Overview

Earlier lectures

- Monte Carlo I: Integration
- Signal processing and sampling

Today: Monte Carlo II

- Noise and variance
- Efficiency
- Stratified sampling
- Importance sampling
Random Sampling Introduces Noise

Center

Random

1 shadow ray
Less Noise with More Rays

1 shadow ray

16 shadow rays
Review: Basic MC Estimator

\[ E \left[ \frac{1}{N} \sum_{i=1}^{N} f(X_i) \right] = \int f(x) \, dx \quad X_i \in [0, 1)^n \]
High-Dimensional Integration

Complete set of samples: $N = n \times n \times \cdots \times n = n^d$

- ‘The curse of dimensionality’

Random sampling error: $E \sim V^{1/2} \sim \frac{1}{\sqrt{N}}$

Numerical integration error: $E \sim \frac{1}{n} = \frac{1}{N^{1/d}}$

In high dimensions, Monte Carlo requires fewer samples than quadrature-based numerical integration for the same error.
A Tale of Two Functions

\[ f(x) = e^{-20(x-\frac{1}{2})^2} \]

\[ f(x) = e^{-5000(x-\frac{1}{2})^2} \]
### Monte Carlo Estimates, n=32

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\[
\int f(x) \, dx = 0.39571 \ldots
\]

\[
f(x) = e^{-20(x-\frac{1}{2})^2}
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\[
\int f(x) \, dx = 0.025067 \ldots
\]

\[
f(x) = e^{-5000(x-\frac{1}{2})^2}
\]
Estimate of integral: $6.1418 \times 10^{-12}$

$$\int f(x)dx = 0.025067\ldots$$
Variance

**Definition**

\[
V[Y] \equiv E[(Y - E[Y])^2]
\]

\[
= E[Y^2] - E[Y]^2
\]

e.g.: 

\[
Y = f(X)
\]

\[
E[Y] = \int f(x) \, dx
\]
Variance

Definition


e.g.:

\[ Y = f(X) \]

\[ E[Y] = \int f(x) \, dx \]

Variance decreases linearly with sample size

\[ V \left[ \frac{1}{N} \sum_{i=1}^{N} Y_i \right] = \frac{1}{N^2} \sum_{i=1}^{N} V[Y_i] = \frac{1}{N^2} NV[Y] = \frac{1}{N} V[Y] \]

\[ V[aY] = a^2 V[Y] \]
(Live Demo of Variance)
Comparing Different Techniques

Efficiency measure

\[
\text{Efficiency} \propto \frac{1}{\text{Variance} \cdot \text{Cost}}
\]

Comparing two sampling techniques A and B

If A has twice the variance as B, then it takes twice as many samples from A to achieve the same variance as B.

If A has twice the cost of B, then it takes twice as much time to reduce the variance using A compared to using B.

The product of variance and cost is a constant independent of the number of samples.

Recall: Variance goes as \(1/N\), time goes as \(C\cdot N\)
Stratified Sampling

Allocate samples per region
Estimate each region separately

\[ F_N = \frac{1}{N} \sum_{i=1}^{N} F_i \]

New variance

\[ V[F_N] = \frac{1}{N^2} \sum_{i=1}^{N} V[F_i] \]

If the variance in each region is the same, then total variance goes as \( 1/N \)
Stratified Sampling

Sample a polygon

\[ V[F_N] = \frac{1}{N^2} \sum_{i=1}^{\sqrt{N}} V[F_j] = \frac{V[F_E]}{N^{1.5}} \]

If the variance in some regions are smaller, then the overall variance will be reduced
Jittered vs. Uniform Supersampling

4x4 Jittered Sampling

4x4 Uniform
Stratified Samples, n=32

\[ \int f(x) \, dx = 0.025067 \ldots \]

<table>
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<tr>
<th>Variance</th>
<th>Uniform</th>
<th>Stratified</th>
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<td>3.99E-04</td>
<td>2.42E-04</td>
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Measuring Power at a Pixel

Film Plane

Lens Aperture

\[
E(p) = \int_{A_{\text{lens}}} L(p' \rightarrow p) \frac{\cos \theta \cos \theta'}{||p' - p||^2} \, dA'
\]

\[
W = \int_{A_{\text{pixel}}} \int_{A_{\text{lens}}} L(p' \rightarrow p) \frac{\cos \theta \cos \theta'}{||p' - p||^2} \, dA' \, dA
\]
Lens Exit Pupils

In Focus

Out of Focus
How to Stratify?

$n$ strata in $d$ dimensions: $O(n^d)$

- 8 strata in 4 dimensions: 4096 samples!
How to Stratify?

$n$ strata in $d$ dimensions: $O(n^d)$

- 8 strata in 4 dimensions: 4096 samples!

