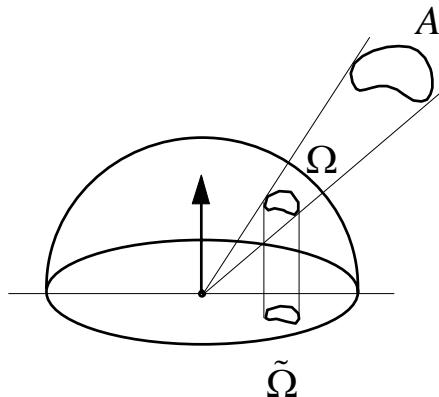
$$E(x) = \int_{H^2} L_i(x, \omega) \cos \theta d\omega$$

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## Uniform Area Source

$$\begin{aligned} E(x) &= \int_{H^2} L \cos \theta d\omega \\ &= L \int_{\Omega} \cos \theta d\omega \\ &= L \tilde{\Omega} \end{aligned}$$

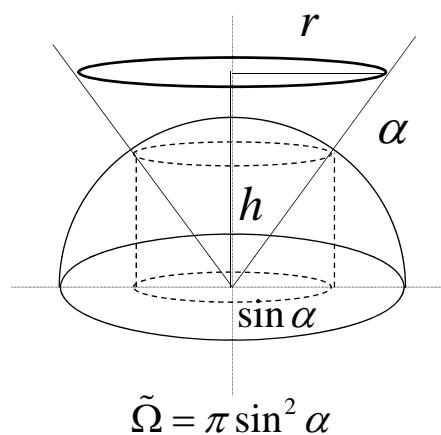


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## Uniform Disk Source

### Geometric Derivation



### Algebraic Derivation

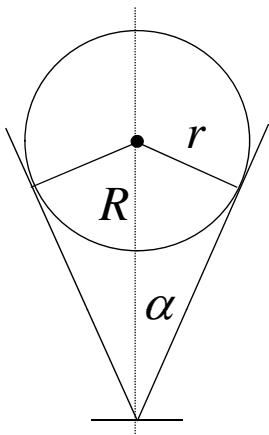
$$\begin{aligned} \tilde{\Omega} &= \int_1^{\cos \alpha} \int_0^{2\pi} \cos \theta d\phi d\cos \theta \\ &= 2\pi \frac{\cos^2 \theta}{2} \Big|_1^{\cos \alpha} \\ &= \pi \sin^2 \alpha \\ &= \pi \frac{r^2}{r^2 + h^2} \end{aligned}$$

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## Uniform Spherical Source

### Geometric Derivation



$$\tilde{\Omega} = \pi \sin^2 \alpha$$

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### Algebraic Derivation

$$\begin{aligned}\tilde{\Omega} &= \int \cos \theta d\omega \\ &= \pi \sin^2 \alpha \\ &= \pi \frac{r^2}{R^2}\end{aligned}$$

