Monte Carlo 1: Integration

Previous lecture: Analytical illumination formula

This lecture: Monte Carlo Integration
- Review random variables and probability
- Sampling from distributions
- Sampling from shapes
- Numerical calculation of illumination
Irradiance from the Environment

\[ d^2 \Phi_i(x, \omega) = L_i(x, \omega) \cos \theta \, dA \, d\omega \]

\[ dE(x, \omega) = L_i(x, \omega) \cos \theta \, d\omega \]

\[ E(x) = \int_{H^2} L_i(x, \omega) \cos \theta \, d\omega \]
Irradiance from a Uniform Area Source

\[ E(x) = \int_{H^2} L \cos \theta \, d\omega \]
\[ = L \int_{\Omega} \cos \theta \, d\omega \]
\[ = L \tilde{\Omega} \]

Direct Illumination
Uniform Triangle Light Source

\[ \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \gamma_i \vec{N}_i \cdot \vec{N} \]
Penumbras and Umbras
Lighting and Soft Shadows

\[ E(x) = \int_{H^2} L_i(x, \omega) \cos \theta \ d\omega \]

Challenges

1 Occluders
   - Complex geometry
   - Number of occluders

2 Non-uniform light sources

Source: Agrawala, Ramamoorthi, Heirich, Moll, 2000
Monte Carlo Illumination Calculation

Center

Random

1 shadow ray per eye ray
Monte Carlo Algorithms

Advantages

■ Easy to implement
■ Easy to think about (but be careful of subtleties)
■ Robust when used with complex integrands (lights, BRDFs) and domains (shapes)
■ Efficient for high dimensional integrals
■ Efficient when only need solution at a few points

Disadvantages

■ Noisy
■ Slow (many samples needed for convergence)
Random Variables

\[ X \text{ is a random variable} \]

A random variable takes on different values
(representing a distribution of potential values)

\[ X \sim p(x) \text{ probability distribution function (PDF)} \]
Discrete Probability Distributions

Discrete values \( x_i \)
with probability \( p_i \)

\[ p_i \geq 0 \]

\[ \sum_{i=1}^{n} p_i = 1 \]
Discrete Probability Distributions

**Cumulative PDF** $P_j$

$$P_j = \sum_{i=1}^{j} p_i$$

$0 \leq P_i \leq 1$

$P_n = 1$
Discrete Probability Distributions

Construction of samples

To randomly select an event,
Select $x_i$ if

$$P_{i-1} < U \leq P_i$$

Uniform random variable
Continuous Probability Distributions

**PDF** \( p(x) \)

\[ p(x) \geq 0 \]

**CDF** \( P(x) \)

\[ P(x) = \int_0^x p(x) \, dx \]

\[ P(x) = \Pr(X < x) \quad P(1) = 1 \]

\[ \Pr(\alpha \leq X \leq \beta) = \int_\alpha^\beta p(x) \, dx \]

\[ = P(\beta) - P(\alpha) \]
Sampling Continuous Distributions

Cumulative probability distribution function

\[ P(x) = \Pr(X < x) \]

Construction of samples

Solve for \( X = P^{-1}(U) \)

Must know the formula for:

1. The integral of \( p(x) \)
2. The inverse function \( P^{-1}(x) \)
Sampling a Circle

\[
A = \int_0^1 \int_0^{2\pi} r \, dr \, d\theta = \int_0^1 r \, dr \int_0^{2\pi} d\theta = \left( \frac{r^2}{2} \right) \left[ \theta \right]_0^{2\pi} = \pi
\]

\[
p(r,\theta) \, dr \, d\theta = \frac{1}{\pi} r \, dr \, d\theta \Rightarrow p(r,\theta) = \frac{r}{\pi}
\]

\[
p(r,\theta) = p(r) \, p(\theta)
\]

\[
p(\theta) = \frac{1}{2\pi}
\]

\[
P(\theta) = \frac{1}{2\pi} \theta
\]

\[
p(r) = 2r
\]

\[
P(r) = r^2
\]

\[
\theta = 2\pi U_1
\]

\[
r = \sqrt{U_2}
\]
Sampling a Circle

WRONG ≠ Equi-Areal

\[ \theta = 2\pi U_1 \]
\[ r = U_2 \]

RIGHT = Equi-Areal

\[ \theta = 2\pi U_1 \]
\[ r = \sqrt{U_2} \]
class Shape {
    public:
        // Shape Interface

        virtual ~Shape();
        virtual Bounds3f ObjectBound() const = 0;
        virtual Bounds3f WorldBound() const;
        virtual bool Intersect(const Ray &ray, Float *tHit,
                        SurfaceInteraction *isect,
                        bool testAlphaTexture = true) const = 0;

        virtual Float Area() const = 0;

        ...

