CS 5643 Homework 1

Problem 1:

$\boxed{\text{Constraint}}$: $C = (Z(x,y) - z)$

$N_c = 1$

$\boxed{\text{Jacobian}}$: $J = \frac{\partial c}{\partial p} = \left(\frac{\partial Z}{\partial x}, \frac{\partial Z}{\partial y}, -1\right)$ Dimension: 1×3.

$$\dot{J} = \frac{dJ}{dt} = \left(\frac{\partial^2 Z}{\partial x^2}\dot{x} + \frac{\partial^2 Z}{\partial x \partial y}\dot{y}, \frac{\partial^2 Z}{\partial x \partial y}\dot{x} + \frac{\partial^2 Z}{\partial y^2}\dot{y}, 0\right)$$

Mass = $M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$

$$\Rightarrow JM^{-1}J^T = \frac{1}{m}\left[\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 + 1\right]$$

$\boxed{\text{Constraint Force}}$:

$$f_c = -J^T(JM^{-1}J^T)^{-1}(\dot{J}v + JM^{-1}f)$$

$$= -\begin{pmatrix} \frac{\partial Z}{\partial x} \\ \frac{\partial Z}{\partial y} \\ -1 \end{pmatrix} \cdot \frac{m}{\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 + 1}\left(\frac{\partial^2 Z}{\partial x^2}\dot{x}^2 + \frac{\partial^2 Z}{\partial y^2}\dot{y}^2 + 2\frac{\partial^2 Z}{\partial x \partial y}\dot{x}\dot{y} + J\cdot\vec{g}\right)$$

$$= -\frac{m}{\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 + 1}\left(\frac{\partial^2 Z}{\partial x^2}v_x^2 + \frac{\partial^2 Z}{\partial y^2}v_y^2 + 2\frac{\partial^2 Z}{\partial x \partial y}v_x v_y + \frac{\partial Z}{\partial x}g_x + \frac{\partial Z}{\partial y}g_y - g_z\right)\begin{pmatrix} \frac{\partial Z}{\partial x} \\ \frac{\partial Z}{\partial y} \\ -1 \end{pmatrix}$$

Problem 2:

$\boxed{\text{Constraints}}$: $C = \begin{pmatrix} \|p_1 - p_2\|^2 - l^2 \\ \|p_2 - p_3\|^2 - l^2 \\ \vdots \\ \|p_{n-1} - p_n\|^2 - l^2 \end{pmatrix}$

$N_c = n - 1$

$\boxed{\text{Jacobian}}$

$$J = \frac{\partial C}{\partial p} = 2\cdot\begin{bmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 & x_2-x_1 & y_2-y_1 & z_2-z_1 & 0 & 0 & 0 & \cdots & & \\ 0 & 0 & 0 & x_2-x_3 & y_2-y_3 & z_2-z_3 & x_3-x_2 & y_3-y_2 & z_3-z_2 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & & 0 & x_{n-1}-x_n & y_{n-1}-y_n & z_{n-1}-z_n & x_n-x_{n-1} & y_n-y_{n-1} & z_n-z_{n-1} & \end{bmatrix}$$

$$= 2\begin{bmatrix} \vec{p}_1^T-\vec{p}_2^T & \vec{p}_2^T-\vec{p}_1^T & 0 & \cdots & 0 & 0 & 0 \\ 0 & \vec{p}_2^T-\vec{p}_3^T & \vec{p}_3^T-\vec{p}_2^T & 0 & \cdots & & \vdots \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & & & & \vec{p}_{n-1}^T-\vec{p}_n^T & \vec{p}_n^T-\vec{p}_{n-1}^T \end{bmatrix}$$

