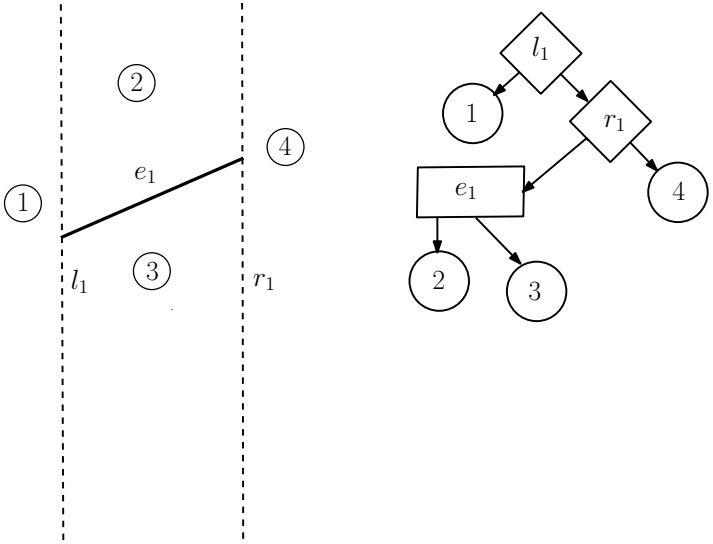
Separating Chains Method

CS368, 5/8/2006

Last Time

- **Arrangements of Line Segments**
  - Special case: **simple polygons**
- Computing the **trapezoidal decomposition**

- **Incremental algorithm**
  - **Decomposition** DS
  - **Point location** DS



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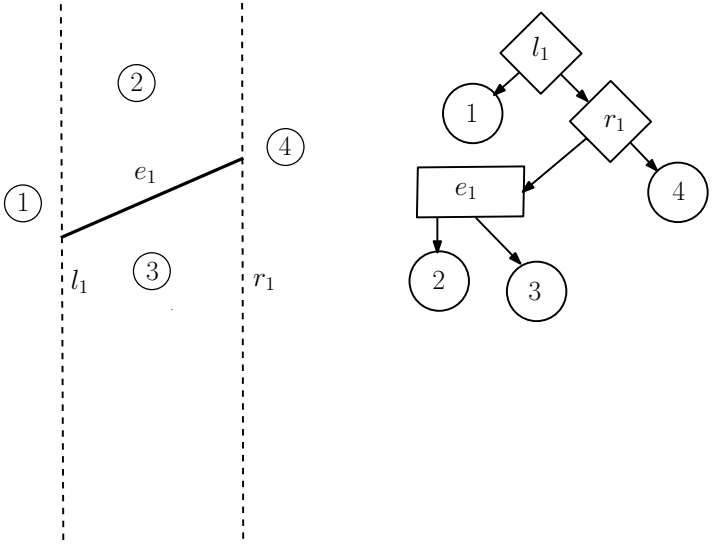
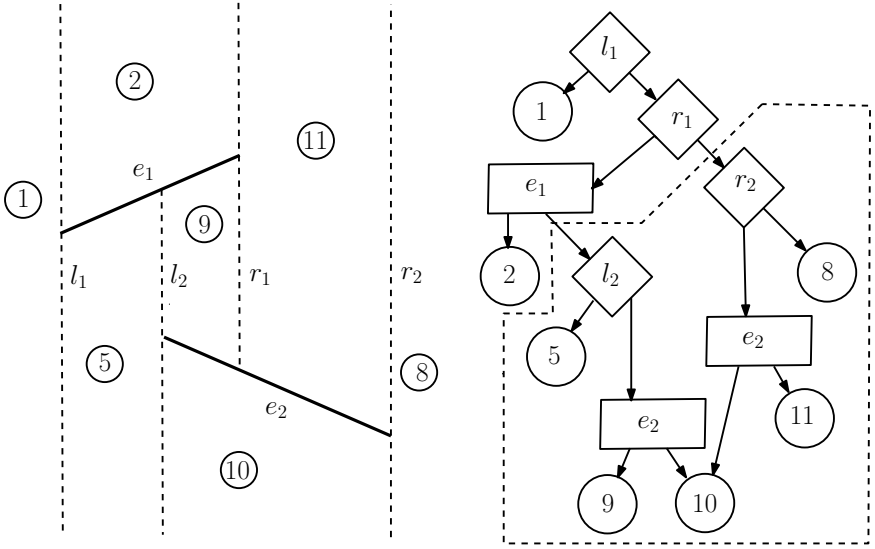


