Overview

Earlier lecture
- Monte Carlo integration

Today
- Variance reduction
- Importance sampling
- Stratified sampling
- Multidimensional sampling patterns
- Discrepancy and Quasi-Monte Carlo

Next lecture
- Signal processing view of sampling

Camera Simulation

<table>
<thead>
<tr>
<th>Effect</th>
<th>Cause</th>
</tr>
</thead>
<tbody>
<tr>
<td>Field of view</td>
<td>Film size, stops and pupils</td>
</tr>
<tr>
<td>Depth of field</td>
<td>Aperture, focal length</td>
</tr>
<tr>
<td>Exposure</td>
<td>Film speed, aperture, shutter</td>
</tr>
<tr>
<td>Motion blur</td>
<td>Shutter</td>
</tr>
</tbody>
</table>

CS348B Lecture 8
Pat Hanrahan, Spring 2011
Cameras (5D integral)

\[ R = \iiint_{T} \iint_{\Omega} \int_{A} L(x, \omega, t) \ P(x) \ S(t) \ \cos \theta \ dA \ d\omega \ dt \]

Motion Blur

Depth of Field

Cook, Porter, Carpenter, 1984

Mitchell, 1991

Lighting (2D integral)

\[ E(x) = \int_{\mu^2} L_i(x, \omega) \cos \theta \ d\omega \]

16 shadow rays per eye ray

Uniform grid

Jittered grid
Unbiased Estimator

\[ E[F_N] = I(f) \]

\[ E[F_N] = E\left[\frac{1}{N} \sum_{i=1}^{N} Y_i\right] = \frac{1}{N} \sum_{i=1}^{N} E[Y_i] = \frac{1}{N} \sum_{i=1}^{N} E[f(X_i)] = \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{1} f(x)p(x) dx = \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{1} f(x)dx = \int_{0}^{1} f(x)dx \]

Properties

\[ E[\sum_{i} Y_i] = \sum_{i} E[Y_i] \]

\[ E[aY] = aE[Y] \]

Assume uniform probability distribution for now

Variance Decreases as N Increases

1 shadow ray per eye ray

16 shadow rays per eye ray
Variance

Definition

\[ V[Y] = E[(Y - E[Y])^2] \]
\[ = E[Y^2] - E[Y]^2 \]

Variance decreases linearly with sample size

\[ V\left[ \frac{1}{N} \sum_{i=1}^{N} Y_i \right] = \frac{1}{N^2} \sum_{i=1}^{N} V[Y_i] = \frac{1}{N^2} NV[Y] = \frac{1}{N} V[Y] \]

\[ V[aY] = a^2 V[Y] \]

“Biasing”

Previously used a uniform probability distribution

Change the distribution - bias the samples

\[ X_i \sim p(x) \]

However, then must change the estimator to

\[ Y_i = \frac{f(X_i)}{p(X_i)} \]

N. B. This use of the term “biasing” is different than the term “unbiased estimator”
Unbiased Estimate

Probability: \( X_i \sim p(x) \)

Estimator: \( Y_i = \frac{f(X_i)}{p(X_i)} \)

Proof:

\[
E[Y_i] = E\left[ \frac{f(x)}{p(x)} \right] = \int \frac{f(x)}{p(x)} p(x) dx = \int f(x) dx
\]
Example of Importance Sampling

Hemispherical Solid Angle
4 eye rays per pixel
100 shadow rays

Light Source Area
4 eye rays per pixel
100 shadow rays

Comparing Different Techniques

Efficiency measure

\[
    \text{Efficiency} \propto \frac{1}{\text{Variance} \cdot \text{Cost}}
\]

If one technique has twice the variance as another technique, then it takes twice as many samples to achieve the same variance.

If one technique has twice the cost of another technique with the same variance, then it takes twice as much time to achieve the same variance.
Stratified Sampling

Allocate samples per region
Estimate each region separately

\[ F_N = \frac{1}{N} \sum_{i=1}^{N} F_i \]

New variance

\[ V[F_N] = \frac{1}{N^2} \sum_{i=1}^{N} V[F_i] \]

If the variance in each region is the same, then total variance goes as 1/N

Stratified Sampling

Sample a polygon

\[ V[F_N] = \frac{1}{N^2} \sum_{i=1}^{N} V[F_i] = \frac{V[F_E]}{N^{1.5}} \]

However, if the variance in some region is smaller, then the variance may be reduced

In this example, the variance in the regions inside and outside is 0; the variance is due to the edge. But there are fewer edge regions
Sampling a Circle

Equi-Areal

\[ \theta = 2\pi U_1 \]
\[ r = \sqrt{U_2} \]

Shirley’s Mapping

\[ r = U_1 \]
\[ \theta = \frac{\pi U_2}{4 U_1} \]
### Space-Time Patterns

**Distribute samples in time**
- Complete in space
- Incomplete in time
- Decorrelate space and time
- Nearby samples in space should differ greatly in time

<table>
<thead>
<tr>
<th>6</th>
<th>10</th>
<th>2</th>
<th>13</th>
</tr>
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<tbody>
<tr>
<td>3</td>
<td>14</td>
<td>12</td>
<td>8</td>
</tr>
<tr>
<td>15</td>
<td>0</td>
<td>7</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