Dimension $= (n-1)\times 3n.$

$$\dot{J} = 2\begin{bmatrix} \dot{x}_1-\dot{x}_2 & \dot{y}_1-\dot{y}_2 & \dot{z}_1-\dot{z}_2 & \dot{x}_2-\dot{x}_1 & \dot{y}_2-\dot{y}_1 & \dot{z}_2-\dot{z}_1 & 0 & \cdots \\ 0 & & \ddots & & & & & \\ \vdots & & & & \ddots & & & \\ 0 & \cdots & 0 & \dot{x}_{n-1}-\dot{x}_n & \dot{y}_{n-1}-\dot{y}_n & \dot{z}_{n-1}-\dot{z}_n & \dot{x}_n-\dot{x}_{n-1} & \dot{y}_n-\dot{y}_{n-1} & \dot{z}_n-\dot{z}_{n-1} \end{bmatrix}$$

$$= 2\begin{bmatrix} \vec{v}_1^T-\vec{v}_2^T & \vec{v}_2^T-\vec{v}_1^T & 0 & \cdots & 0 \\ 0 & \vec{v}_2^T-\vec{v}_3^T & \vec{v}_3^T-\vec{v}_2^T & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & \vec{v}_{n-1}^T-\vec{v}_n^T & \vec{v}_n^T-\vec{v}_{n-1}^T \end{bmatrix}$$

Mass $=\; M = m\cdot I_{3n\times 3n}. \Rightarrow M^{-1} = \frac{1}{m} I_{3n\times 3n}.$

$$JM^{-1}J^T = \frac{4}{m}\begin{bmatrix} \vec{p}_1^T-\vec{p}_2^T & \vec{p}_2^T-\vec{p}_1^T & & & \\ & \vec{p}_2^T-\vec{p}_3^T & \vec{p}_3^T-\vec{p}_2^T & & \\ & & \ddots & & \\ & & & \vec{p}_{n-1}^T-\vec{p}_n^T & \vec{p}_n^T-\vec{p}_{n-1}^T \end{bmatrix}_{(n-1)\times 3n}\begin{bmatrix} \vec{p}_1-\vec{p}_2 & & & \\ \vec{p}_2-\vec{p}_1 & \vec{p}_2-\vec{p}_3 & & \\ & \vec{p}_3-\vec{p}_2 & & \\ & & \ddots & \\ & & & \vec{p}_{n-1}-\vec{p}_n \\ & & & \vec{p}_n-\vec{p}_{n-1} \end{bmatrix}_{3n\times(n-1)}.$$

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$$JM^{-1}J^T = \frac{4}{m}\begin{bmatrix} (\vec{p}_1^T-\vec{p}_2^T)(\vec{p}_1-\vec{p}_2)+(\vec{p}_2^T-\vec{p}_1^T)(\vec{p}_2-\vec{p}_1) & (\vec{p}_2^T-\vec{p}_1^T)(\vec{p}_2-\vec{p}_3) & 0 & \cdots & \\ (\vec{p}_2^T-\vec{p}_3^T)(\vec{p}_2-\vec{p}_1) & (\vec{p}_2^T-\vec{p}_3^T)(\vec{p}_2-\vec{p}_3)+(\vec{p}_3^T-\vec{p}_2^T)(\vec{p}_3-\vec{p}_2) & (\vec{p}_3^T-\vec{p}_2^T)(\vec{p}_3-\vec{p}_4) & 0 & \cdots \\ 0 & & \ddots & & \\ \vdots & & & & \\ 0 & \cdots & & (\vec{p}_{n-1}^T-\vec{p}_n^T)(\vec{p}_{n-1}-\vec{p}_{n-2}) & (\vec{p}_{n-1}^T-\vec{p}_n^T)(\vec{p}_{n-1}-\vec{p}_n)+(\vec{p}_n^T-\vec{p}_{n-1}^T)(\vec{p}_n-\vec{p}_{n-1}) \end{bmatrix}$$