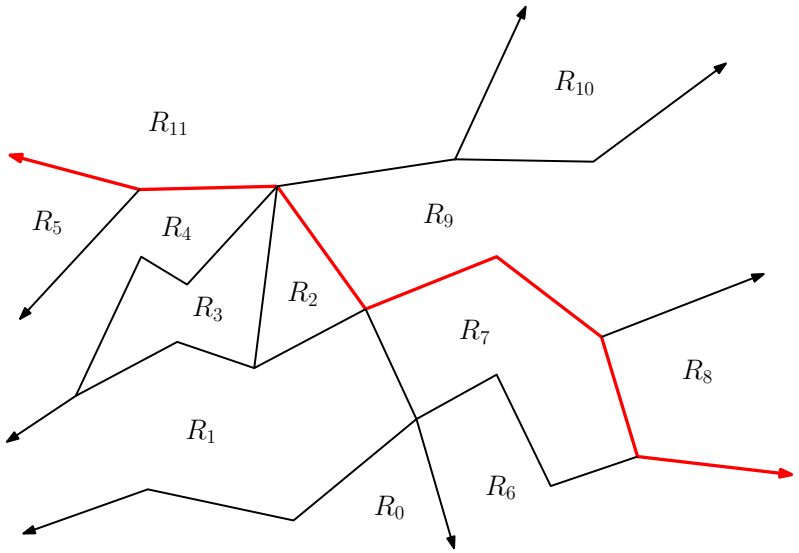
- **Update**
  - **Point location** step
  - **Tracing** step



- Solution specific to trapezoidal decompositions

Today

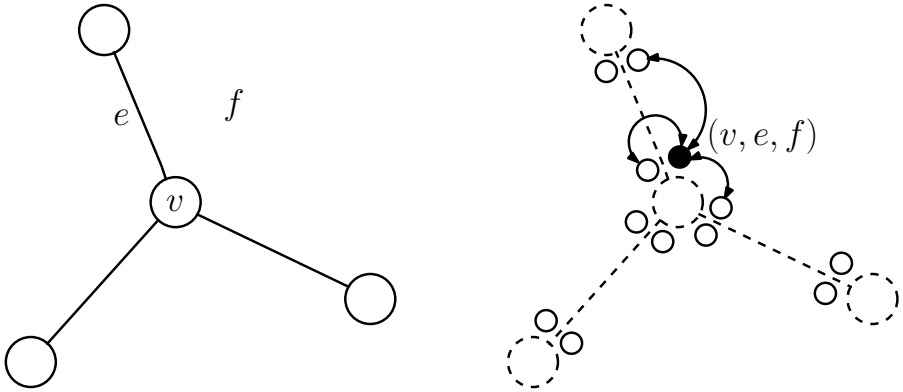
- **Point location in polygonal subdivisions in 2D**



