Quality Improves with More Rays

Area
1 shadow ray

Area
16 shadow rays
pixelsamples = 1

jaggies
pixelsamples = 16

anti-aliased
Sampling and Reconstruction

Basic signal processing

■ Fourier transforms
■ The convolution theorem
■ The sampling theorem

Aliasing and antialiasing

■ Uniform supersampling
■ Stochastic sampling
Review: Basic Signal Processing
Sines and Cosines

\[
\cos 2\pi x
\]

\[
\sin 2\pi x
\]
Frequencies $\cos 2\pi f x$

$f = \frac{1}{T}$

$f = 1$

$\cos 2\pi x$

$f = 2$

$\cos 4\pi x$
Spatial Domain | Frequency Domain
\[
\sin\left(\frac{2\pi}{32}\right) x
\]

32 pixels per cycle

Spatial Domain

Frequency Domain
\[ \sin\left(\frac{2\pi}{16}\right)y \]

16 pixels per cycle

Spatial Domain

Frequency Domain
\[
\sin\left(\frac{2\pi}{16}\right)y
\]

16 pixels per cycle

Spatial Domain

Frequency Domain
\[ \sin\left(\frac{2\pi}{32}\right)x \times \sin\left(\frac{2\pi}{16}\right)y \]
$e^{-r^2/16^2}$

Spatial Domain  
Frequency Domain
\[ e^{-\frac{x^2}{32^2}} \times e^{-\frac{y^2}{16^2}} \]
Rotate 45  \[ e^{-\frac{x^2}{32^2}} \times e^{-\frac{y^2}{16^2}} \]
Fourier Transforms

The Fourier transform converts between the spatial and frequency domain

\[ F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx \quad \leftrightarrow \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} \, d\omega \]

Figures generated using fft2d.py
Pat’s Frequencies

Spatial Domain

Frequency Domain
Filtering: Low Pass Filter

Spatial Domain

Frequency Domain
Filtering: Low Pass Filter

Keep low frequencies

Spatial Domain  Frequency Domain
Filtering: Band Pass Filter

Keep band of frequencies

Spatial Domain

Frequency Domain
Filtering: Band Pass Filter

Keep band of frequencies

Spatial Domain

Frequency Domain
Filtering: High Pass Filter

Keep high frequencies

Spatial Domain

Frequency Domain
Filtering by Convolution

\[ h(x) = f \otimes g = \int f(x')g(x - x') \, dx' \]
Convolution Theorem

Convolution Theorem: Multiplication in the frequency domain is equivalent to convolution in the space domain.

\[ f \otimes g \leftrightarrow F \times G \]

Symmetric Theorem: Multiplication in the space domain is equivalent to convolution in the frequency domain.

\[ f \times g \leftrightarrow F \otimes G \]
Spatial and Frequency Domain

Spatial Domain

Frequency Domain
Math: Box and Sinc Functions

\[ \Pi_T(x) = \begin{cases} 
1 & |x| \leq \frac{T}{2} \\
0 & |x| > \frac{T}{2} 
\end{cases} \]

\[ \text{sinc}(x) \equiv \frac{\sin \pi x}{\pi x} \]
The Sampling Theorem
Simple Function

\[ f(x) \quad F(\omega) \]
Sampling: Multiply in Spatial Domain

\[ f(x) \times \]
The Fourier transform of a sequence of spikes is a sequence of spikes

\[ III_1/T(\omega) = \sum_{n=-\infty}^{n=\infty} \delta(\omega - n/T) \]
Sampling: Convolve in Freq Domain

\[ F(\omega) \]

\[ \otimes \]

\[ = \]

\[
\begin{align*}
\end{align*}
\]
Reconstruction: Frequency Domain

\[ \times \]

\[ = \]
Recall

\[ \Pi_T(x) = \begin{cases} 
1 & |x| \leq \frac{T}{2} \\
0 & |x| > \frac{T}{2}
\end{cases} \]

\[ \text{sinc}(x) \equiv \frac{\sin \pi x}{\pi x} \]
Reconstruction: Spatial Domain

\[
\begin{align*}
\bullet & \quad \bullet & \quad \bullet \\
\times & \quad \bullet & \quad \bullet \\
\bullet & \quad \bullet & \quad \bullet \\
\end{align*}
\]

\[
= 
\]

\[
\begin{align*}
\text{Graph 1} & \quad \text{Graph 2}
\end{align*}
\]
Recovering a Sampled Signal
Sampling Theorem

This result is known as the Sampling Theorem, and Claude Shannon is credited with discovering it in 1949.