$$= \frac{4}{m}\begin{bmatrix} 2l^2 & (\vec{p}_2^T-\vec{p}_1^T)(\vec{p}_2-\vec{p}_3) & 0 & \cdots & 0 \\ (\vec{p}_2^T-\vec{p}_3^T)(\vec{p}_2-\vec{p}_1) & 2l^2 & (\vec{p}_3^T-\vec{p}_2^T)(\vec{p}_3-\vec{p}_4) & 0 \cdots & 0 \\ 0 & & \ddots & & \vdots \\ \vdots & & & & 0 \\ & & & & (\vec{p}_{n-1}^T-\vec{p}_{n-2}^T)(\vec{p}_{n-1}-\vec{p}_n) \\ 0 & \cdots & & (\vec{p}_{n-1}^T-\vec{p}_n^T)(\vec{p}_{n-1}-\vec{p}_{n-2}) & 2l^2 \end{bmatrix}_{(n-1)\times(n-1)}$$

$$= \frac{4l^2}{m}\begin{bmatrix} 2 & \cos\theta_2 & 0 & 0 \\ \cos\theta_2 & 2 & \cos\theta_3 & \vdots \\ 0 & \cos\theta_3 & 2 & 0 \\ \vdots & & \ddots & \cos\theta_{n-1} \\ 0 & \cdots & \cos\theta_{n-1} & 2 \end{bmatrix}_{(n-1)\times(n-1)}$$

where $\theta_i = \angle P_{i-1} P_i P_{i+1}$

$$\dot{J}\vec{v} = 2\begin{bmatrix} \vec{v}_1^T-\vec{v}_2^T & \vec{v}_2^T-\vec{v}_1^T & 0 & \cdots & 0 \\ 0 & \vec{v}_2^T-\vec{v}_3^T & \vec{v}_3^T-\vec{v}_2^T & \cdots & \vdots \\ & & & & \vdots \\ 0 & \cdots & & \vec{v}_{n-1}^T-\vec{v}_n^T & \vec{v}_n^T-\vec{v}_{n-1}^T \end{bmatrix}\begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{pmatrix}$$

$$= 2\begin{pmatrix} (\vec{v}_1^T-\vec{v}_2^T)(\vec{v}_1-\vec{v}_2) \\ (\vec{v}_2^T-\vec{v}_3^T)(\vec{v}_2-\vec{v}_3) \\ \vdots \\ (\vec{v}_{n-1}^T-\vec{v}_n^T)(\vec{v}_{n-1}-\vec{v}_n) \end{pmatrix} = 2\begin{pmatrix} \|\vec{v}_1-\vec{v}_2\|^2 \\ \|\vec{v}_2-\vec{v}_3\|^2 \\ \vdots \\ \|\vec{v}_{n-1}-\vec{v}_n\|^2 \end{pmatrix}_{(n-1)\times 1}$$

$$JM^{-1}f = 2\begin{bmatrix} \vec{p}_1^T-\vec{p}_2^T & \vec{p}_2^T-\vec{p}_1^T & 0 & \cdots & & 0 \\ 0 & \vec{p}_2^T-\vec{p}_3^T & \vec{p}_3^T-\vec{p}_2^T & 0 & \cdots & \vdots \\ & & & & & 0 \\ 0 & \cdots & & 0 & \vec{p}_{n-1}^T-\vec{p}_n^T & \vec{p}_n^T-\vec{p}_{n-1}^T \end{bmatrix}_{(n-1)\times 3n} \begin{pmatrix} \vec{g} \\ \vec{g} \\ \vdots \\ \vec{g} \end{pmatrix}_{3n\times 1}$$