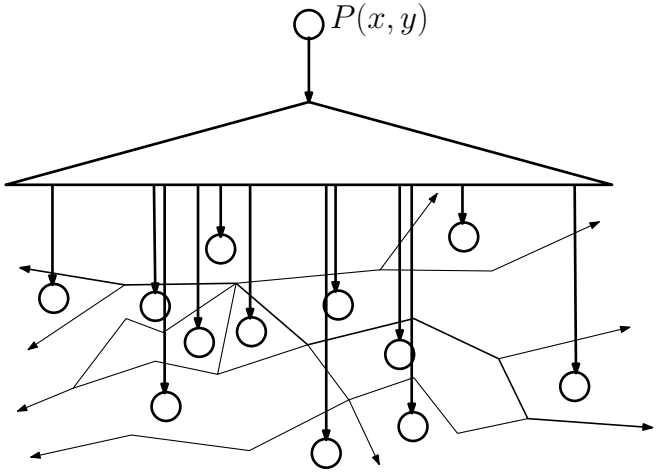
- Important examples
  - **Triangulations**
  - **Line arrangements**
  - **Voronoi diagrams**
- Based on the notion of **separators** (**separating chains**)
- References
  [Lee, Preparata 1977] [Edelsbrunner, Guibas, Stolfi 1986]
- Several **simple** algorithmic ideas that come together nicely

Point Location in 2D Polygonal Subdivisions

- **Input**: A partition of the Euclidean plane $\mathbb{E}^2$ into polygonal regions
  - For example, given as a **vertex-edge-face** data structure



- **Output**: **Point location data structure** and **algorithm**
  - Preprocess to build an **index** structure
  - Support queries: given a point, **find the element** that contains it



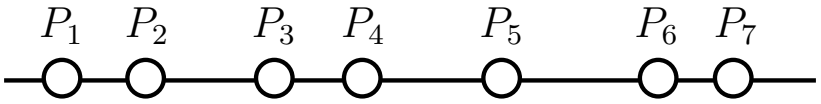
Measures of Performance

- **Input size**
  - Number of edges m
  - Number of faces (regions) n
  - Euler: $n = O(m)$
- **Space,** $S(m)$
- **Preprocessing time,** $P(m)$
- **Query time,** $Q(m)$
- **Separating chains method**

$$S(m) = O(m) \qquad P(m) = O(m) \qquad Q(m) = O(\log m)$$

Separating Chains Method

- **1D intuition**
  - Subdivision of a **line**
  - Partitioned by **points**



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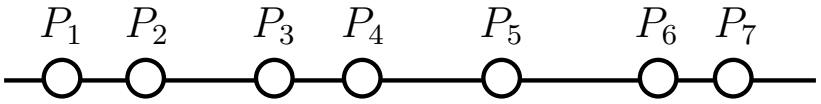

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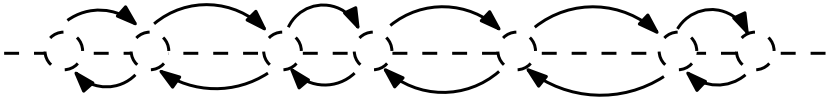
- **Output**: A query DS and algorithm that reports an element of the subdivision that contains a given point



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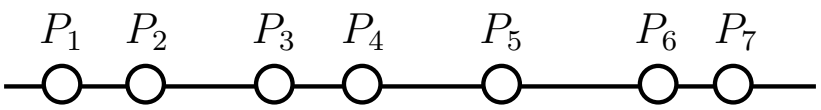

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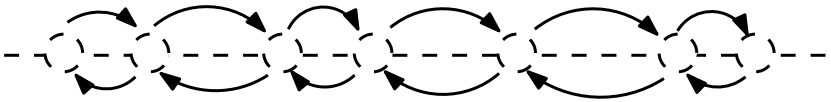
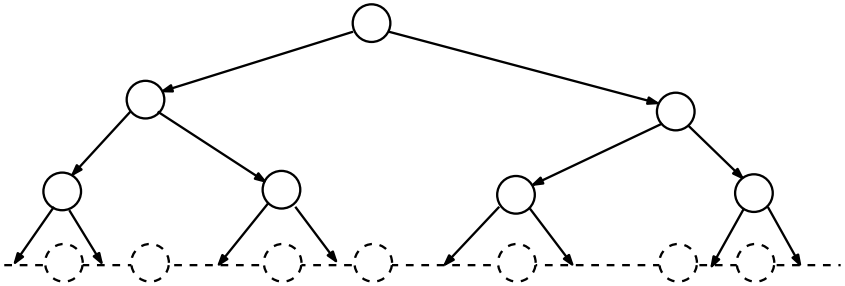


