Delaunay Triangulation

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slides courtesy of O. Devillers
MST
MST
→ use Kruskal's algorithm with Del as input → $O(n \log n)$
Last: lower bound for Delaunay

Let $x_1, x_2, \ldots, x_n \in \mathbb{R}$, to be sorted
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$(x_1, x_1^2), \ldots, (x_n, x_n^2)$  $n$ points

Last: lower bound for Delaunay
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Let \( x_1, x_2, \ldots, x_n \in \mathbb{R} \), to be sorted

\[
(x_1, x_1^2), \ldots, (x_n, x_n^2)
\]

\( n \) points

Delaunay

\[ \rightarrow \text{ order in } x \]
Let $x_1, x_2, \ldots, x_n \in \mathbb{R}$, to be sorted 

$(x_1, x_1^2), \ldots, (x_n, x_n^2)$ \hspace{1cm} \text{n points}

\text{Delaunay} \hspace{1cm} \downarrow \hspace{1cm} f(n)

\downarrow \hspace{1cm} O(n)

\rightarrow \text{order in } x

O(n) + f(n) \in \Omega(n \log n)
Last: lower bound for Delaunay

$\Omega(n \log n)$
Optimal algorithm for computing Delaunay Division – Fusion

Division-Fusion

Classical approach  example: sort

Problem of size $n$

→ division into 2 pbs of size $O(\frac{n}{2})$

→ recursive call on sub-problems

→ fusion
Division-Fusion

Classical approach  
example: sort

Problem of size $n$

→ division into 2 pbs of size $O\left(\frac{n}{2}\right)$

$O(n)$

→ recursive call on sub-problems

$2 f \left(\frac{n}{2}\right)$

→ fusion

$O(n)$
Division-Fusion

Classical approach example: sort

Problem of size \( n \)

\[
f(n) = O(n) + 2f\left(\frac{n}{2}\right)
\]

\[
= O(n \log n)
\]

\( \rightarrow \) division into 2 pbs of size \( O\left(\frac{n}{2}\right) \)

\( O(n) \)

\( \rightarrow \) recursive call on sub-problems

\( 2f\left(\frac{n}{2}\right) \)

\( \rightarrow \) fusion

\( O(n) \)
Division
Division
Division
Fusion
Division
Division Fusion
Fusion
Division

Diagram showing the process of division and fusion.
Division
Division
Division
Division
Fusion
Division
Fusion
Division
Division

Sort in $x$
Division

Sort in $x$

store all the medians
Division

Sort in $\mathcal{O}(n \log n)$

$\rightarrow$ store all the medians

query in $\mathcal{O}(1)$
Fusion

Monochromatic triangles to be deleted
Fusion

Bi-chromatic triangles to be constructed
Constructing bi-chromatic edges from top to bottom next edge?
Constructing bi-chromatic edges from top to bottom

rising bubble: set of circumscribed circles

next edge?
Constructing bi-chromatic edges from top to bottom

rising bubble: set of circumscribed circles

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Constructing bi-chromatic edges from top to bottom

rising bubble: set of circumscribed circles

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rising bubble: set of circumscribed circles
Constructing bi-chromatic edges from top to bottom

next edge?
first blue vertex crossed by set of circles
Only one is Delaunay
first red vertex crossed by set of circles

\[ r_2 \in \text{circle}(b, r, r_1) \]

Always look at first neighbor of \( r \) ccw
first red vertex crossed by set of circles

\[ r_3 \in \text{circle}(b, r, r_2) \]

Always look at first neighbor of \( r \) ccw
$r_4 \in \text{circle}(b, r, r_3)$

Always look at first neighbor of $r$ ccw
first red vertex crossed by set of circles

Always look at first neighbor of \( r \) ccw

\[ r_5 \notin \text{circle}(b, r, r_4) \]
first red vertex crossed by set of circles

\[ r_5 \notin \text{circle}(b, r, r_4) \]

\[ \forall \text{red}, \text{red} \notin \text{circle}(b, r, r_4) \]
first blue vertex crossed by set of circles

\[ b_2 \in \text{circle}(b, r, b_1) \]

Always look at first neighbor of \( b \) cw
first blue vertex crossed by set of circles

\[ b_3 \notin \text{circle}(b, r, b_2) \]
first blue vertex crossed by set of circles

\[ b_3 \notin \text{circle}(b, r, b_2) \]

\[ \forall \text{blue, blue } \notin \text{circle}(b, r, b_2) \]
no point

$p$

$b$

$b_{next}$

$r_{next}$
no red

\( p \)

\( b \)

\( b_{next} \)

\( r_{next} \)
Complexity of Fusion
Complexity of Fusion

At each step of the search for $r_{next}$
Complexity of Fusion

At each step of the search for $r_{next}$

A red edge is deleted
Complexity of Fusion

At each step of the search for \( r_{next} \)