$$= 0$$

$\Rightarrow$ $\boxed{\text{Constraint Force}}$ =

$$f_c = -J^T(JM^{-1}J^T)^{-1}(\dot{J}\vec{v} + JM^{-1}f)$$

$$= -\frac{m}{l^2}\begin{bmatrix} \vec{p}_1-\vec{p}_2 & & & \\ \vec{p}_2-\vec{p}_1 & \vec{p}_2-\vec{p}_3 & & \\ & \vec{p}_3-\vec{p}_2 & \vec{p}_3-\vec{p}_4 & \\ & & \ddots & \vec{p}_{n-1}-\vec{p}_n \\ & & & \vec{p}_n-\vec{p}_{n-1} \end{bmatrix}_{3n\times(n-1)} \begin{bmatrix} 2 & \cos\theta_2 & & \\ \cos\theta_2 & 2 & \cos\theta_3 & \\ & \cos\theta_3 & 2 & \ddots \\ & & \ddots & \cos\theta_{n-1} \\ & & \cos\theta_{n-1} & 2 \end{bmatrix}^{-1}_{(n-1)\times(n-1)} \begin{pmatrix} \|\vec{v}_1-\vec{v}_2\|^2 \\ \|\vec{v}_2-\vec{v}_3\|^2 \\ \vdots \\ \|\vec{v}_{n-1}-\vec{v}_n\|^2 \end{pmatrix}_{(n-1)\times 1}$$

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Problem 3.

$\boxed{\text{Constraints}}$: $$C = \begin{pmatrix} Z(x_1, y_1) - z_1 \\ \dfrac{\|\vec{p}_2 - \vec{p}_1\|^2 - l^2}{2} \\ \dfrac{\|\vec{p}_3 - \vec{p}_2\|^2 - l^2}{2} \end{pmatrix}$$

$$N_C = 3$$

$\boxed{\text{Jacobian}}$:

$$J = \frac{\partial C}{\partial p} = \begin{bmatrix} \frac{\partial Z}{\partial x_1} & \frac{\partial Z}{\partial y_1} & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_1 - x_2 & y_1 - y_2 & z_1 - z_2 & x_2 - x_1 & y_2 - y_1 & z_2 - z_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 - x_3 & y_2 - y_3 & z_2 - z_3 & x_3 - x_2 & y_3 - y_2 & z_3 - z_2 \end{bmatrix}$$

Dimension: 3×9

$$\dot{J} = \begin{bmatrix} \frac{\partial^2 Z}{\partial x^2}\dot{x} + \frac{\partial^2 Z}{\partial x \partial y}\dot{y} & \frac{\partial^2 Z}{\partial x \partial y}\dot{x} + \frac{\partial^2 Z}{\partial y^2}\dot{y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \dot{x}_1 - \dot{x}_2 & \dot{y}_1 - \dot{y}_2 & \dot{z}_1 - \dot{z}_2 & \dot{x}_2 - \dot{x}_1 & \dot{y}_2 - \dot{y}_1 & \dot{z}_2 - \dot{z}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dot{x}_2 - \dot{x}_3 & \dot{y}_2 - \dot{y}_3 & \dot{z}_2 - \dot{z}_3 & \dot{x}_3 - \dot{x}_2 & \dot{y}_3 - \dot{y}_2 & \dot{z}_3 - \dot{z}_2 \end{bmatrix}$$

Mass: $M = m I_{9\times9} \Rightarrow M^{-1} = \frac{1}{m} I_{9\times9}$

$$JM^{-1}J^T = \frac{1}{m}\begin{bmatrix} \left(\frac{\partial Z}{\partial x_1}\right)^2 + \left(\frac{\partial Z}{\partial y_1}\right)^2 + 1 & (x_1 - x_2)\frac{\partial Z}{\partial x_1} + (y_1 - y_2)\frac{\partial Z}{\partial y_1} - (z_1 - z_2) & 0 \\ (x_1 - x_2)\frac{\partial Z}{\partial x_1} + (y_1 - y_2)\frac{\partial Z}{\partial y_1} - (z_1 - z_2) & 2\|\vec{p}_1 - \vec{p}_2\|^2 & (\vec{p}_2^T - \vec{p}_1^T)(\vec{p}_2 - \vec{p}_3) \\ 0 & (\vec{p}_2^T - \vec{p}_1^T)(\vec{p}_2 - \vec{p}_3) & 2\|\vec{p}_3 - \vec{p}_2\|^2 \end{bmatrix}$$

