Reflection Models

Last lecture
- Reflection models
- The reflection equation and the BRDF
- Ideal reflection, refraction and diffuse

Today
- Phong model
- Microfacet models
- Torrance-Sparrow model
- Self-shadowing

Phong Model
Glossy Surfaces

Mirror

Diffuse

Reflection Geometry

\[ \hat{H} = \frac{\hat{L} + \hat{E}}{\hat{L} + \hat{E}} \]

\[ \cos \theta_i = \hat{L} \cdot \hat{N} \]
\[ \cos \theta_r = \hat{E} \cdot \hat{N} \]

\[ \cos \theta_t = \hat{E} \cdot \mathbf{R}_N(\hat{L}) = \mathbf{R}_N(\hat{E}) \cdot \hat{L} \]
\[ \cos \theta_g = \hat{E} \cdot \hat{L} \]
\[ \cos \theta_f = \hat{H} \cdot \hat{N} \]
Phong Model

\[ (\hat{E} \cdot R_\hat{N}(\hat{L}))^\gamma \]

\[ (\hat{L} \cdot R_\hat{N}(\hat{E}))^\gamma \]

Reciprocity:
\[ (\hat{E} \cdot R(\hat{L}))^\gamma = (\hat{L} \cdot R(\hat{E}))^\gamma \]

Distributed light source!

Energy Normalization

Energy normalize Phong Model

\[
\rho(H^2 \rightarrow \omega_r) = \int_{H^2(\hat{N})} \left(\hat{L} \cdot R_\hat{N}(\hat{E})\right)^s \cos \theta_i \, d\omega_i \\
\leq \int_{H^2(\hat{N})} \left(\hat{L} \cdot R_\hat{N}(\hat{E})\right)^s \, d\omega_i \\
\leq \int_{H^2(\hat{N})} \left(\hat{L} \cdot R_\hat{N}(\hat{E})\right)^\gamma \, d\omega_R \\
= \int_{H^2} \cos^s \theta \, d\omega = \frac{2\pi}{s + 1}
\]
Microfacet Model

Bouguer’s “little faces”

P. Bouguer, *Treatise on Optics*, 1760
Reflection of the Sun from the Sea

Minnaert, *Light and Color in the Outdoors*, p. 28

Reflection Angles

Assume \( L \) and \( E \) are at the same height \( h \)

Calculate the angular height \( \gamma \)

\[ h = r \cos \theta \]
Reflection Angles

\[ \alpha + \beta = \gamma + \delta \]

Reflection Angles

\[ \alpha + \beta = \gamma + \delta \]
\[ \beta - \alpha = \delta \]
Reflection Angles

\[ \alpha + \beta = \gamma + \delta \]
\[ \beta - \alpha = \delta \quad \Rightarrow \quad \gamma = 2\alpha \]

Reflection Angles

\[ h = r \cos \theta \]
Reflection Angles

Calculate the angular width

\[ \tan \alpha = \frac{b}{h} \]

\[ h = r \cos \theta \]

Front view

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Reflection Angles

Calculate the angular width

\[ \tan \psi = \frac{b}{r} \]

Front view
Reflection Angles

\[ \tan \alpha = \frac{b}{h} \]
\[ h = r \cos \theta \]
\[ \tan \psi = \frac{b}{r} = \frac{h}{r \tan \alpha} = \tan \alpha \cos \theta \]

Analysis on the Sphere

\[ \gamma = 2\alpha \]
\[ \tan \psi = \tan \alpha \cos \theta \]
**Microfacet Distributions**

Microfacet

Total projected area
\[ \int_{\mathcal{H}^2} dA(\omega_h) \cos \theta_h \, d\omega_h = dA \]

Probability distribution
\[ \int_{\mathcal{H}^2} D(\omega_h) \cos \theta_h \, d\omega_h = 1 \]

Area distribution \( dA(\omega_h) \)

Microfacet distribution \( D(\omega_h) \equiv dA(\omega_h) / dA \)

**Gaussian Rough Surface**

Gaussian distribution of heights
\[ p(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2\sigma^2}} \]

Gaussian distribution of slopes
\[ D(\alpha) = \frac{1}{\sqrt{\pi m^2 \cos^2 \alpha}} e^{\frac{\tan^2 \alpha}{m^2}} \]

Beckmann

\[ m = \frac{2\sigma}{\tau} \]
Microfacet Distribution Functions

Isotropic distributions

Characterize by half-angle $\beta$

Examples:

- **Blinn**
  
  $D_1(\alpha) = \cos^5 \alpha$

- **Torrance-Sparrow**
  
  $D_2(\alpha) = e^{-(c_2 \alpha^2)}$

- **Trowbridge-Reitz**
  
  $D_3(\alpha) = \frac{c_3^2}{(1 - c_3^2 \cos^2 \alpha - 1)}$

\[ c_3 = \left( \frac{\cos^2 \beta - 1}{\cos^2 \beta - \sqrt{2}} \right)^{\sqrt{2}} \]

Torrance-Sparrow Model
Torrance-Sparrow Comparison

Aluminum

Found an off-specular peak

Magnesium Oxide

Found an off-specular peak
Explanation: Fresnel Term

\[ f_r(\omega_i \rightarrow \omega_r) \approx F(\theta_i')D(\alpha) \]

Torrance-Sparrow Model

\[ \cos \theta_i = \hat{L} \cdot \hat{N} \]
\[ \cos \theta_i' = \hat{L} \cdot \hat{H} \]
\[ d\Phi_h = L_i(\omega_i) \cos \theta_i' d\omega_i' dA(\omega_h) \]
\[ dA(\omega_h) = D(\omega_h) d\omega_h dA \]
\[ d\Phi_h = L_i(\omega_i) \cos \theta_i' d\omega_i' D(\omega_h) d\omega_h dA \]
\[ d\Phi_r = dL_r(\omega_i \rightarrow \omega_r) \cos \theta_r d\omega_r dA \]
\[ d\Phi_r = d\Phi_h \]

\[ dL_r(\omega_i \rightarrow \omega_r) \cos \theta_r d\omega_r dA = L_i(\omega_i) \cos \theta_i' d\omega_i' D(\omega_h) d\omega_h dA \]

Prime indicates wrt H
Torrance-Sparrow Model

\[ dL_r(\omega_i \rightarrow \omega_r) \cos \theta_r \, d\omega_r \, dA \]
\[ = L_i(\omega_i) \cos \theta_i' \, d\omega_i' \, D(\omega_h) \, d\omega_h \, dA \]

\[ f_r(\omega_i \rightarrow \omega_r) = \frac{dL_r(\omega_i \rightarrow \omega_r)}{dE(\omega_i)} \]
\[ = \frac{L_i(\omega_i) \cos \theta_i' \, d\omega_i' \, D(\omega_h) \, d\omega_h \, dA}{(\cos \theta_i \, d\omega_i \, dA)(L_i(\omega_i) \cos \theta_i \, d\omega_i)} \]
\[ = \frac{D(\omega_h)}{\cos \theta_i \cos \theta_r} \frac{\cos \theta_i' \, d\omega_h}{d\omega_r} \]
\[ = \frac{D(\omega_h)}{4 \cos \theta_i \cos \theta_r} \]

Solid Angle Distributions

\[ d\omega_r = \sin \theta_r \, d\theta_r \, d\varphi_r \]
\[ = (\sin 2\theta_r')2 \, d\theta_r' \, d\varphi_r \]
\[ = (2 \, \sin \theta_i' \cos \theta_i') \, d\theta_r' \, d\varphi_r \]
\[ = 4 \, \cos \theta_i' \sin \theta_i' \, d\theta_r' \, d\varphi_r \]
\[ = 4 \, \cos \theta_i' \, d\omega_h \]

\[ \frac{d\omega_h}{d\omega_r} = \frac{1}{4 \cos \theta_i'} \]
Self-Shadowing

Problem: Conservation of Energy

Without self-shadowing  With self-shadowing
Self-Shadowing Function

\[ W = m \]

From Smith, 1967

\[ S(\theta) = \frac{1 - \frac{1}{2} \text{erfc} \left( \frac{\mu}{\sqrt{2m}} \right)}{1 + \Lambda(\mu)} \]

\[ 2\Lambda(\mu) = \left( \frac{2}{\sqrt{\pi}} \right) \frac{m}{\mu} e^{-\mu^2 / 2m^2} - \text{erfc} \left( \frac{\mu}{\sqrt{2m}} \right) \]
Self-Consistency Condition

\[ \int S(\theta)D(\alpha)\cos \theta' d\omega_\alpha = \cos \theta \]

The sum of the areas of the illuminated surface projected onto the plane normal to the direction of incidence is independent of the roughness of the surface, and equal to the projected area of the underlying mean plane.

Torrance-Sparrow Theory

\[ f_r(\omega_i \rightarrow \omega_r) = \frac{F(\theta'_i)S(\theta'_i)S(\theta_r)D(\alpha)}{4 \cos \theta_i \cos \theta_r} \]

*Fig. 6.* Fresnel reflectance.

*Fig. 7.* The factor \( G(\phi)/\cos \psi \) in the plane of incidence for various incidence angles \( \phi \).