A red edge is deleted

At each step of the search for \( b_{next} \)
Complexity of Fusion

At each step of the search for $r_{next}$

A red edge is deleted

At each step of the search for $b_{next}$

A blue edge is deleted
Complexity of Fusion

At each step of the search for $r_{next}$

A red edge is deleted

At each step of the search for $b_{next}$

A blue edge is deleted

After the choice between $r_{next}$ and $b_{next}$
Complexity of Fusion

At each step of the search for $r_{next}$

A red edge is deleted

At each step of the search for $b_{next}$

A blue edge is deleted

After the choice between $r_{next}$ and $b_{next}$

A black edge is created
Complexity of Fusion

\[ \text{Complexity} \leq \# \text{ red edges} + \# \text{ blue edges} + \# \text{ black edges} \]
Complexity of Fusion

Complexity $\leq \# \text{ red edges} + \# \text{ blue edges} + \# \text{ black edges}$

$\leq 3 \frac{n}{2} + 3 \frac{n}{2} + 3n = O(n)$
Overall Complexity

Division = $O(n)$ for any sub-problem

$+ O(n \log n)$ (preprocessing)

Fusion = $O(n)$ on sub-pb of size $n$

Division-Fusion $\Rightarrow O(n \log n)$
Generalizations
Voronoi diagram
Voronoï diagram

$Q$ Nearest neighbor of $q$ among $S$
Voronoi diagram

Q Nearest neighbor of $q$ among $S$

Change
Voronoi diagram

\[ \text{Q} \quad \text{Nearest neighbor of } q \text{ among } S \]

Change

ambient space (for \( q \))

\[ \mathbb{R}^2 \quad \mathbb{R}^3 \quad \mathbb{R}^d \]
Voronoi diagram

Q Nearest neighbor of $q$ among $S$

Change metrics

Euclidean $L_2$
$L_1, L_\infty, L_p$
hyperbolic

additive weights
multiplicative weights
Voronoi diagram

Nearest neighbor of $q$ among $S$

Change

universal set $\supset S$

points of $\mathbb{R}^d$ segments of $\mathbb{R}^d$
spheres of $\mathbb{R}^d$
Points in $\mathbb{R}^3$
Points in $\mathbb{R}^3$
query
Points in $\mathbb{R}^3$

query
Points in $\mathbb{R}^3$

query
Points in $\mathbb{R}^3$

query
Points in $\mathbb{IR}^3$

query
Points in $\mathbb{R}^3$ query

Nearest neighbor

Voronoi diagram
Points in $\mathbb{R}^3$

Voronoi vertex

Voronoi diagram
Points in $\mathbb{R}^3$

Voronoi vertex

empty sphere

Voronoi diagram
Points in $\mathbb{R}^3$

- empty sphere
- tetrahedron
- Delaunay triangulation
Delaunay in 3D

Similar to 2D:
- Delaunay test (empty sphere)
- Incremental algorithm
- Randomized algorithm
- Duality with convex hull in 4D
Delaunay in 3D

Similar to 2D:
different from 2D

variable size (linear — quadratic)
Quadratic example
Quadratic example

empty sphere
Quadratic example

$\Omega(n^2)$
Size of Delaunay

Θ(n^2) worst case

Θ(n) uniformly-sampled points in unit cube

O(n \log n) uniformly-sampled points on a surface
Exotic metrics
Norm $L_\infty$: $\max(|x|, |y|)$
Norm $L_\infty$: $\max(|x|, |y|)$
Norm $L_\infty$: $\max(|x|, |y|)$

query
Norm $L_\infty$: $\max(|x|, |y|)$

query
Norm $L_\infty$: $\max(|x|, |y|)$

query
Norm $L_\infty$: $\max(|x|, |y|)$

query
Norm $L_\infty$: $\max(|x|, |y|)$

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Norm $L_{\infty}$: $\max(|x|, |y|)$
Norm $L_\infty$: $\max(|x|, |y|)$
multiplicatively-weighted points
multiplicatively-weighted points

1

2
multiplicatively-weighted points
multiplicatively-weighted points
multiplicatively-weighted points
multiplicatively-weighted points
multiplicatively-weighted points
multiplicatively-weighted points

circular bisector
multiplicatively-weighted points

circular bisector
multiplicatively-weighted points
multiplicatively-weighted points
multiplicatively-weighted points

disconnected cell
multiplicatively-weighted points
multiplicatively-weighted points
multiplicatively-weighted points

quadratic size
Voronoi diagram of segments
Voronoi diagram of segments

nearest segment
Voronoi diagram of segments

parabolic bisector
Voronoi diagram of segments

angle bisector
Voronoi diagram of segments

points bisector
Voronoi diagram of segments

Voronoi diagram
Voronoi diagram of segments
Voronoi diagram of segments
Voronoi diagram of segments

Dual complex
That’s all for today