Direct Lighting

Earlier lectures

- Reflection Models I, II
- Monte Carlo I, II, III

Today

- MC estimation of the reflection equation
- Sampling lights and BRDFs
- Improving efficiency: splitting
- Scenes with 1000s of light sources
The Reflection Equation

\[
L_o(p, \omega_o) = \int_{H^2} f_r(p, \omega_i \rightarrow \omega_o) L_i(p, \omega_i) \cos \theta_i \, d\omega_i
\]
Directional Light Sources

Incident radiance in parallel rays

\[ L_i(ω) = L \delta(ω - ω_{light}) \]

\[ L_o(p, ω_o) = \int_{H^2} f_r(p, ω_i \to ω_o) L_i(p, ω_i) \cos θ_i \, dω_i \]

\[ = \int_{H^2} f_r(p, ω_i \to ω_o) L \delta(ω - ω_{light}) \cos θ_i \, dω_i \]

\[ = f_r(p, ω_{light} \to ω_o) L V(p, ω_{light}) \cos θ_{light} \]

Binary visibility function
Specular BRDFs

\[ L_o(p, \omega_o) = \int_{H^2} f_r(p, \omega_i \rightarrow \omega_o) L_i(p, \omega_i) \cos \theta_i \, d\omega_i \]

**Perfect Specular BRDF:**

\[ f_r(\omega_i \rightarrow \omega_o) = \frac{\delta(\omega_i - R(\omega_o, \vec{n}))}{\cos \theta_i} \]

\( R(\omega_r, \vec{n}) \) is specular direction direction

\[ L_o(p, \omega_o) = L_i(p, R(\omega_o, \vec{n})) \]
Monte Carlo Estimate

\[ L_o(p, \omega_o) = \int_{H^2} f_r(p, \omega_i \rightarrow \omega_o) L_i(p, \omega_i) \cos \theta_i \, d\omega_i \]

**Sample:** \( \omega_j \sim p(\omega) \)

**Estimate:**

\[ \frac{1}{N} \sum_{j=1}^{N} \frac{f_r(p, \omega_j \rightarrow \omega_o) L_i(p, \omega_j) \cos \theta_j}{p(\omega_j)} \]
Uniform Sampling by Direction
Sampling Area Light Sources

Sample uniformly by area on the light’s surface
Convert to solid angle, compute estimator

\[ L_0(p, \omega_o) = \int_A f_r(p, \omega_i \rightarrow \omega_o) L_i(p, \omega_i) \cos \theta_i \frac{\cos \theta}{r^2} dA \]
Sampling Area Light Sources

Sample uniformly by area on the light’s surface
Convert to solid angle, compute estimator

\[ L_o(p, \omega_o) = \int_A f_r(p, \omega_i \rightarrow \omega_o) L_e V(p, p') \cos \theta_i \frac{\cos \theta}{r^2} \, dA \]
Sample uniformly by area on the light’s surface.
Convert to solid angle, compute estimator.

\[ L_o(p, \omega_o) \approx \frac{f_r(p, \omega_i \rightarrow \omega_o) L_e V(p, p') \cos \theta_i \cos \theta}{p(p') r^2} \]
Sampling a Unit Sphere

Differential measure on the sphere:
\[ d\omega = \sin \theta \, d\theta \, d\phi \]

Integrate to get normalization term:
\[ \int_{S^2} d\omega = 4\pi \]

PDF for directions:
\[ p(\omega) = \frac{1}{4\pi} \]

PDF w.r.t. \((\theta, \phi)\):
\[ p(\theta, \phi) = \frac{\sin \theta}{4\pi} \]
Sampling a Unit Sphere

PDF w.r.t. \((\theta, \phi)\):
\[
p(\theta, \phi) = \frac{\sin \theta}{4\pi} = p(\theta)p(\phi) = \frac{\sin \theta}{2} \frac{1}{2\pi}
\]

CDF for phi:
\[
P(\phi) = \int_0^\phi \frac{1}{2\pi} \, d\phi = \frac{\phi}{2\pi}
\]

Sampling phi:
\[
\xi_1 = P(\phi) = \frac{\phi}{2\pi}
\]

\[
\phi = 2\pi \xi_1
\]
Sampling a Unit Sphere

**PDF w.r.t. \((\theta, \phi)\):**
\[
p(\theta, \phi) = \frac{\sin \theta}{4\pi} = p(\theta)p(\phi) = \frac{\sin \theta}{2} \frac{1}{2\pi}
\]

**CDF for theta:**
\[
P(\theta) = \int_{0}^{\theta} \frac{\sin \theta}{2} \, d\theta = \frac{1 - \cos \theta}{2}
\]

**Sampling theta:**
\[
\xi_{2} = \frac{1 - \cos \theta}{2}
\]
\[
\cos \theta = 1 - 2\xi_{2}
\]
\[
\theta = \arccos(1 - 2\xi_{2})
\]
Sampling Spherical Lights

Recipes:  
\[ \cos \theta = 1 - 2\xi \]
\[ \theta = \arccos(1 - 2\xi) \]
\[ \phi = 2\pi\xi \]

