The Rendering Equation

Direct illumination (local computation)
- Light directly from light sources
- No shadows

Direct illumination (global computation)
- Hard and soft shadows

Indirect global illumination (GI)
- Diffuse inter-reflections (radiosity)
- Glossy inter-reflections (caustics)
Lighting Effects

- Hard Shadows
- Soft Shadows
- Caustics
- Indirect Illumination
Radiosity
Challenge

To evaluate the reflection equation the incoming radiance must be known

\[ L_r(x, \omega_r) = \int_{H^2} f_r(x, \omega_i \rightarrow \omega_r) L_i(x, \omega_i) \cos \theta_i \, d\omega_i \]

The incoming radiance may result from emission and reflection from other surfaces
To The Rendering Equation

Questions

1. How is light measured?

2. How is the spatial distribution of light energy described?

3. How is reflection from a surface characterized?

4. What are the conditions for equilibrium flow of light in an environment?
The Grand Scheme

Light and Radiometry

Energy Balance

Surface Rendering Equation

Volume Rendering Equation

Radiosity Equation
Energy Balance
Balance Equation

Accountability

[outgoing] - [incoming] = [emitted] - [absorbed]

■ Macro level

The total light energy put into the system must equal the energy leaving the system (usually, via heat).

\[ \Phi_o - \Phi_i = \Phi_e - \Phi_a \]

■ Micro level

The energy flowing into a small region of phase space must equal the energy flowing out.

\[ B(x) - E(x) = B_e(x) - E_a(x) \]
Surface Balance Equation

\[ \text{[outgoing]} = \text{[emitted]} + \text{[reflected]} \]

\[
L_o(x, \omega_o) = L_e(x, \omega_o) + L_r(x, \omega_o) \\
= L_e(x, \omega_o) + \int_{H^2} f_r(x, \omega_i \rightarrow \omega_o) L_i(x, \omega_i) \cos \theta_i \, d\omega_i
\]
Direction Conventions

BRDF

Field radiance \( L(x, \omega) \)

Surface radiance \( L_i(x, \omega_i) L_o(x, \omega_o) \)
Two-Point Geometry

$$\omega(x, x') = \omega(x \rightarrow x') = \frac{x' - x}{|x' - x|}$$

Ray Tracing $$x^*(x, \omega)$$

$$\omega_i = \omega(x, x')$$

$$\omega_o = \omega(x', x)$$

$$x' = x^*(x, \omega_i)$$

$$x = x^*(x', \omega_o)$$
Coupling Equations

Invariance of radiance

\[
L_i(x, \omega_i) = L(x, -\omega_i) \quad \text{and} \quad L(x, \omega) = L(x', \omega')
\]
The Rendering Equation

**Directional form**

\[ L(x, \omega) = L_e(x, \omega) + \int_{H^2} f_r(x, \omega' \rightarrow \omega) L(x^*(x, \omega'), -\omega') \cos \theta' \, d\omega' \]

**Scatter light**

**Transport light**
The Rendering Equation

Surface form

\[ L(x', x) = L_e(x', x) + \int_{M^2} f_r(x'', x', x) L(x'', x') G(x'', x') dA''(x'') \]

Integrate over all surfaces

Geometry term

\[ G(x'', x') = \frac{\cos \theta_i \cos \theta_o}{\|x'' - x'\|^2} V(x'', x') \]

Visibility term

\[ V(x'', x') = \begin{cases} 1 & \text{visible} \\ 0 & \text{not visible} \end{cases} \]
The Radiosity Equation

Assume diffuse reflection

1. \( f_r(x, \omega_i \rightarrow \omega_o) = f_r(x) \Rightarrow \rho(x) = \pi f_r(x) \)

2. \( L(x, \omega) = B(x) / \pi \)

\[
B(x) = B_e(x) + \rho(x)E(x)
\]

\[
B(x) = B_e(x) + \rho(x) \int_{M^2} F(x, x')B(x') \, dA'(x')
\]

\[
F(x, x') = \frac{G(x, x')}{\pi}
\]
Integral Equations
Integral Equations

Integral equations of the 1st kind

\[ f(x) = \int k(x, x') g(x') \, dx' \]

Integral equations of the 2nd kind

\[ f(x) = g(x) + \int k(x, x') f(x') \, dx' \]
Linear Operators

Linear operators act on functions like matrices act on vectors

\[ h(x) = (L \circ f)(x) \]

They are linear in that

\[ L \circ (af + bg) = a(L \circ f) + b(L \circ g) \]

Types of linear operators

\[ (K \circ f)(x) \equiv \int k(x, x') f(x') dx' \]

\[ (D \circ f)(x) \equiv \frac{df}{dx}(x) \]
Solving the Rendering Equation

Rendering Equation

\[ L = L_e + K \circ L \]
\[ (I - K) \circ L = L_e \]

Solution

\[ L = (I - K)^{-1} \circ L_e \]
Formal Solution

Neumann series

\[(I - K)^{-1} = \frac{1}{I - K} = I + K + K^2 + ...\]

Verify

\[(I - K) \circ (I - K)^{-1} = (I - K) \circ (I + K + K^2 + ...)\]

\[= (I + K + ...) - (K + K^2 + ...)\]

\[= I\]
Successive approximations

\[ L^1 = L_e \]
\[ L^2 = L_e + K \circ L^1 \]
\[ \ldots \]
\[ L^n = L_e + K \circ L^{n-1} \]

Converged

\[ L^n = L^{n-1} \quad \therefore \quad L^n = L_e + K \circ L^n \]
Successive Approximation

\[ L_e \]
\[ K \circ L_e \]
\[ K \circ K \circ L_e \]
\[ K \circ K \circ K \circ L_e \]
\[ L_e \]
\[ L_e + K \circ L_e \]
\[ L_e + \cdots K^2 \circ L_e \]
\[ L_e + \cdots K^3 \circ L_e \]
Paths
\[ S(x_0, x_1) = L_e(x_0, x_1) \]

\[ L_S(x_0, x_1, x_2, x_3) \]
Light Path

\[
L_S(x_0, x_1, x_2, x_3) = S(x_0, x_1) G(x_o, x_1) f_r(x_0, x_1, x_2) G(x_1, x_2) f_r(x_1, x_2, x_3)
\]
Light Paths

\[ L(x_2, x_3) = \int_{A_0} \int_{A_1} L_S(x_0, x_1, x_2, x_3) dA(x_0) dA(x_1) \]
Light Transport

Integrate over all paths of all lengths

\[ L(x_{k-1}, x_k) = \sum_{k=1}^{\infty} \int_{M^2} \cdots \int_{M^2} L_S(x_0, \ldots, x_{k-2}, x_{k-1}, x_k) dA(x_0) \cdots dA(x_{k-2}) \]

Questions:

- How to generate all the possible paths?
- How to sample space of paths efficiently?
Classic Ray Tracing

Forward (from eye): $E \ S^* \ (D|G) \ L$

From Heckbert
How to Solve It?

Finite element methods

- Classic radiosity
  - Mesh surfaces
  - Piecewise constant basis functions
  - Solve matrix equation
- Not practical for rendering equation

Monte Carlo methods

- Path tracing (distributed ray tracing)
- Bidirectional ray tracing
- Photon mapping