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## The Sun

**Solar constant (normal incidence at zenith)**

**Irradiance 1353 W/m<sup>2</sup>**

**Illuminance 127,500 lm/m<sup>2</sup> = 127.5 kilolux**

**Solar angle**

**$\alpha = .25 \text{ degrees} = .004 \text{ radians (half angle)}$**

**$\tilde{\Omega} = \pi \sin^2 \alpha \approx \pi \alpha^2 = 6 \times 10^{-5} \text{ steradians}$**

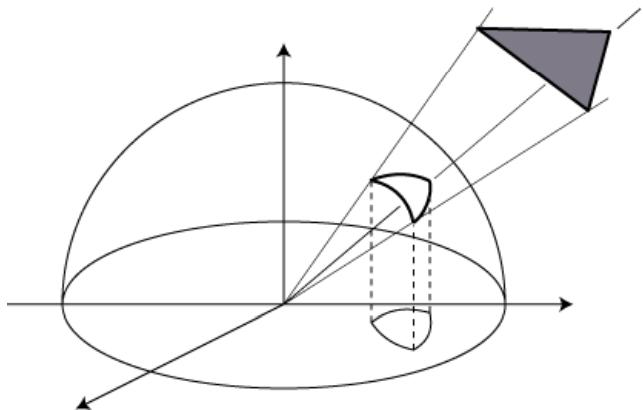
**Solar radiance**

$$L = \frac{E}{\tilde{\Omega}} = \frac{1.353 \times 10^3 \text{ W/m}^2}{6 \times 10^{-5} \text{ sr}} = 2.25 \times 10^7 \frac{\text{W}}{\text{m}^2 \cdot \text{sr}}$$

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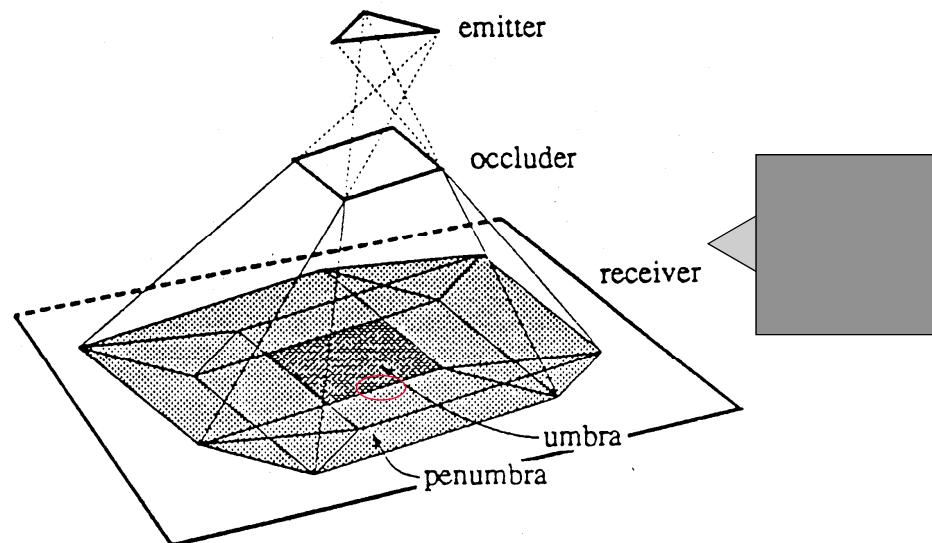
## Polygonal Source



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## Penumbras and Umbras

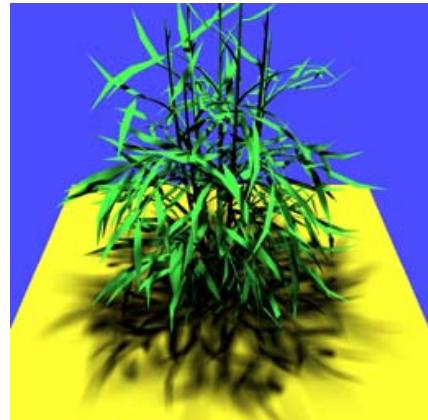


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## **Lighting and Soft Shadows**

$$E(x) = \int_{H^2} L_i(x, \omega) \cos \theta d\omega$$



### **Challenges**

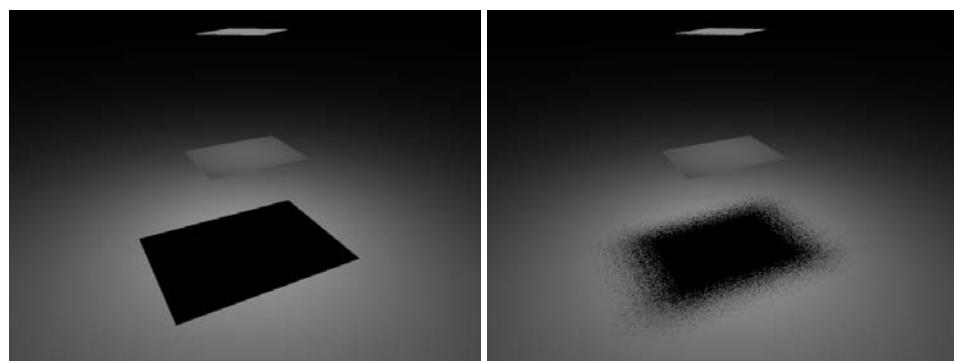
- **Visibility and blockers**
- **Varying light distribution**
- **Complex geometry**