Spherical coordinates:  
\( (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \)

Sample locations:  
\[ x = \cos(2\pi\xi_2) \sqrt{1 - z^2} \]
\[ y = \sin(2\pi\xi_2) \sqrt{1 - z^2} \]
\[ z = 1 - 2\xi_1 \]
Direct Lighting With Spherical Lights

1. Choose point randomly on sphere surface

2. PDF over area of sphere:

\[ p_A(x, y, z) = \frac{1}{4\pi r^2} \]

3. Convert area PDF to solid angle PDF:

\[ p(\omega) = \frac{1}{4\pi r^2} \frac{\cos \theta'}{|p' - p|^2} \]
4. Trace ray to see if $p$ and $p'$ are mutually visible
5. If so, exitant radiance from $p'$ gives incident radiance at $p$
6. Can now evaluate the value of the estimator:

$$L_o(p, \omega_o) \approx \frac{f_r(p, \omega_i \rightarrow \omega_o) \cdot L_i(p, \omega_i) \cdot \cos \theta_i}{p(\omega_i)}$$
Problems with sampling uniformly by area?
Sampling Spherical Lights: Directions

Problems with sampling uniformly by area?
Uniform Cone Sampling

\[ \cos \theta' = (1 - \xi) + \xi \cos \theta \quad \phi = 2\pi \xi \]

\[ p(\omega) = \frac{1}{2\pi (1 - \cos \theta)} \]
Sampling Area vs. Solid Angle

Solid angle reflection equation:

\[ L_o(p, \omega_o) = \int_{H^2} f_r(p, \omega_i \rightarrow \omega_o) \, L_i(p, \omega_i) \, \cos \theta_i \, d\omega_i \]

Area reflection equation:

\[ L_o(p, \omega_o) = \int_A f_r(p, \omega_i \rightarrow \omega_o) \, L_e V(p, p') \, \cos \theta_i \, \frac{\cos \theta}{r^2} \, dA \]
Sampling Area vs. Solid Angle

Solid angle reflection estimator:

\[ L_o(p, \omega_o) \approx \frac{f_r(p, \omega_i \rightarrow \omega_o) L_i(p, \omega_i) \cos \theta_i}{p(\omega_i)} \]

Area reflection estimator:

\[ L_o(p, \omega_o) \approx \frac{f_r(p, \omega_i \rightarrow \omega_o) L_e \cos \theta_i \cos \theta}{p(A) r^2} \]
**Sampling Area vs. Solid Angle**

**Solid angle estimator (diffuse, uniform direction)**

\[ L_o(p, \omega_o) \approx \left[ \frac{f_r L_i}{p} \right] V(p, \omega_i) \cos \theta_i \]

**Area estimator (diffuse, uniform area)**

\[ L_o(p, \omega_o) \approx \left[ \frac{f_r L_i}{p} \right] \frac{V(p, \omega_i) \cos \theta_i \cos \theta}{r^2} \]
Area vs. Solid Angle Sampling

Area

Solid Angle
(1.81x less variance)
Sampling Spherical Lights: Directions

Problems with sampling uniformly by solid angle?
Sampling Spherical Lights: Directions

Problems with sampling uniformly by solid angle?
Sampling Spherical Caps

[Peters and Dachsbacher 2019]
Solid Angle vs. Projected Solid Angle

Solid angle estimator (diffuse, uniform direction):

\[ L_o(p, \omega_o) \approx \left[ \frac{f_r L_i}{p} \right] V(p, \omega_i) \cos \theta_i \]

Projected solid angle estimator (diffuse, cosine-weighted direction):

\[ L_o(p, \omega_o) \approx \left[ \frac{f_r L_i}{p} \right] V(p, \omega_i) \]
Environment Map Light Sources
Environment Map Light Sources
Capturing Environment Light Sources
Rendering with Environment Maps

[ILM 2009]
Rendering with Environment Maps
Sampling Environment Map Lights

Luminance Map
Sampling Environment Map Lights

Sample row’s 1D PDF to choose a column

Sample 1D PDF to choose a row
Uniform Sampling by Direction
Light Source Sampling
**Diffuse BRDF + Environment Light**

\[
L_o(p, \omega_o) = f_r \int_{H^2} L_i(\omega_i) \cos \theta_i \, d\omega_i
\]

\[
= f_r \int_{H^2} L_e(\omega_i) \, V(p, \omega_i) \cos \theta_i \, d\omega_i
\]

\[
\approx f_r \frac{1}{N} \sum_j^N \frac{L_e(\omega_j) \, V(p, \omega_j) \cos \theta_j}{p(\omega_j)}
\]

**Low variance if** \( p(\omega) \propto L_e(\omega) \)
Sampling a Diffuse BRDF

Malley’s Method:
1. Generate uniform samples on the unit disk.
2. Project to hemisphere
3. Resulting distribution is cosine-weighted,

\[ p(\omega) = \frac{\cos \theta}{\pi} \]