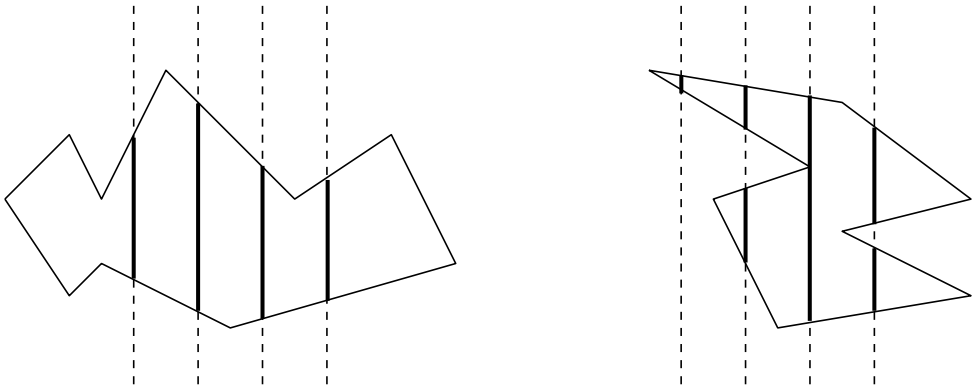
- **Solution**: Binary search tree



$$S(m) = O(m) \qquad P(m) = O(m) \qquad Q(m) = O(\log m)$$

Monotone Polygons

- **Def**: A polygon is **monotone** if it is intersected by any vertical line either along **exactly one line segment** or **not at all**

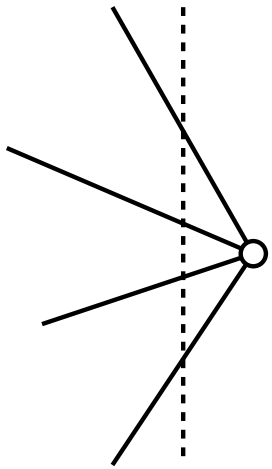


- **Def**: A polygonal subdivision is **monotone** if all its polygons are monotone
- Assume no vertical edges

- Any polygon boundary is **properly partitioned** into a **"floor"** and a **"ceiling"**
  - Starting and ending points of intersections with vertical lines

Regularization

- **Observation**: A subdivision is monotone iff **no vertex has all adjacent edges on the same side** of the vertical line through it



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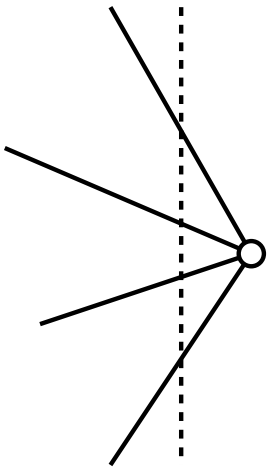
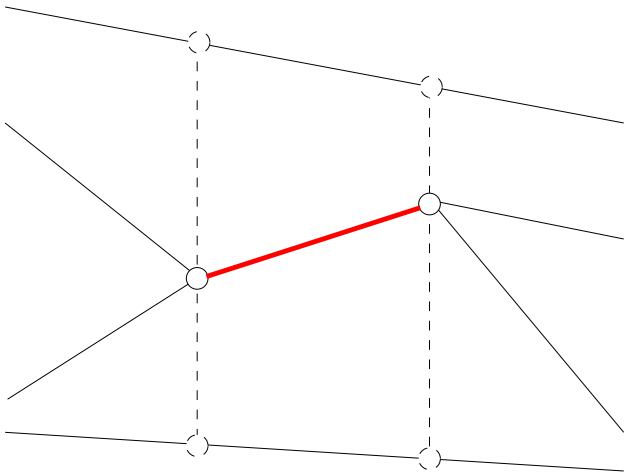




- **Triangulation**
- **Trapezoidal decomposition**

[Chazelle]	$O(m)$ worst case
[Seidel]	$O(m \log^* m)$ expected
[Bentley, Ottman] + diagonals	$O(m \log m)$ worst case

- **Note**: effects preprocessing time!

