The Rendering Equation

Direct *(local)* illumination
- Light directly from light sources
- No shadows

Indirect *(global)* illumination
- Hard and soft shadows
- Diffuse interreflections *(radiosity)*
- Glossy interreflections *(caustics)*

Radiosity
Challenge

To evaluate the reflection equation
the incoming radiance must be known

\[ L_r(x, \omega_r) = \int_{H^2} f_i(x, \omega_i \rightarrow \omega_r) L_i(x, \omega_i) \cos \theta_i \, d\omega_i \]

To evaluate the incoming radiance
the reflected radiance must be known
To The Rendering Equation

Questions
1. How is light measured?
2. How is the spatial distribution of light energy described?
3. How is reflection from a surface characterized?
4. What are the conditions for equilibrium flow of light in an environment?

The Grand Scheme

- Light and Radiometry
  - Energy Balance
  - Surface Rendering Equation
  - Radiosity Equation
  - Volume Rendering Equation
Energy Balance

Balance Equation

Accountability

\[ \text{[outgoing]} - \text{[incoming]} = \text{[emitted]} - \text{[absorbed]} \]

- Macro level

*The total light energy put into the system must equal the energy leaving the system (usually, via heat).*

\[ \Phi_o - \Phi_i = \Phi_e - \Phi_a \]

- Micro level

*The energy flowing into a small region of phase space must equal the energy flowing out.*

\[ B(x) - E(x) = B_e(x) - E_a(x) \]
Surface Balance Equation

\[ \text{[outgoing]} = \text{[emitted]} + \text{[reflected]} \]

\[ L_o(x, \omega_o) = L_e(x, \omega_o) + L_r(x, \omega_o) \]

\[ = L_e(x, \omega_o) + \int_{H^2} f_t(x, \omega_t \rightarrow \omega_o) L_t(x, \omega_t) \cos \theta_t \; d\omega_t \]

Direction Conventions

\( f_t(\omega_i \rightarrow \omega_o) \)

BRDF

Surface vs. Field Radiance

\( L(x, \omega) = L(x, -\omega) \)

\( L(x, \omega) = L_o(x, \omega_o) \)
**Surface Balance Equation**

\[
\text{[outgoing]} = \text{[emitted]} + \text{[reflected]} + \text{[transmitted]}
\]

\[
L_e(x, \omega_o) = L_e(x, \omega_o) + L_r(x, \omega_o) + L_i(x, \omega_i)
\]

\[
L_r(x, \omega_o) = \int_{n_i^2} f_r(x, \omega_i \rightarrow \omega_o) L_i(x, \omega_i) \cos \theta_i \, d\omega_i
\]

\[
L_i(x, \omega_o) = \int_{n_i^2} f_i(x, \omega_i \rightarrow \omega_o) L_i(x, \omega_i) \cos \theta_i \, d\omega_i
\]

\[
H^2_+(n) \cdot \omega_o \cdot n(x) > 0
\]

\[
H^2_-(n) \cdot \omega_o \cdot n(x) < 0
\]

**Two-Point Geometry**

\[
\omega(x, x') = \omega(x \rightarrow x') = \frac{x' - x}{\sqrt{(x' - x)^2 + (x' - x)^2}}
\]

Ray Tracing

\[
x^*(x, \omega)
\]

\[
\omega_o = \omega(x', x)
\]

\[
x = x^*(x, \omega_i)
\]

\[
x' = x^*(x, \omega_o)
\]
### Coupling Equations

Invariance of radiance

\[ L(x', \omega') = L_e(x', \omega) \]

\[ \omega' = \omega_e \]

\[ n_1 \]

\[ n' \]

\[ \omega_i \]

\[ x \]

\[ x' \]

\[ L_i(x, \omega_i) = L(x, -\omega_i) \quad L(x, \omega) = L(x', \omega') \]

### The Rendering Equation

Directional form

\[ L(x, \omega) = L_e(x, \omega) + \int f_r(x, \omega' \rightarrow \omega) L(x^*(x, \omega'), -\omega') \cos \theta' \, d\omega' \]

Integrate over hemisphere of directions

Transport operator i.e. ray tracing
The Rendering Equation

Surface form
\[ L(x', x) = L_e(x', x) + \int_{M^2} f_r(x'', x', x) L(x'', x') G(x'', x') dA''(x'') \]

