Delaunay Triangulation

Steve Oudot

slides courtesy of O. Devillers
Outline

1. Definition and Examples
2. Applications
3. Basic properties
4. Construction
Definition
Classical example

looking for nearest neighbor
Classical example

looking for nearest neighbor
Classical example

looking for nearest neighbor
Classical example

looking for nearest neighbor
\[ V_i = \{ q; \forall j \neq i \left| q_i \right| \leq \left| q_j \right| \} \]

Voronoi

Classical example

Georgy F. Voronoi
(1868–1908)
Voronoi

Delaunay

Classical example

Boris N. Delaunay
(1890–1980)
Voronoi ↔ geometry Delaunay ↔ topology

Boris N. Delaunay (1890–1980)
faces of the Voronoi diagram
faces of the Voronoi diagram
faces of the Voronoi diagram
faces of the Voronoi diagram
Voronoi is everywhere
THE Delaunay property
Voronoi

Empty sphere
Voronoi
Delaunay
Empty sphere
Several applications
nearest neighbor graph
nearest neighbor graph

$q$ nearest neighbor of $p$

$\Rightarrow pq$ Delaunay edge
nearest neighbor graph
$k$ nearest neighbors

$k - 1$ nearest neighbors

$k^{th}$ nearest neighbor

query point
$k$ nearest neighbors

$k - 1$ nearest neighbors

query point

$k^{th}$ nearest neighbor
$k$ nearest neighbors

$k - 1$ nearest neighbors

$k$th nearest neighbor

query point
Largest empty circle
Largest empty circle
MST
Other applications

Reconstruction
Other applications

Reconstruction

Meshing
Other applications

Reconstruction

Meshing / Remeshing
Other applications

Reconstruction

Meshing / Remeshing
Other applications

Reconstruction

Meshing  /  Remeshing

Path planning
Other applications

Reconstruction

Meshing / Remeshing

Path planning and others...
Main properties of Delaunay
point / sphere duality

\[ p^* = (x_p, y_p, x_p^2 + y_p^2) \]

\[ P : x^2 + y^2 = z \]

\[ p = (x_p, y_p) \]
point / sphere duality

\[ C^* : z - 2ax - 2by + c = 0 \]

\[ C : x^2 + y^2 - 2ax - 2by + c = 0 \]
point / sphere duality

\[ p^* \in C^* \]

\[ p \in \partial C \]
point / sphere duality

$p^* \text{ below } C^*$

$p \in C$

\( \mathbb{R}^2 \)
point / sphere duality

$p^* \text{ above } C^*$

$p \notin C$

$p \notin C$
point / sphere duality

orientation predicate

in-sphere predicate
point / sphere duality

Delaunay is a triangulation
Euler formula

\[ f: \text{number of facets (except } \infty) \]
\[ e: \text{number of edges} \]
\[ v: \text{number of vertices} \]

\[ f - e + v = 1 \]
Euler formula

\[ f - e + v = 1 \]

- \( f \): number of facets (except \( \infty \))
- \( e \): number of edges
- \( v \): number of vertices

\[ 1 - 3 + 3 = 1 \]
Euler formula

\[ f: \text{number of facets (except } \infty \text{)} \]
\[ e: \text{number of edges} \]
\[ v: \text{number of vertices} \]

\[ f - e + v = 1 \]

\[ +1 - 2 + 1 = +0 \]
number of oriented edges
in a triangulation: $2e = 3f + k$

$k$: size of $\infty$ facet
Euler formula
\[ f - e + v = 1 \]

Triangulation
\[ 2e = 3f + k \]

\[ f = 2v - 2 - k = O(v) \]
\[ e = 3v - 3 - k = O(v) \]
Delaunay maximizes the smallest angle
Delaunay maximizes the sequence of angles in lexicographical order
Local optimality vs global optimality

locally Delaunay... but not globally Delaunay
Theorem

Locally Delaunay everywhere

\iff \iff

Globally Delaunay
Proof:

Let $t_0$ be locally Delaunay, but not globally Delaunay

Let $v \in \text{circle}(t) \ (v \notin t)$
Proof:
Let $t_0$ be locally Delaunay, but not globally Delaunay
Let $v \in \text{circle}(t) \ (v \notin t)$
Proof:

Let $t_0$ be locally Delaunay, but not globally Delaunay

Let $v \in \text{circle}(t)$ ($v \notin t$)
Proof:

Let $t_0$ be locally Delaunay, but not globally Delaunay

Let $v \in \text{circle}(t)$ ($v \notin t$)
Proof:
Let $t_0$ be locally Delaunay, but not globally Delaunay
Let $v \in \text{circle}(t) \ (v \notin t)$
Proof:

Let \( t_0 \) be locally Delaunay, but not globally Delaunay. Let \( v \in \text{circle}(t) \) (\( v \notin t \)).

Since \( \exists \) finitely many triangles, at some point \( v \) is a vertex of \( t_i \).
Local optimality and smallest angle

Case of 4 points

Lemma:
For any 4 points in convex position,
Delaunay $\iff$ smallest angle maximized
Local optimality and smallest angle
Case of 4 points

Let $\delta$ be the smallest angle
Local optimality and smallest angle
Case of 4 points

Let $\delta$ be the smallest angle $\leq \delta$ iff $r \notin \text{circle}(pq)$.
Local optimality and smallest angle

Theorem
Delaunay $\iff$ maximum smallest angle

Proof:
Local optimality and smallest angle

Theorem
Delaunay $\iff$ maximum smallest angle

Proof:
$T$ triangulation w/ max. smallest angle
Local optimality and smallest angle

Theorem
Delaunay $\iff$ maximum smallest angle

Proof:

$T$ triangulation w/ max. smallest angle

$\implies$ max. in each quadrilateral
Local optimality and smallest angle

Theorem
Delaunay $\iff$ maximum smallest angle

Proof:

$T$ triangulation w/ max. smallest angle

$\implies$ max. in each quadrilateral

$\implies$ locally Delaunay
Local optimality and smallest angle

Theorem
Delaunay $\iff$ maximum smallest angle

Proof:
$T$ triangulation w/ max. smallest angle
$\implies$ max. in each quadrilateral
$\implies$ locally Delaunay
$\implies$ globally Delaunay
Computing Delaunay

Lower bound
Lower bound for Delaunay

Delaunay can be used to sort numbers
Lower bound for Delaunay

Delaunay can be used to sort numbers

Take an instance of sort
Assume one can compute Delaunay in $\mathbb{R}^2$
Use Delaunay to solve this instance of sort
Let $x_1, x_2, \ldots, x_n \in \mathbb{R}$, to be sorted.
Let $x_1, x_2, \ldots, x_n \in \mathbb{R}$, to be sorted

$$(x_1, x_1^2), \ldots, (x_n, x_n^2) \quad n \text{ points}$$
Let $x_1, x_2, \ldots, x_n \in \mathbb{R}$, to be sorted

$$(x_1, x_1^2), \ldots, (x_n, x_n^2) \quad n \text{ points}$$

Delaunay

$\rightarrow$ order in $x$
Let $x_1, x_2, \ldots, x_n \in \mathbb{R}$, to be sorted

$(x_1, x_1^2), \ldots, (x_n, x_n^2)$  $n$ points

Delaunay

$O(n)$

$f(n)$

$O(n)$

$\rightarrow$ order in $x$

$O(n) + f(n) \in \Omega(n \log n)$
Lower bound for Delaunay

$\Omega(n \log n)$
Computing Delaunay
Incremental algorithm
(SHORT OVERVIEW)
Find triangles in conflict
Delete triangles in conflict
Triangulate hole
That’s all for today