Direct Lighting

Earlier lectures
- Reflection Models I, II
- Monte Carlo I, II, III

Today
- MC estimation of the reflection equation
- Sampling lights and BRDFs
- Improving efficiency: splitting
- Scenes with 1000s of light sources
\[ L_i(p, \omega_i) \]

\[ L_o(p, \omega_o) \]
The Reflection Equation

\[ L_o(p, \omega_o) = \int_{H^2} f_r(p, \omega_i \rightarrow \omega_o) L_i(p, \omega_i) \cos \theta_i \, d\omega_i \]
Directional Light Sources

Incident radiance in parallel rays

\[ L_i(\omega) = L \, \delta(\omega - \omega_{\text{light}}) \]

\[ L_o(p, \omega_o) = \int_{H^2} f_r(p, \omega_i \rightarrow \omega_o) \, L_i(p, \omega_i) \, \cos \theta_i \, d\omega_i \]

\[ = \int_{H^2} f_r(p, \omega_i \rightarrow \omega_o) \, L \, \delta(\omega - \omega_{\text{light}}) \, \cos \theta_i \, d\omega_i \]

\[ = f_r(p, \omega_{\text{light}} \rightarrow \omega_o) \, L \, V(p, \omega_{\text{light}}) \, \cos \theta_{\text{light}} \]

Binary visibility function
Specular BRDFs

\[ L_o(p, \omega_o) = \int_{H^2} f_r(p, \omega_i \rightarrow \omega_o) L_i(p, \omega_i) \cos \theta_i \, d\omega_i \]

**Perfect Specular BRDF:**

\[ f_r(\omega_i \rightarrow \omega_o) = \frac{\delta(\omega_i - R(\omega_o, \vec{n}))}{\cos \theta_i} \]

\( R(\omega_r, \vec{n}) \) is specular direction direction

\[ L_o(p, \omega_o) = L_i(p, R(\omega_o, \vec{n})) \]
Monte Carlo Estimate

\[ L_0(p, \omega_0) = \int_{H^2} f_r(p, \omega_i \rightarrow \omega_0) L_i(p, \omega_i) \cos \theta_i \, d\omega_i \]

Sample: \( \omega_j \sim p(\omega) \)

Estimate: \[ \frac{1}{N} \sum_{j=1}^{N} \frac{f_r(p, \omega_j \rightarrow \omega_0) L_i(p, \omega_j) \cos \theta_j}{p(\omega_j)} \]
Uniform Sampling by Direction
Why is Area Better than Hemisphere?

Hemisphere
16 shadow rays

Area
16 shadow rays
Sampling Area Light Sources

Sample uniformly by area on the light’s surface
Convert to solid angle, compute estimator

\[ L_o(p, \omega_o) = \int_A f_r(p, \omega_i \rightarrow \omega_o) L_i(p, \omega_i) \cos \theta_i \frac{\cos \theta}{r^2} \, dA \]
Sampling Area Light Sources

Sample uniformly by area on the light’s surface
Convert to solid angle, compute estimator

\[ L_o(p, \omega_o) = \int_A f_r(p, \omega_i \rightarrow \omega_o) L_e V(p, p') \cos \theta_i \frac{\cos \theta}{r^2} dA \]
Sampling a Unit Sphere

Differential measure on the sphere:

\[ d\omega = \sin \theta \, d\theta \, d\phi \]

Integrate to get normalization term:

\[ \int_{S^2} d\omega = 4\pi \]

PDF for directions:

\[ p(\omega) = \frac{1}{4\pi} \]

PDF w.r.t. \((\theta, \phi)\):

\[ p(\theta, \phi) = \frac{\sin \theta}{4\pi} \]
Sampling a Unit Sphere

PDF w.r.t. \((\theta, \phi)\):
\[
p(\theta, \phi) = \frac{\sin \theta}{4\pi} = p(\theta)p(\phi) = \frac{\sin \theta}{2} \frac{1}{2\pi}
\]

CDF for \(\phi\):
\[
P(\phi) = \int_{0}^{\phi} \frac{1}{2\pi} \, d\phi = \frac{\phi}{2\pi}
\]

Sampling \(\phi\):
\[
\xi_1 = P(\phi) = \frac{\phi}{2\pi}
\]
\[
\phi = 2\pi \xi_1
\]
Sampling a Unit Sphere

PDF w.r.t. \((\theta, \phi)\):

\[
p(\theta, \phi) = \frac{\sin \theta}{4\pi} = p(\theta)p(\phi) = \frac{\sin \theta}{2} \frac{1}{2\pi}
\]

CDF for theta:

\[
P(\theta) = \int_{0}^{\theta} \frac{\sin \theta}{2} \, d\theta = \frac{1 - \cos \theta}{2}
\]

Sampling theta:

\[
\xi_2 = \frac{1 - \cos \theta}{2}
\]

\[
\cos \theta = 1 - 2\xi_2
\]

\[
\theta = \arccos(1 - 2\xi_2)
\]
Sampling Spherical Lights

Recipes:

\[
\cos \theta = 1 - 2\xi \\
\theta = \arccos(1 - 2\xi) \\
\phi = 2\pi\xi
\]