**Source: Agrawala. Ramamoorthi, Heirich, Moll, 2000**

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## **Monte Carlo Lighting**



**1 shadow ray per eye ray**

**Center**

**Random**

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# Monte Carlo Algorithms

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## Advantages

- **Easy to implement**
- **Easy to think about (but be careful of subtleties)**
- **Robust when used with complex integrands (lights, ...) and domains (shapes)**
- **Efficient for high dimensional integrals**
- **Efficient solution method for a few selected points**

## Disadvantages

- **Noisy**
- **Slow (many samples needed for convergence)**

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# Random Variables

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$X$  is chosen by some random process

$X \sim p(x)$  **probability distribution function (PDF)**

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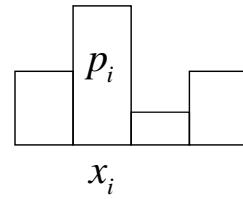
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## Discrete Probability Distributions

**Discrete events**  $X_i$   
**with probability**  $p_i$

$$p_i \geq 0$$

$$\sum_{i=1}^n p_i = 1$$



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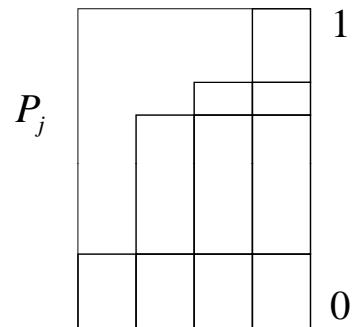
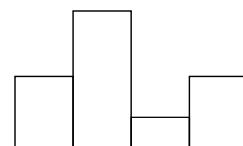
## Discrete Probability Distributions

**Cumulative PDF**

$$P_j = \sum_{i=1}^j p_i$$

$$0 \leq P_i \leq 1$$

$$P_n = 1$$



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## Discrete Probability Distributions

**Construction of samples**

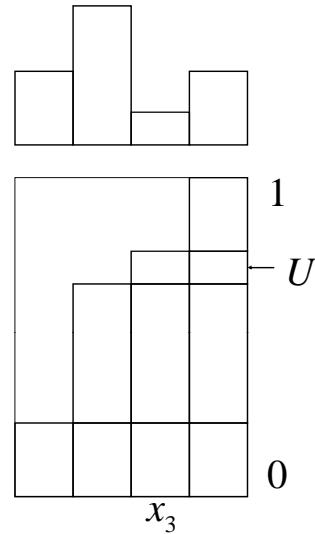
To randomly select an event,

Select  $x_i$  if

$$P_{i-1} < U \leq P_i$$

↓

**Uniform random variable**



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## Continuous Probability Distributions

**PDF**  $p(x)$

$$p(x) \geq 0$$

**Uniform**



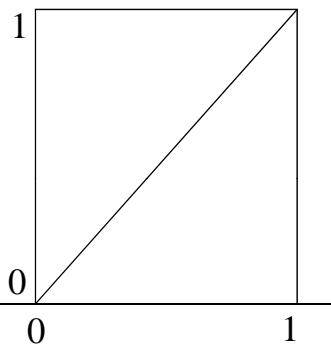
**CDF**  $P(x)$

$$P(x) = \int_0^x p(x) dx$$

$$P(x) = \Pr(X < x) \quad P(1) = 1$$

$$\Pr(\alpha \leq X \leq \beta) = \int_{\alpha}^{\beta} p(x) dx$$

$$= P(\beta) - P(\alpha)$$



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# Sampling Continuous Distributions

