Particle systems with holonomic constraints

Due: Thurs Oct 13, 2016 (at start of class)

The following problems consider a 3D particle system (of \(n\) particles of mass \(m\)) subject to gravitational acceleration \(g\), and specified holonomic constraints of the form \(C(p) = 0\). In each case, derive specific mathematical expressions for the following:

1. the constraint(s), \(C\), and their total number, \(N_c\);
2. the constraint Jacobian, \(J\), its derivative \(\dot{J}\), and the matrix dimensions; and
3. an analytical expression for the constraint force \(f_c\) expressed as the product of three quantities:

\[
f_c = J^\top \left( JWJ^\top \right)^{-1} \left( -\dot{J}v - JWf \right) \lambda
\]

where \(W = M^{-1}\), and \(f\) is the external force due to gravity. Be as explicit as possible when deriving the two quantities in ( ) brackets, but do not multiply the full expression, or calculate the matrix inverse.

Note: there are different values depending on how you formulate your constraints, but the same physical \(f_c\).

**PROBLEM 1. Particle on a height field:** Consider a particle with position \(p = (x, y, z)^\top\), and a surface specified by a heightfield function, \(z = Z(x, y)\). The particle is constrained to be attached to the heightfield surface at all times.

**PROBLEM 2. Inextensible n-particle chain:** Consider \(n\) particles \(p_1, p_2, \ldots, p_n\) with massless rods of length \(\ell\) attached between each consecutive particle, i.e., the \(n - 1\) line segments \(\overrightarrow{p_1p_2}, \ldots, \overrightarrow{p_{n-1}p_n}\). (Assume that \(n \geq 2\).)

**PROBLEM 3. Double pendulum on a height field:** In this problem, you will combine the constraints from the previous two problems for the case of \(n = 3\) particles with positions \(p_1, p_2, p_3\) and the following constraints:

- particle \(p_1\) is constrained to the height field from question 1, and
- there are rigid link constraints (of length \(\ell\)) on the line segments \(\overrightarrow{p_1p_2}\) and \(\overrightarrow{p_2p_3}\).