Spherical coordinates:

\((\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)\)

Sample locations:

\[
x = \cos(2\pi\xi_2) \sqrt{1 - z^2} \\
y = \sin(2\pi\xi_2) \sqrt{1 - z^2} \\
z = 1 - 2\xi_1
\]
Direct Lighting With Spherical Lights

1. Choose point randomly on sphere surface

2. PDF over area of sphere:

\[ p_A(x, y, z) = \frac{1}{4\pi r^2} \]

3. Convert area PDF to solid angle PDF:

\[ p(\omega) = \frac{1}{4\pi r^2} \frac{\cos \theta'}{|p' - p|^2} \]
Direct Lighting With Spherical Lights

4. Trace ray to see if \( p \) and \( p' \) are mutually visible

5. If so, exitant radiance from \( p' \) gives incident radiance at \( p \)

6. Can now evaluate the value of the estimator!
Sampling Spherical Lights: Directions

Problems with sampling by area?
Sampling Spherical Lights: Directions

Problems with sampling by area? 
Uniform Cone Sampling

\[
\cos \theta' = (1 - \xi) + \xi \cos \theta \quad \phi = 2\pi \xi
\]

\[
p(\omega) = \frac{1}{2\pi(1 - \cos \theta)}
\]
Environment Map Light Sources
Environment Map Light Sources
Capturing Environment Light Sources

http://www.aarondabelow.com/v1/pages/resources/resources-hdr_probes.html
Rendering with Environment Maps

http://magazine.creativecow.net/article/making-big-plans
Sampling Environment Map Lights

Luminance Map
Sampling Environment Map Lights

Sample 1D PDF to choose a row
Sampling Environment Map Lights

Sample row’s 1D PDF to choose a column

Sample 1D PDF to choose a row
Uniform Sampling by Direction
Light Source Sampling
Diffuse BRDF + Environment Light

\[
L_0(p, \omega_0) = f_r \int_{H^2} L_i(\omega_i) \cos \theta_i \, d\omega_i
\]

\[
= f_r \int_{H^2} L_e(\omega_i) V(p, \omega_i) \cos \theta_i \, d\omega_i
\]

\[
\approx f_r \frac{1}{N} \sum_{j}^{N} \frac{L_e(\omega_j) V(p, \omega_j) \cos \theta_j}{p(\omega_j)}
\]

**Low variance if** \(p(\omega) \propto L_e(\omega)\)
Sampling a Diffuse BRDF

Malley’s Method:
1. Generate uniform samples on the unit disk.
2. Project to hemisphere
3. Resulting distribution is cosine-weighted,

\[ p(\omega) = \frac{\cos \theta}{\pi} \]

Estimator:
\[
\frac{f_r(p, \omega_j \rightarrow \omega_r) L_i(p, \omega_j) \cos \theta_j}{p(\omega_j)} = \pi f_r L_i(p, \omega_j) = \rho L_i(p, \omega_j)
\]
Sampling a Phong Lobe

(Normalized) Phong BRDF:

\[ f_r(\omega_i \rightarrow \omega_r) = \frac{2\pi}{e + 1} (\omega_i \cdot R(\vec{n}, \omega_r))^e = \frac{2\pi}{e + 1} \cos^e \theta \]

\[ p(\phi) = \frac{1}{2\pi} \]

\[ p(\theta) = \frac{\cos^e \theta}{e + 1} \]

\[ \phi = 2\pi \xi_1 \]
Sampling a Phong Lobe

(Normalized) Phong BRDF:

\[
f_r(\omega_i \to \omega_r) = \frac{2\pi}{e+1} (\omega_i \cdot R(\vec{n}, \omega_r))^e = \frac{2\pi}{e+1} \cos^e \theta
\]

\[
p(\theta) = \frac{\cos^e \theta}{e+1} \quad P(\theta) = \int_0^\theta \frac{\cos^e \theta}{e+1} \sin \theta \, d\theta = \cos^{e+1} \theta
\]

\[
\xi_2 = P(\theta) = \cos^{e+1} \theta
\]

\[
\cos \theta = \frac{e+1}{\sqrt{\xi_2}}
\]
Uniform Sampling by Direction
Light Source Sampling
BSDF Sampling
Light Sampling Problem Cases
Light Sampling Problem Cases

BRDF small; low contribution
Light Sampling Problem Cases

BRDF small; low contribution

PDF small; too-high contribution
BRDF Sampling Problem Cases
BRDF Sampling Problem Cases

Misses light; zero contribution
BRDF Sampling Problem Cases

PDF small; surprise large contribution

Misses light; zero contribution
Multiple Importance Sampling

Use multiple sampling distributions \( p_i(x) \)