Properties of Monotone Subdivisions

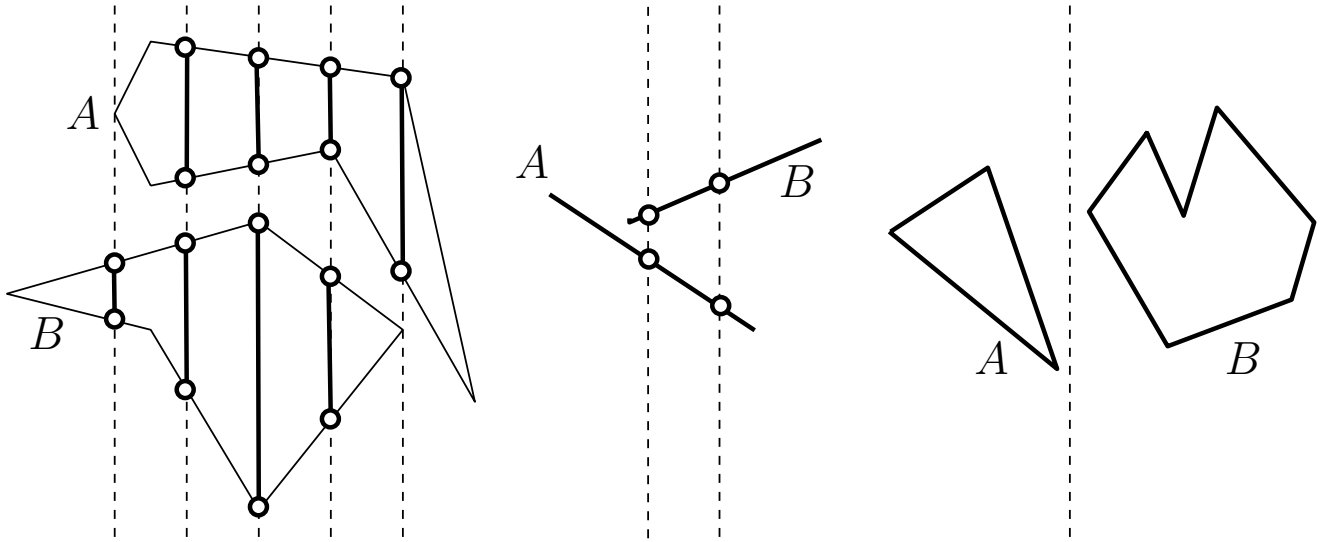
- **Def**: For all $A, B \subseteq \mathbb{E}^2$, we say that

  $$A \textbf{ is above } B \textbf{ } (A \gg B)$$

  or

  $$B \textbf{ is below } A \textbf{ } (B \ll A)$$

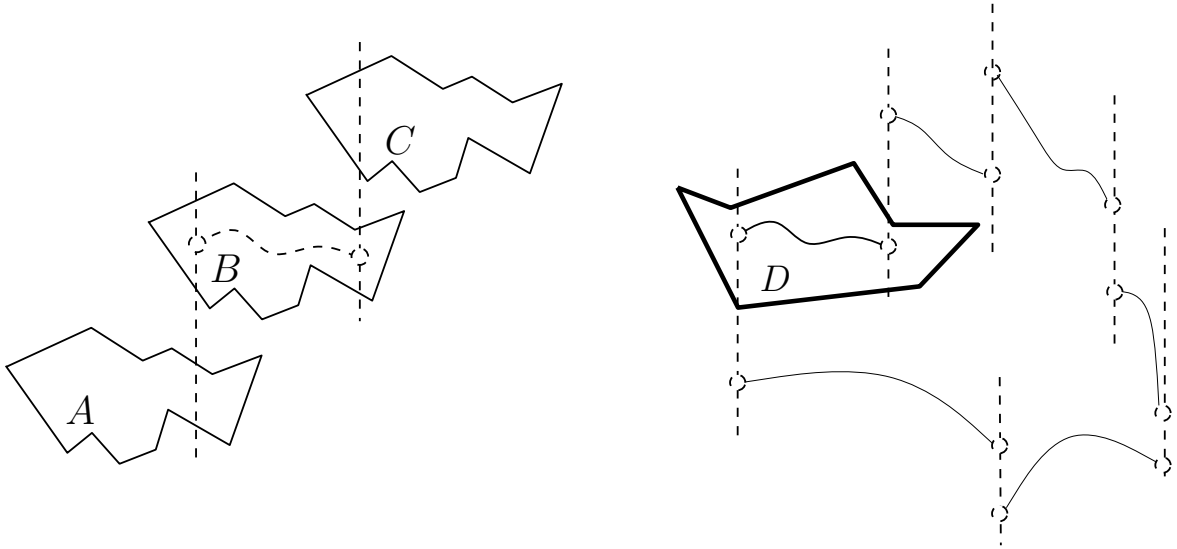
  if any vertical line intersects A strictly above B (if the intersections are not empty)