## Cumulative probability distribution function

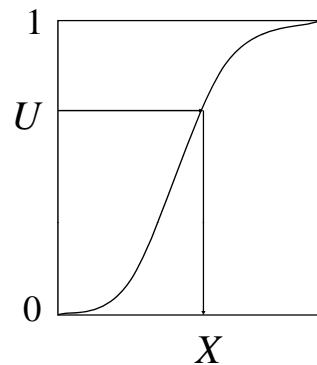
$$P(x) = \Pr(X < x)$$

### Construction of samples

$$\text{Solve for } X = P^{-1}(U)$$

### Must know:

1. The integral of  $p(x)$
2. The inverse function  $P^{-1}(x)$



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# Sampling a Circle

$$A = \int_0^{2\pi} \int_0^1 r dr d\theta = \int_0^1 r dr \int_0^{2\pi} d\theta = \left( \frac{r^2}{2} \right) \Big|_0^1 \cdot 2\pi = \pi$$

$$p(r, \theta) dr d\theta = \frac{1}{\pi} r dr d\theta \Rightarrow p(r, \theta) = \frac{r}{\pi}$$

$$p(r, \theta) = p(r)p(\theta)$$

$$p(\theta) = \frac{1}{2\pi}$$

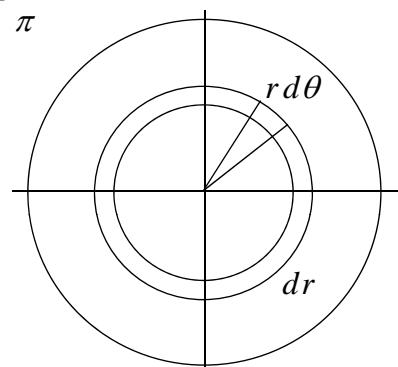
$$P(\theta) = \frac{1}{2\pi} \theta$$

$$p(r) = 2r$$

$$P(r) = r^2$$

$$\theta = 2\pi U_1$$

$$r = \sqrt{U_2}$$

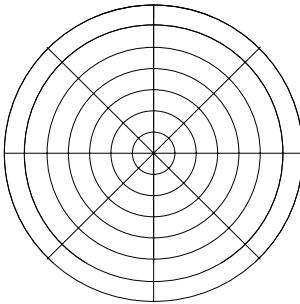


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## Sampling a Circle

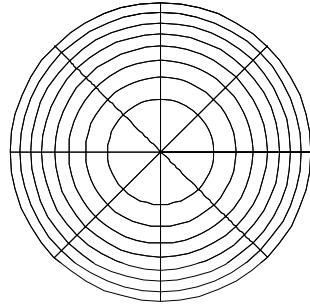
**WRONG  $\neq$  Equi-Areal**



$$\theta = 2\pi U_1$$

$$r = U_2$$

**RIGHT = Equi-Areal**



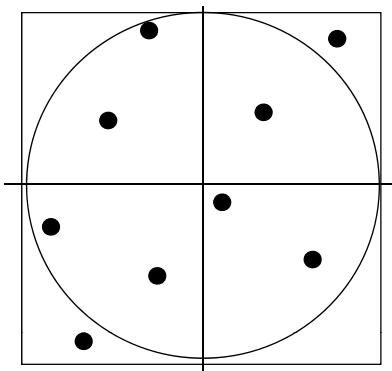
$$\theta = 2\pi U_1$$

$$r = \sqrt{U_2}$$

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## Computing Area of a Circle



```
A = 0
for i=0 to N
    X=1-2*U1
    Y=1-2*U2
    if(X*X+Y*Y < 1)
        A += 1
A *= 4/N
```