A signal can be reconstructed from its samples without loss of information, if the original signal has no frequencies above 1/2 the sampling frequency.

For a given band-limited function, the rate it must be sampled is called the Nyquist Frequency.
Aliasing
Aliasing
Undersampling: Overlaps/Aliases

\[ \otimes = \]
Sampling a "Zone Plate"

Zone plate: \( \sin x^2 + y^2 \)

Left rings: signal
Right rings: aliasing
Antialiasing
Preventing Aliasing
Antialiasing by Prefiltering

\[
\begin{align*}
\text{Frequency Space}
\end{align*}
\]
Uniform Supersampling

Take $n$ samples

Average them together (filter = weighted average)

$$\text{Pixel} = \sum_s w_s \cdot \text{Sample}_s$$
Point vs. Supersampled

Point

4x4 Uniform

Fewer aliases, but still present
Area vs. Supersampled

Exact Area
Fewer aliases, but still present

4x4 Uniform
Uniform Supersampling

Increasing the number of samples moves each copy of the spectra further apart, thus there is less overlap

This reduces, but does not eliminate, aliasing

\[ \text{Pixel} = \sum_s w_s \cdot \text{Sample}_s \]
Add uniform random jitter to each sample
Distribution of Extrafoveal Cones

Monkey eye cone distribution

Fourier transform

Yellot
Jittered vs. Uniform Supersampling

4x4 Jittered Sampling

4x4 Uniform
**Theory: Analysis of Jitter**

**Non-uniform sampling**

\[
s(x) = \sum_{n=-\infty}^{n=\infty} \delta (x - x_n)
\]

\[
x_n = nT + j_n
\]

**Jittered sampling**

\[
j_n \sim j(x)
\]

\[
j(x) = \begin{cases} 
1 & |x| \leq 1/2 \\
0 & |x| > 1/2 
\end{cases}
\]

\[
J(\omega) = \text{sinc} \omega
\]

\[
S(\omega) = \frac{1}{T} \left[ 1 - |J(\omega)|^2 \right] + \frac{2\pi}{T^2} |J(\omega)|^2 \sum_{n=-\infty}^{n=\infty} \delta (\omega - \frac{2\pi n}{T})
\]

\[
= \frac{1}{T} \left[ 1 - \text{sinc}^2 \omega \right] + \delta (\omega)
\]
Jittered Sampling
Poisson Disk Sampling
Integration Error when Sampling

Integral

\[ I(f) = \int f(x) \, dx = F(0) \]

Sampled integral

\[ I_s(f) = \int f(x) s(x) \, dx = \frac{1}{N} \sum f(x_i) \]

\[ f(x) s(x) = F(\omega) \oplus S(\omega) \]

\[ I_s(f) = \int f(x) s(x) \, dx = F(\omega) \oplus S(\omega) \mid_{\omega=0} \]

Error

\[ \Delta = F(0) - F(\omega) \oplus S(\omega) \mid_{\omega=0} \]
Non-uniform Sampling

Uniform sampling

- The spectrum of uniformly spaced samples is also a set of uniformly spaced spikes.
- Multiplying the signal by the sampling pattern corresponds to placing a copy of the spectrum at each spike (in freq. space).
- Aliases are coherent, and very noticeable.

Non-uniform sampling

- Samples at non-uniform locations have a different spectrum; a single spike plus noise.
- Sampling a signal in this way converts aliases into broadband noise.
- Noise is incoherent, and much less objectionable.
- May cause error in the integral.