- Not every two elements are comparable

Properties of Monotone Subdivisions

- **Lemma**: The relation is **acyclic**
  - The relation graph is a **DAG**
  - Extend by reflexivity
  - It is a **partial order**
- **Proof**: Consider the shortest cycle, prove that it can be shortened



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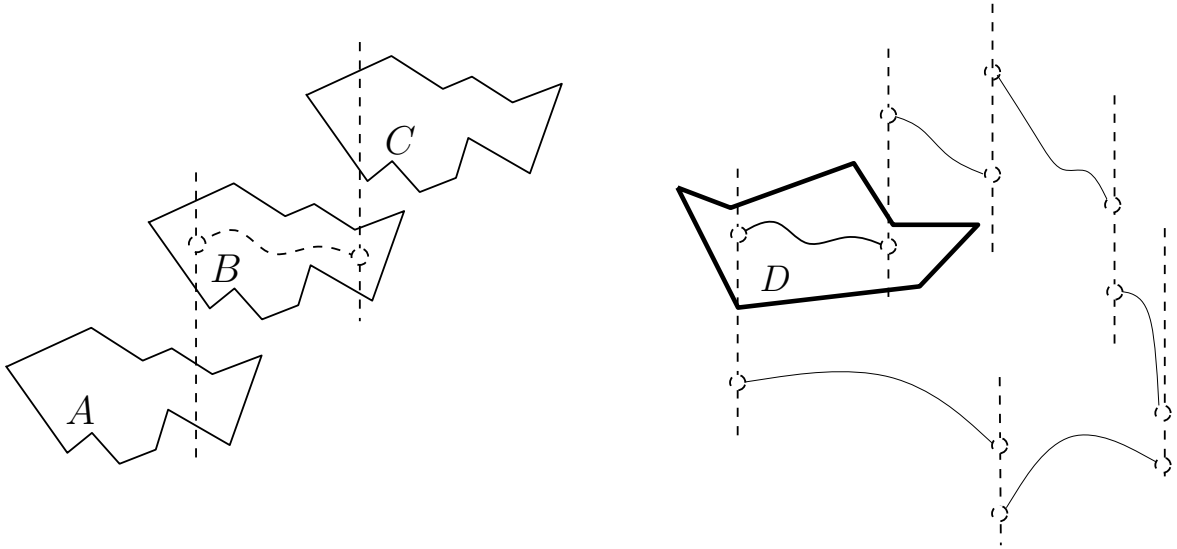
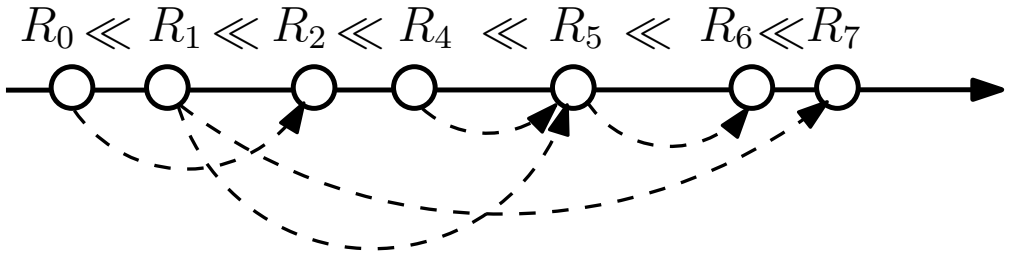