Estimator:
\[
\frac{f_r(p, \omega_j \rightarrow \omega_r) \ L_i(p, \omega_j) \ \cos \theta_j}{p(\omega_j)} = \pi \ f_r \ L_i(p, \omega_j) = \rho \ L_i(p, \omega_j)
\]
Uniform Sampling by Direction
Light Source Sampling
Light Sampling Problem Cases

BRDF small; low contribution

PDF small; too-high contribution
BRDF Sampling Problem Cases

PDF small; surprise large contribution

Misses light; zero contribution
Multiple Importance Sampling

Use multiple sampling distributions $p_i(x)$

New Monte Carlo estimator:

$$\int f(x) \, dx \approx \frac{1}{N} \left[ \sum_{i=1}^{N_a} w_1(x_{1,i}) \frac{f(x_{1,i})}{p_1(x_{1,i})} + \sum_{i=1}^{N_b} w_2(x_{2,i}) \frac{f(x_{2,i})}{p_2(x_{2,i})} + \cdots \right]$$

$w_i(x)$ are weighting terms

$N_i$ samples taken from the i’th distribution

$$N = \sum_i N_i$$
Multiple Importance Sampling

Balance heuristic:

\[
\int f(x) \, dx \approx \frac{1}{N} \left[ \sum_{i=1}^{N_a} w_1(x_{1,i}) \frac{f(x_{1,i})}{p_1(x_{1,i})} + \sum_{i=1}^{N_b} w_2(x_{2,i}) \frac{f(x_{2,i})}{p_2(x_{2,i})} + \cdots \right]
\]

Estimator with balance heuristic:

\[
\int f(x) \, dx \approx \frac{1}{N} \left[ \sum_{i=1}^{N_a} \sum_{j} \frac{f(x_{1,i})}{N_j p_j(x)} + \sum_{i=1}^{N_b} \sum_{j} \frac{f(x_{2,i})}{N_j p_j(x)} + \cdots \right]
\]

**Samples from** \(p_1\)  

**Weighted average** of all PDFs
BSDF Sampling
Light Source Sampling
Multiple Importance Sampling
Splitting
“Splitting”

Standard estimator:

$$\int_A \int_B f(a, b) \, da \, db$$

Estimator: $\frac{1}{N} \sum_{i=1}^{N} \frac{f(a_i, b_i)}{p(a_i, b_i)}$

Alternative: multiple samples of $b$ for each $a$

New estimator: $\frac{1}{N_a} \frac{1}{N_b} \sum_{i=1}^{N_a} \sum_{j=1}^{N_b} \frac{f(a_i, b_{i,j})}{p(a_i, b_{i,j})}$
“Splitting”

Can improve efficiency if the evaluation of

\[
\frac{1}{N_a} \frac{1}{N_b} \sum_{i=1}^{N_a} \sum_{j=1}^{N_b} \frac{f(a_i, b_{i,j})}{p(a_i, b_{i,j})}
\]

can share computation over terms

\[f(a_i, b_{i,1}), f(a_i, b_{i,2}), \ldots\]
1 image sample x 64 light samples
7.0 seconds, Variance 0.000397, Efficiency 359.71
8 image samples x 8 light samples
7.7 seconds, Variance 0.000198, Efficiency 657.89
64 image samples x 1 light sample
13.2 seconds, Variance 0.000187, Efficiency 405.12
\[
\frac{1}{N_{\text{pixel}}} \sum_{i=1}^{N_{\text{pixel}}} \left[ \frac{1}{N_{\text{light}}} \sum_{j=1}^{N_{\text{light}}} \frac{f_r(p_i, \omega_o, \omega_{i,j}) L_i(p_i, \omega_{i,j}) \cos \theta_{i,j}}{p(\omega_{i,j})} \right]
\]
Multiple Light Sources

Monte Carlo estimate of a sum $\sum_{i=1}^{N} f_i$

- Define a discrete probability over terms $p_i$

$$\sum_{i} p_i = 1$$

- Draw $N_{\text{samples}}$ samples $s_j \sim p_i$

- Estimator:

$$\frac{1}{N_{\text{samples}}} \sum_{j=1}^{N_{\text{samples}}} \frac{f_{s_j}}{p_{s_j}}$$
Multiple Light Sources

Consider single sample, equal probabilities:

- Draw one sample \( s \sim p_i \)
- Compute \( f_s \)
- Estimator: \( f_s / p_s \)

Expected value:

\[
E \left[ \frac{f_s}{p_s} \right] = \sum_i p_i \frac{f_i}{p_i} = \sum_i f_i
\]
Multiple Light Source Strategies

**Uniform:** \( p_i = \frac{1}{N_{\text{lights}}} \)

**Power:** \( p_i = \frac{\Phi_i}{\sum_j \Phi_j} \)

**Spatially-varying:** \( p_i(p) = \frac{\tilde{L}_i(p)}{\sum_j \tilde{L}_j(p)} \)
Power
Spatially-Varying
Measure One (beeple@)

>8k light sources
7M Light Sources

[Disney/Pixar Coco]