One solution: padding

- Generate stratified lower dimensional point sets, randomly associate pairs of samples

$(p_0, p_1, p_2, p_3)$
Random Sampling
Stratified Sampling
"Biased" Sampling is "Unbiased"

**Probability**
\[ X_i \sim p(x) \]

**Estimator**
\[ Y_i = \frac{f(X_i)}{p(X_i)} \]

**Proof**
\[
E[Y_i] = E\left[ \frac{f(x)}{p(x)} \right] \\
= \int \frac{f(x)}{p(x)} p(x) dx \\
= \int f(x) dx
\]
Importance Sampling

Sample according to $f$

$$\tilde{p}(x) = \frac{f(x)}{E[f]}$$

$$\tilde{f}(x) = \frac{f(x)}{\tilde{p}(x)}$$
Importance Sampling

Sample according to $f$

$$\tilde{p}(x) = \frac{f(x)}{E[f]}$$

$$\tilde{f}(x) = \frac{f(x)}{\tilde{p}(x)}$$

Variance

$$V[f] = E[f^2] - E^2[f]$$
Importance Sampling

Sample according to \( f \)

\[
\tilde{p}(x) = \frac{f(x)}{E[f]}
\]

\[
\tilde{f}(x) = \frac{f(x)}{\tilde{p}(x)}
\]

Variance

\[
V[f] = E[f^2] - E^2[f]
\]

\[
P[f^2] = \int \left[ \frac{f(x)}{\tilde{p}(x)} \right]^2 \tilde{p}(x) \, dx
\]

\[
= \int \left[ \frac{f(x)}{f(x) / E[f]} \right]^2 \frac{f(x)}{E[f]} \, dx
\]

\[
= E[f] \int f(x) \, dx
\]

\[
= E^2[f]
\]
Importance Sampling

Sample according to $f$

$$\tilde{p}(x) = \frac{f(x)}{E[f]}$$

$$\tilde{f}(x) = \frac{f(x)}{\tilde{p}(x)}$$

Variance

$$V[f] = E[f^2] - E^2[f]$$

Zero variance!

$$V[\tilde{f}^2] = 0$$

$$E[\tilde{f}^2] = \int \left[ \frac{f(x)}{\tilde{p}(x)} \right]^2 \tilde{p}(x) \, dx$$

$$= \int \left[ \frac{f(x)}{f(x)/E[f]} \right]^2 \frac{f(x)}{E[f]} \, dx$$

$$= E[f] \int f(x) \, dx$$

$$= E^2[f]$$
Importance Sampling

Sample according to $f$

$$\tilde{p}(x) = \frac{f(x)}{E[f]}$$

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$$= E[f] \int f(x) \, dx$$

$$= E^2[f]$$

Gotcha?
Importance Sampling

Sample distribution

\[ f(x) = e^{-5000(x - \frac{1}{2})^2} \]

MC Estimates, n=16

<table>
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<th>Method</th>
<th>Estimate</th>
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<td>Importance</td>
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\[ \int f(x) \, dx = 0.025067 \ldots \]
Importance Sampling

Sample distribution?

\[ f(x) = e^{-5000(x - \frac{1}{2})^2} \]
Importance Sampling: Area

Solid Angle
100 shadow rays

Area
100 shadow rays
**Importance Sampling**

**Uniform hemisphere sampling:**

\[
\begin{align*}
f(\omega) &= L_i(\omega) \cos \theta & \quad p(\omega) &= \frac{1}{2\pi} \\
\int_{\Omega} (f(\omega) \omega) d\omega & \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f(\omega_i)}{p(\omega_i)} \cos(2\pi \xi_2) \left( \frac{1}{N} \sum_{i} L_i(\omega_i) \cos \theta \right) \sqrt{\frac{1}{1/2\pi} \sin(2\pi \xi_2) \sum_{i} \sqrt{4_i^2 (\omega \xi_2^2) \cos \theta} \right)
\end{align*}
\]
Importance Sampling

Cosine-weighted hemisphere sampling:

\[ f(\omega) = L_i(\omega) \cos \theta \quad \quad \quad \quad p(\omega) = \frac{\cos \theta}{\pi} \]

\[ \int_{\Omega} f(\omega) \, d\omega \approx \frac{1}{N} \sum_{i}^{N} \frac{f(\omega_i)}{p(\omega_i)} = \frac{1}{N} \sum_{i}^{N} \frac{L_i(\omega_i) \cos \theta_i}{\cos \theta_i / \pi} = \frac{\pi}{N} \sum_{i}^{N} L_i(\omega_i) \]
Ambient Occlusion

\[ \int_{\Omega} V(\omega) \cos \theta \, d\omega \]

Landis, McGaugh, Koch @ ILM
\[ \int_{\Omega} V(\omega) \cos \theta \, d\omega \]
Reference: 4096 samples
Uniform, 32 samples
Variance 0.0160
Cosine, 32 samples
Variance 0.0103
Uniform, 72 samples
Variance 0.00931
Sampling a Circle

Equi-Areal

\[ \theta = 2\pi U_1 \]

\[ r = \sqrt{U_2} \]
Shirley’s Mapping: Better Strata

\[ r = U_1 \]
\[ \theta = \frac{\pi}{4} \frac{U_2}{U_1} \]
Equi-Areal
Shirley’s Mapping
Views of Sampling

1. Numerical integration
   ■ Quadrature/Integration rules
   ■ Efficient for smooth functions
   ■ “Curse of dimensionality”

2. Statistical sampling (Monte Carlo integration)
   ■ Unbiased estimate of integral
   ■ High dimensional sampling: $1/N^{1/2}$

3. Signal processing
   ■ Sampling and reconstruction
   ■ Aliasing and antialiasing
   ■ Blue noise good

4. Quasi Monte Carlo
   ■ Bound error using discrepancy
   ■ Asymptotic efficiency in high dimensions