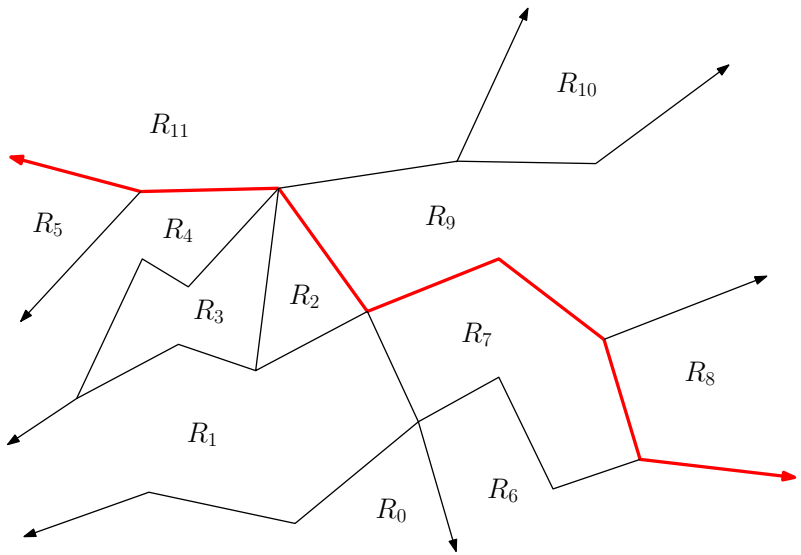
- **Consequence**: There exists **total ordering**
  - **Consistent** with the partial order
  - **Topological sorting** of the relation DAG



Separators

- **Def**: A **separator** is a polygonal line which
  - consists of the edges of the subdivision
  - intersects any vertical line in **exactly one point**
- Separator is a "horizontal" line
- **Def**: A **complete family of separators** is a collection of $n-1$ distinct separators

$$S_1 \ll S_2 \ll \cdots \ll S_{n-1}$$



Existence of Separators

- **Theorem**: Every monotone subdivision admits a complete family of separators

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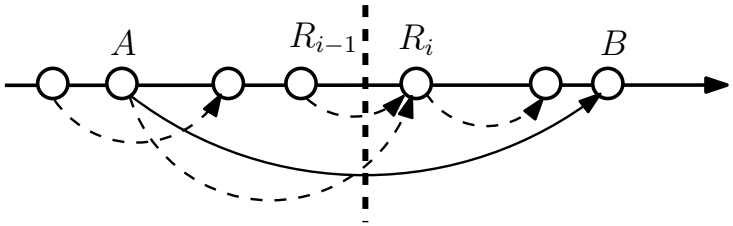
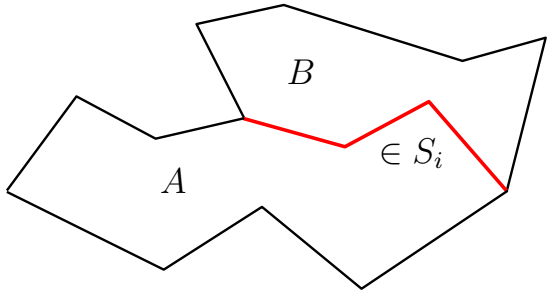
- **Proof**: Explicit construction
- Linear ordering of the faces $R_0 \ll R_2 \cdots \ll R_{n-1}$