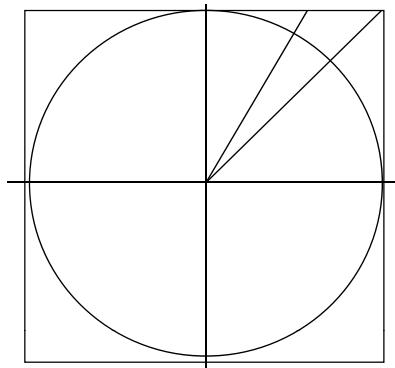
**Efficiency?**

**Area of circle / Area of square**

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## Sampling 2D Directions



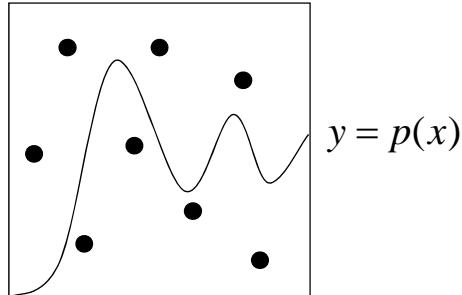
```
do {  
    X=1-2*U1  
    Y=1-2*U2  
    while(X*X+Y*Y>1)  
  
    R = sqrt(X*X+Y*Y)  
    dx = X/R  
    dy = Y/R
```

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## Rejection Methods

$$\int_0^1 p(x) dx = \iint_{y < p(x)} dx dy$$



### Algorithm

**Pick**  $U_1$  and  $U_2$

**Accept**  $U_1$  if  $U_2 < f(U_1)$

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## Monte Carlo Integration

**Definite integral**  $I(f) \equiv \int_0^1 f(x) dx$

**Expectation of  $f$**   $E[f] \equiv \int_0^1 f(x) p(x) dx$

**Random variables**  $X_i \sim p(x)$

$$Y_i = f(X_i)$$

**Estimator**  $F_N = \frac{1}{N} \sum_{i=1}^N Y_i$

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## Unbiased Estimator

$$\begin{aligned} E[F_N] &= E\left[\frac{1}{N} \sum_{i=1}^N Y_i\right] \\ E[F_N] &= \frac{1}{N} \sum_{i=1}^N E[Y_i] = \frac{1}{N} \sum_{i=1}^N E[f(X_i)] \end{aligned}$$

**Properties**

$$E\left[\sum_i Y_i\right] = \sum_i E[Y_i]$$

$$E[aY] = aE[Y]$$

$$\begin{aligned} &= \frac{1}{N} \sum_{i=1}^N \int_0^1 f(x) p(x) dx \\ &= \frac{1}{N} \sum_{i=1}^N \int_0^1 f(x) dx \end{aligned}$$

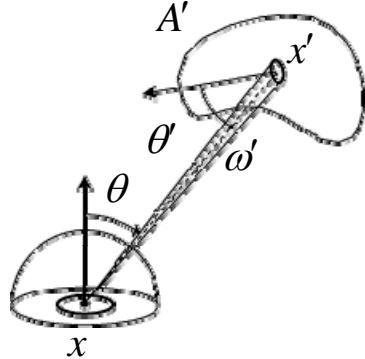
**Assume uniform probability distribution for now**

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## Direct Lighting – Area Sampling

$$E(x) = \int_{H^2} L(x, \omega) \cos \theta d\omega = \int_{A'} L_o(x', \omega') V(x, x') \frac{\cos \theta \cos \theta'}{|x - x'|^2} dA'$$



**Ray direction**  $\omega' = x - x'$

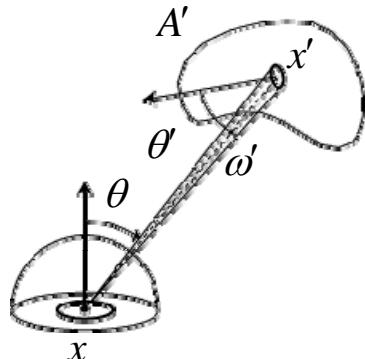
$$V(x, x') = \begin{cases} 0 & \text{blocked} \\ 1 & \text{visible} \end{cases}$$