Integrate over all surfaces

Geometry term
\[ G(x'', x') = \frac{\cos \theta'' \cos \theta'}{\|x'' - x\|^2} V(x'', x') \]

Visibility term
\[ V(x'', x') = \begin{cases} 1 & \text{visible} \\ 0 & \text{not visible} \end{cases} \]

The Radiosity Equation

Assume diffuse reflection

1. \( f_r(x, \omega_i \rightarrow \omega_o) = f_r(x) \Rightarrow \rho(x) = \pi f_r(x) \)

2. \( L(x, \omega) = B(x) / \pi \)

\[ B(x) = B_s(x) + \rho(x) E(x) \]
\[ B(x) = B_s(x) + \rho(x) \int_{M^2} F(x, x') B(x') dA'(x') \]

\[ F(x, x') = \frac{G(x, x')}{\pi} \]
Integral Equations

Integral equations of the 1st kind

\[ f(x) = \int k(x,x') g(x') \, dx' \]

Integral equations of the 2nd kind

\[ f(x) = g(x) + \int k(x,x') f(x') \, dx' \]
Linear Operators

*Linear operators act on functions like matrices act on vectors*

\[ h(x) = (L \circ f)(x) \]

They are linear in that

\[ L \circ (af + bg) = a(L \circ f) + b(L \circ g) \]

**Types of linear operators**

\[ (K \circ f)(x) = \int k(x, x') f(x') dx' \]

\[ (D \circ f)(x) = \frac{df}{dx}(x) \]

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Solving the Rendering Equation

**Rendering Equation**

\[ L = L_e + K \circ L \]

\[ (I - K) \circ L = L_e \]

**Solution**

\[ L = (I - K)^{-1} \circ L_e \]
Formal Solution

Neumann series

\[
(I - K)^{-1} = \frac{1}{I - K} = I + K + K^2 + \ldots
\]

Verify

\[
(I - K) \circ (I - K)^{-1} = (I - K) \circ (I + K + K^2 + \ldots)
\]

\[
= (I + K + \ldots) - (K + K^2 + \ldots)
\]

\[
= I
\]

Successive Approximations

Successive approximations

\[
L^1 = L_e
\]

\[
L^2 = L_e + K \circ L^1
\]

\[
\ldots
\]

\[
L^n = L_e + K \circ L^{n-1}
\]

Converged

\[
L^n = L^{n-1} \quad \therefore \quad L^n = L_e + K \circ L^n
\]
Successive Approximation

Paths
**Light Path**

\[ S(x_0, x_1) = L_e(x_0, x_1) \]

\[ L_s(x_0, x_1, x_2, x_3) \]

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**Light Path**

\[ S(x_0, x_1) \]

\[ f_r(x_1, x_2, x_3) \]

\[ G(x_0, x_1) \]

\[ G(x_1, x_2) \]

\[ x_0 \]

\[ x_1 \]

\[ x_2 \]

\[ x_3 \]

\[ L_s(x_0, x_1, x_2, x_3) \]

\[ L_s(x_0, x_1, x_2, x_3) = S(x_0, x_1)G(x_0, x_1)f_r(x_0, x_1, x_2)G(x_1, x_2)f_r(x_1, x_2, x_3) \]
Light Paths

\[ L(x_2, x_3) = \int \int L_S(x_0, x_1, x_2, x_3) dA(x_0) dA(x_1) \]

Light Transport

Integrate over all paths of all lengths

\[ L(x_{k-1}, x_k) = \sum_{k=1}^{\infty} \int \cdots \int M^2 M^2 L_S(x_0, \ldots, x_{k-2}, x_{k-1}, x_k) dA(x_0) \cdots dA(x_{k-2}) \]

Question:
- How to sample space of paths?
Classic Ray Tracing

Forward (from eye): \( E S^* (D|G) L \)

Photon Paths

From Heckbert

CS348B Lecture 13  Pat Hanrahan, Spring 2009
How to Solve It?

Finite element methods
- Classic radiosity
  - Mesh surfaces
  - Piecewise constant basis functions
  - Solve matrix equation
- Not practical for rendering equation

Monte Carlo methods
- Path tracing (distributed ray tracing)
- Bidirectional ray tracing
- Photon mapping