**Cook Pattern**

<table>
<thead>
<tr>
<th>15</th>
<th>8</th>
<th>5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
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</tr>
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<td>1</td>
<td>6</td>
<td>11</td>
<td>12</td>
</tr>
</tbody>
</table>

**Pan-diagonal Magic Square**

### Path Tracing

- **4 eye rays per pixel**
  - 16 shadow rays per eye ray
  - Complete

- **64 eye rays per pixel**
  - 1 shadow ray per eye ray
  - Incomplete
Block Design

Latin Square

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>d</th>
<th>c</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>b</td>
<td>a</td>
<td>d</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>d</td>
</tr>
<tr>
<td>c</td>
<td>d</td>
<td>c</td>
<td>b</td>
<td>a</td>
</tr>
</tbody>
</table>

Alphabet of size \( n \)
Each symbol appears exactly once in each row and column
Rows and columns are stratified

N-Rook Pattern

Incomplete block design
Replaced \( n^2 \) samples with \( n \) samples
Permutations: \( (\pi_x(i), \pi_y(i), \ldots, \pi_d(i)) \)
Generalizations: N-queens, 2D projection

\( (\pi_x = \{1, 2, 3, 4\}, \pi_y = \{4, 2, 3, 1\}) \)
High-dimensional Sampling

Complete set of samples \( N = n \times n \times \cdots \times n = n^d \)

Random sampling

Error … \( E \sim V^{1/2} \sim \frac{1}{N^{1/2}} \)

Numerical integration

Error … \( E \sim \frac{1}{n} = \frac{1}{N^{1/d}} \)

*In high dimensional space, Monte Carlo requires fewer samples than numerical integration for the same error*

Discrepancy

\[ \Delta(x, y) = \frac{n(x, y)}{N} - xy \]

\( A = xy \)

\( n(x, y) \) number of samples in \( A \)

\[ D_N = \max_{x,y} |\Delta(x, y)| \]
Quasi-Monte Carlo Patterns

Radical inverse (digit reverse)

of integer $i$ in integer base $b$

\[ i = d_i \cdots d_2 d_1 d_0 \]
\[ \phi_b(i) \equiv 0.d_0 d_1 d_2 \cdots d_i \]

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\phi_b(i)$</th>
<th>$\phi_b(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>.1</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>.01</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>.11</td>
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<tr>
<td>4</td>
<td>100</td>
<td>.001</td>
</tr>
<tr>
<td>5</td>
<td>101</td>
<td>.101</td>
</tr>
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</table>

Hammersley points

\( i/N, \phi_2(i), \phi_3(i), \phi_3(i), \cdots \)

\[ D_N = O\left(\frac{\log^{d-1} N}{N}\right) \]

Halton points (sequential)

\( (\phi_2(i), \phi_3(i), \phi_3(i), \cdots) \)

\[ D_N = O\left(\frac{\log^d N}{N}\right) \]

Hammersly Points

\( i/N, \phi_2(i), \phi_3(i), \phi_3(i), \cdots \)
Edge Discrepancy

Note: SGI IR Multisampling extension: 8x8 subpixel grid; 1,2,4,8 samples

Low-Discrepancy Patterns

<table>
<thead>
<tr>
<th>Process</th>
<th>16 points</th>
<th>256 points</th>
<th>1600 points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zaremba</td>
<td>0.0504</td>
<td>0.00478</td>
<td>0.00111</td>
</tr>
<tr>
<td>Jittered</td>
<td>0.0538</td>
<td>0.00595</td>
<td>0.00146</td>
</tr>
<tr>
<td>Poisson-Disk</td>
<td>0.0613</td>
<td>0.00767</td>
<td>0.00241</td>
</tr>
<tr>
<td>N-Rooks</td>
<td>0.0637</td>
<td>0.0123</td>
<td>0.00488</td>
</tr>
<tr>
<td>Random</td>
<td>0.0924</td>
<td>0.0224</td>
<td>0.00866</td>
</tr>
</tbody>
</table>

Discrepancy of random edges, From Mitchell (1992)

Random sampling converges as $N^{-1/2}$
Zaremba converges faster and has lower discrepancy
Zaremba has a relatively poor blue noise spectra
Jittered and Poisson-Disk recommended
Theorem on Total Variation

Theorem:
\[ \left| \frac{1}{N} \sum_{i=1}^{N} f(X_i) - \int f(x) \, dx \right| \leq V(f) D_N \]

Proof: Integrate by parts
\[
\int f(x) \left[ \frac{\delta(x-x_i)}{N} - 1 \right] dx = \int f(x) \frac{\partial \Delta(x)}{\partial x} dx = \frac{\delta(x-x_i)}{N} - 1
\]
\[
= f \Delta^i_0 - \int \frac{\partial f(x)}{\partial x} \Delta(x) \, dx = - \int \frac{\partial f(x)}{\partial x} \Delta(x) \, dx
\]
\[
\leq D_N \int \left| \frac{\partial f(x)}{\partial x} \right| \, dx = V(f) D_N
\]

Views of Integration

1. Numerical
   - Quadrature/Integration rules
   - Smooth functions
2. Statistical sampling (Monte Carlo)
   - Sampling like polling
   - Variance reduction techniques
   - High dimensional sampling: $1/N^{1/2}$
3. Quasi Monte Carlo
   - Discrepancy
   - Asymptotic efficiency in high dimensions
4. Signal processing
   - Sampling and reconstruction
   - Aliasing and antialiasing
   - Blue noise good