        // Sample a point on the surface of the shape
        // and return the PDF with respect to area on the surface.
        virtual Interaction Sample(const Point2f &u, Float *pdf) const = 0;
        virtual Float Pdf(const Interaction &) const { return 1 / Area(); } 
};
Rejection Sampling

do {
  X=Uniform(-1,1)
  Y=Uniform(-1,1)
} while (X*X+Y*Y>1);

Efficiency?

Area of circle / Area of square
do { 
    X=Uniform(-1,1);
    Y=Uniform(-1,1);
} while (X*X+Y*Y>1);

R = sqrt(X*X+Y*Y)
dx = X/R
dy = Y/R
Computing Area of a Circle

A = 0
for( i=0; i<N; i++ ) {
    X=Uniform(-1,1);
    Y=Uniform(-1,1);
    if(X*X+Y*Y < 1)
        A += 1;
}
A = 4*A/N
Monte Carlo Integration

Definite integral
\[ I(f) = \int_{0}^{1} f(x) \, dx \]

Expectation of \( f \)
\[ E[f] = \int_{0}^{1} f(x) p(x) \, dx \]

Random variables
\[ X_i \sim p(x) \]
\[ Y_i = f(X_i) \]

Estimator
\[ F_N = \frac{1}{N} \sum_{i=1}^{N} Y_i \]
Unbiased Estimator

\[ E[F_N] = I(f) \]

Properties

\[ E[\sum_i Y_i] = \sum_i E[Y_i] \]
\[ E[aY] = aE[Y] \]

\[ E[F_N] = E\left[\frac{1}{N} \sum_{i=1}^{N} Y_i\right] = \frac{1}{N} \sum_{i=1}^{N} E[Y_i] = \frac{1}{N} \sum_{i=1}^{N} E[f(X_i)] = \frac{1}{N} \sum_{i=1}^{N} \int_0^1 f(x) p(x) \, dx \]
\[ = \frac{1}{N} \sum_{i=1}^{N} \int_0^1 f(x) \, dx \]
\[ = \int_0^1 f(x) \, dx \]

Assume uniform probability distribution for now
Direct Lighting: Hemispherical Integral

\[ E(x) = \int_{H^2} L(x, \omega) \cos \theta \, d\omega \]
Direct Lighting: Solid Angle Sampling

Sample hemisphere uniformly

\[ L_i(x,\omega) \]

\[ \int_{H^2} p(\omega) \, d\omega = 1 \]

\[ p(\omega) = \frac{1}{2\pi} \]
Direct Lighting: Solid Angle Sampling

\[ E(x) = 2\pi \int L(x, \omega) \cos \theta \frac{1}{2\pi} d\omega \]

\[ = 2\pi \int L(x, \omega) \cos \theta p(\omega) d\omega \]

Estimator

\[ Y_i = 2\pi L(x, \omega_i) \cos \theta_i \]
Direct Lighting: Hemisphere Sampling

Hemisphere

16 shadow rays
Direct Lighting: Area Integral

\[ E(x) = \int_{H^2} L_i(x, \omega) \cos \theta \, d\omega = \int_{A'} L_o(x', \omega') V(x, x') \frac{\cos \theta \cos \theta'}{|x - x'|^2} \, dA' \]

Integral

\[ d\omega = \frac{\cos \theta'}{|x - x'|^2} \, dA' \]

Visibility

\[ V(x, x') = \begin{cases} 0 & \text{blocked} \\ 1 & \text{visible} \end{cases} \]

Radiance

\[ L_i(x, \omega) = L_o(x', \omega') \]
Direct Lighting: Area Integral

\[ E(x) = \int_{A'} L_o(x', \omega') V(x, x') \frac{\cos \theta \cos \theta'}{|x - x'|^2} dA' \]
Direct Lighting: Area Sampling

\[ E(x) = \int_{A'} L_o(x', \omega') V(x, x') \frac{\cos \theta \cos \theta'}{|x - x'|^2} dA' \]

Sample shape uniformly by area
Direct Lighting: Area Sampling

\[ E(x) = \int_{A'} L_o(x', \omega') V(x, x') \frac{\cos \theta \cos \theta'}{|x - x'|^2} dA' \]

Estimator

\[ Y_i = L_o(x'_i, \omega'_i) V(x, x'_i) \frac{\cos \theta_i \cos \theta'_i}{|x - x'_i|^2} A' \]
Direct Lighting: Area Sampling

Area

16 shadow rays
Random Sampling Introduces Noise

Center

Random

1 shadow ray per eye ray
Quality Improves with More Rays

Area
1 shadow ray

Area
16 shadow rays
Why is Area Better than Hemisphere?

Hemisphere
16 shadow rays

Area
16 shadow rays