New Monte Carlo estimator:

\[
\int f(x) \, dx \approx \frac{1}{N} \left[ \sum_{i=1}^{N_a} w_1(x_{1,i}) \frac{f(x_{1,i})}{p_1(x_{1,i})} + \sum_{i=1}^{N_b} w_2(x_{2,i}) \frac{f(x_{2,i})}{p_2(x_{2,i})} + \cdots \right]
\]

\( w_i(x) \) are weighting terms

\( N_i \) samples taken from the \( i \)'th distribution

\[
N = \sum_i N_i
\]
Multiple Importance Sampling

Balance heuristic:

\[
\int f(x) \, dx \approx \frac{1}{N} \left[ \sum_{i=1}^{N_a} w_1(x_{1,i}) \frac{f(x_{1,i})}{p_1(x_{1,i})} + \sum_{i=1}^{N_b} w_2(x_{2,i}) \frac{f(x_{2,i})}{p_2(x_{2,i})} + \cdots \right]
\]

Estimator with balance heuristic:

\[
\int f(x) \, dx \approx \frac{1}{N} \left[ \sum_{i=1}^{N_a} w_1(x_{1,i}) \frac{f(x_{1,i})}{p_1(x_{1,i})} + \sum_{i=1}^{N_b} w_2(x_{2,i}) \frac{f(x_{2,i})}{p_2(x_{2,i})} + \cdots \right]
\]

\[
\frac{w_i(x)}{\sum_j \frac{N_j}{N} p_j(x)} = \frac{p_i(x)}{\sum_j \frac{N_j}{N} p_j(x)}
\]

Samples from \(p_1\) and Weighted average of all PDFs
BSDF Sampling
Light Source Sampling
Multiple Importance Sampling
“Splitting”

Standard estimator:

\[
\int_A \int_B f(a, b) \, da \, db
\]

Estimator:

\[
\frac{1}{N} \sum_{i=1}^{N} \frac{f(a_i, b_i)}{p(a_i, b_i)}
\]

Alternative: multiple samples of \(b\) for each \(a\)

New estimator:

\[
\frac{1}{N_a} \frac{1}{N_b} \sum_{i=1}^{N_a} \sum_{j=1}^{N_b} \frac{f(a_i, b_{i,j})}{p(a_i, b_{i,j})}
\]
“Splitting”

Can improve efficiency if the evaluation of

\[
\frac{1}{N_a} \frac{1}{N_b} \sum_{i=1}^{N_a} \sum_{j=1}^{N_b} \frac{f(a_i, b_{i,j})}{p(a_i, b_{i,j})}
\]

can share computation over terms

\[
f(a_i, b_{i,1}), f(a_i, b_{i,2}), \ldots
\]
\[
\frac{1}{N_{\text{pixel}}} \sum_{i=1}^{N_{\text{pixel}}} \left[ \frac{1}{N_{\text{light}}} \sum_{j=1}^{N_{\text{light}}} \frac{f_r(p_i, \omega_o, \omega_{i,j})L_i(p_i, \omega_{i,j}) \cos \theta_{i,j}}{p(\omega_{i,j})} \right]
\]
64 image samples x 1 light sample, 13.2 seconds
1 image sample x 64 light samples, 7.0 seconds
8 image samples x 8 light samples, 7.7 seconds
64 image samples x 1 light sample, 13.2 seconds
Multiple Light Sources

\[ L_o(p, \omega_o) = \sum_{j=1}^{N_{\text{lights}}} \int_{H^2} f_r(p, \omega_i \rightarrow \omega_o) L_{i,j}(p, \omega_i) \cos \theta_i \, d\omega_i \]
Multiple Light Sources

Monte Carlo estimate of a sum \( \sum_{i=1}^{N} f_i \)

- Define a discrete probability over terms \( p_i \)

\[ \sum_i p_i = 1 \]

- Draw \( N_{\text{samples}} \) samples \( s_j \sim p_i \)

- Estimator:

\[ \frac{1}{N_{\text{samples}}} \sum_{j=1}^{N_{\text{samples}}} \frac{f_{s_j}}{p_{s_j}} \]
Multiple Light Sources

Consider single sample, equal probabilities:

- **Draw one sample** \( s \sim p_i \)
- **Compute** \( f_s \)
- **Estimator:** \( f_s / p_s \)

**Expected value:**

\[
E \left[ \frac{f_s}{p_s} \right] = \sum_i p_i \frac{f_i}{p_i} = \sum_i f_i
\]
Uniform: \( p_i = \frac{1}{N_{\text{lights}}} \)

Power: \( p_i = \frac{\Phi_i}{\sum_j \Phi_j} \)

Spatially-varying: \( p_i(p) = \frac{\tilde{L}_i(p)}{\sum_j \tilde{L}_j(p)} \)
Measure One (beeple@)

>8k light sources
Clustering Far-Away Lights
Clustering Far-Away Lights
Light cuts

Light Tree

Clusters

Individual Lights

Representative Light

4 - Intensity

Three Lightcuts

Walter et al., 2005