$$JM^{-1}J^T = \frac{1}{m}\begin{bmatrix} \left(\frac{\partial z}{\partial x_1}\right)^2+\left(\frac{\partial z}{\partial y_1}\right)^2+1 & (x_1-x_2)\frac{\partial z}{\partial x_1}+(y_1-y_2)\frac{\partial z}{\partial y_1}-(z_1-z_2) & 0 \\ (x_1-x_2)\frac{\partial z}{\partial x_1}+(y_1-y_2)\frac{\partial z}{\partial y_1}+(z_1-z_2) & 2l^2 & l^2\cos\theta \\ 0 & l^2\cos\theta & 2l^2 \end{bmatrix}$$

where $\theta = \angle P_1P_2P_3$.

$$\dot{J}v = \begin{pmatrix} \frac{\partial^2 z}{\partial x_1^2}\dot{x}_1^2 + \frac{\partial^2 z}{\partial y_1^2}\dot{y}_1^2 + 2\frac{\partial^2 z}{\partial x_1 \partial y_1}\dot{x}_1\dot{y}_1 \\ \|\vec{v}_1-\vec{v}_2\|^2 \\ \|\vec{v}_2-\vec{v}_3\|^2 \end{pmatrix}$$

$$JM^{-1}f = \begin{pmatrix} \frac{\partial z}{\partial x_1} & \frac{\partial z}{\partial y_1} & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_1-x_2 & y_1-y_2 & z_1-z_2 & x_2-x_1 & y_2-y_1 & z_2-z_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2-x_3 & y_2-y_3 & z_2-z_3 & x_3-x_2 & y_3-y_2 & z_3-z_2 \end{pmatrix}\begin{pmatrix} g_x \\ g_y \\ g_z \\ g_x \\ g_y \\ g_z \\ g_x \\ g_y \\ g_z \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial z}{\partial x_1}g_x + \frac{\partial z}{\partial y_1}g_y - g_z \\ 0 \\ 0 \end{pmatrix}$$

$\boxed{\text{Constraint Force}}$:

$$f_c = -J^T(JM^{-1}J^T)^{-1}(\dot{J}v + JM^{-1}f)$$

$$= -m\begin{bmatrix} \begin{matrix}\frac{\partial z}{\partial x_1} \\ \frac{\partial z}{\partial y_1} \\ -1\end{matrix} & \vec{p}_1-\vec{p}_2 & \vec{0}_{3\times1} \\ \vec{0}_{3\times1} & \vec{p}_2-\vec{p}_1 & \vec{p}_2-\vec{p}_3 \\ \vec{0}_{3\times1} & \vec{0}_{3\times1} & \vec{p}_3-\vec{p}_2 \end{bmatrix}_{9\times3} \begin{bmatrix} \left(\frac{\partial z}{\partial x_1}\right)^2+\left(\frac{\partial z}{\partial y_1}\right)^2+1 & (x_1-x_2)\frac{\partial z}{\partial x_1}+(y_1-y_2)\frac{\partial z}{\partial y_1}+(z_1-z_2) & 0 \\ (x_1-x_2)\frac{\partial z}{\partial x_1}+(y_1-y_2)\frac{\partial z}{\partial y_1}-(z_1-z_2) & 2l^2 & l^2\cos\theta \\ 0 & l^2\cos\theta & 2l^2 \end{bmatrix}^{-1}_{3\times3} \cdot$$

$$\begin{pmatrix} \frac{\partial z}{\partial x_1}g_x + \frac{\partial z}{\partial y_1}g_y - g_z + \frac{\partial^2 z}{\partial x_1^2}\dot{x}_1^2 + \frac{\partial^2 z}{\partial y_1^2}\dot{y}_1^2 + \frac{\partial^2 z}{\partial x_1\partial y_1}\dot{x}_1\dot{y}_1 \\ \|\vec{v}_1-\vec{v}_2\|^2 \\ \|\vec{v}_2-\vec{v}_3\|^2 \end{pmatrix}_{3\times1}$$