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## Direct Lighting – Area Sampling

$$E(x) = \int_{H^2} L(x, \omega) \cos \theta d\omega = \int_{A'} L_o(x', \omega') V(x, x') \frac{\cos \theta \cos \theta'}{|x - x'|^2} dA'$$



**Ray direction**  $\omega' = x - x'$

**Sample shape uniformly by area A**

$$\int_A p(u, v) dA = 1$$

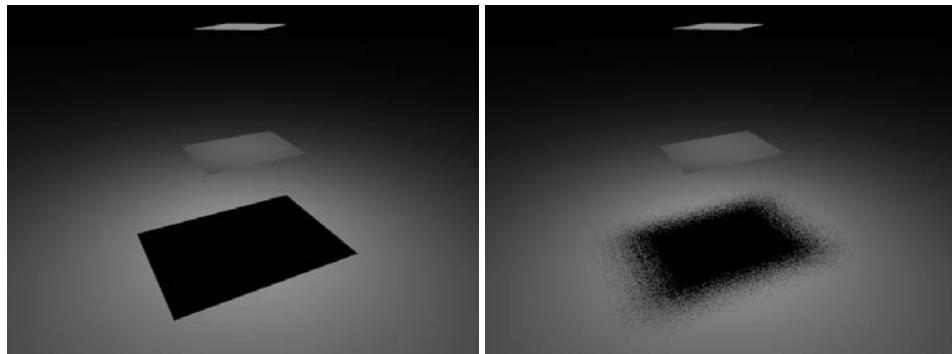
**Estimator**

$$Y_i = L_o(x'_i, \omega'_i) V(x, x'_i) \frac{\cos \theta_i \cos \theta'_i}{|x - x'_i|^2} A$$

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## Example – Area Sampling



**1 shadow ray per eye ray**

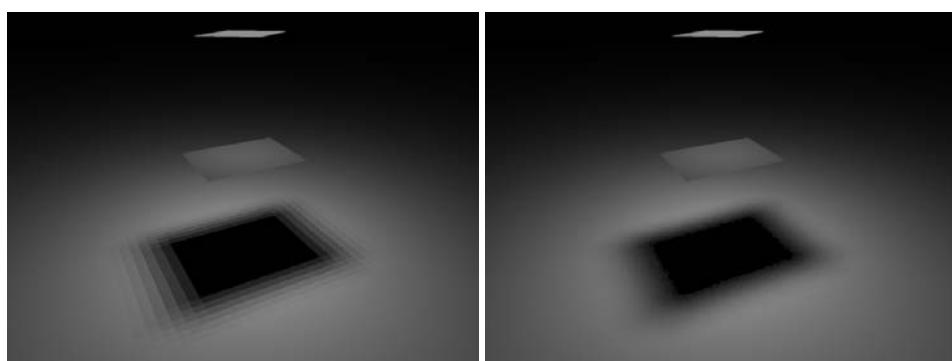
**Center**

**Random**

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## Example – Area Sampling



**16 shadow rays per eye ray**

**Uniform grid**

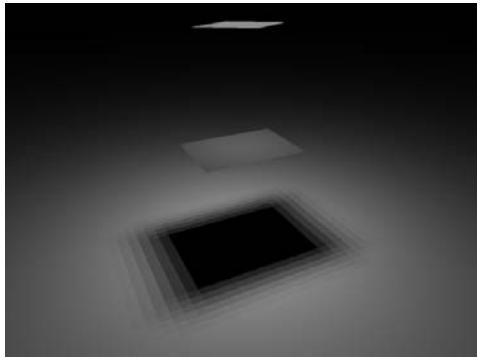
**Stratified random**

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## **Example – Area Sampling**

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**64 shadow rays per eye ray**

**Uniform grid**

**Stratified random**

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