- Separator S_i $(1 \leq i \leq n-1)$ consists of all segments on the **common boundary of any two faces** A, B **such that**

$$A \ll R_{i-1} \ll R_i \ll B$$

Existence of Separators

- Verifying the definition for arbitrary S_i: need to prove
  - Consists of **polygon edges**
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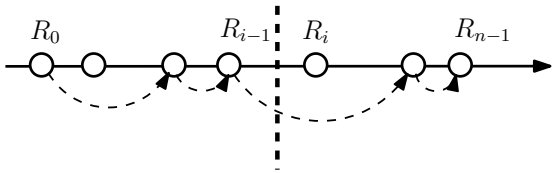
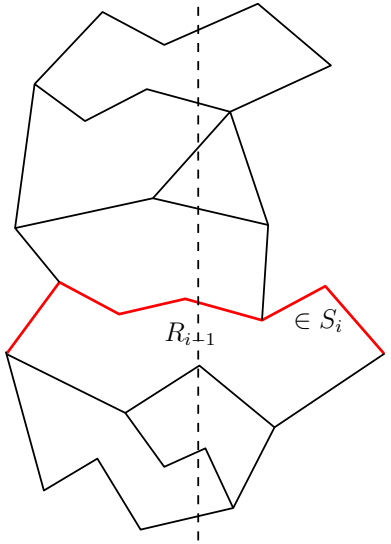
  - Consider the sequence of intersected faces





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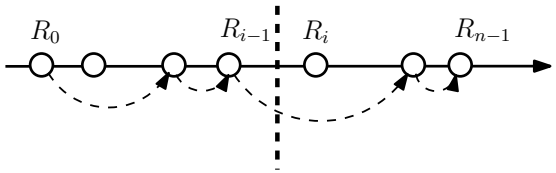

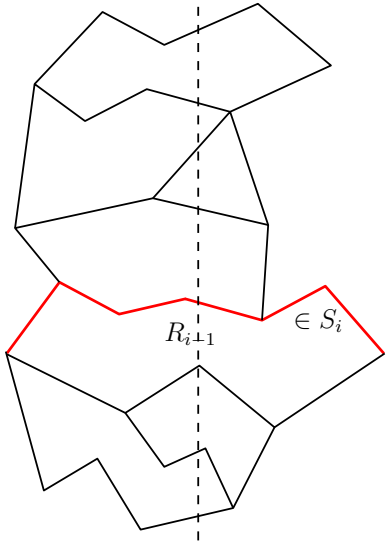

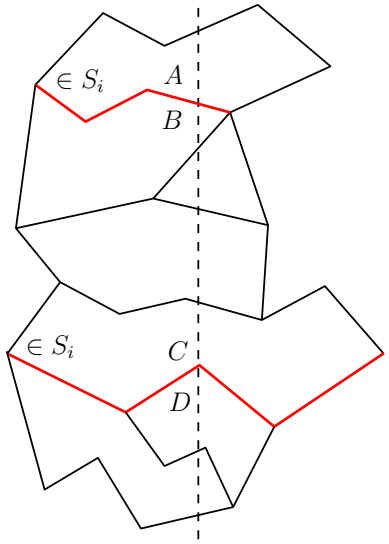

- **At most one** intersection
  - **Contradiction**: $R_i \ll C \ll B \ll R_{i-1}$
- Also implies that S_i is a **polygonal line**

Existence of a Separator System

- Verifying the definition for $S_1, S_2, \ldots, S_{n-1}$. Need to prove
  - Ordering $S_1 \ll S_2 \ll \cdots \ll S_{n-1}$
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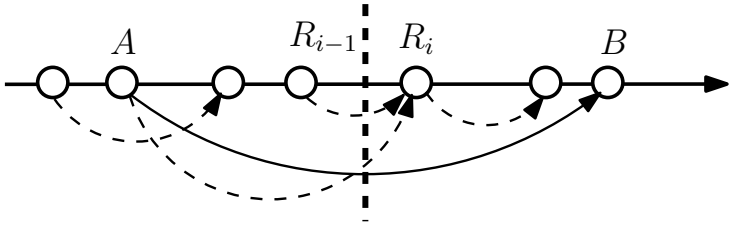
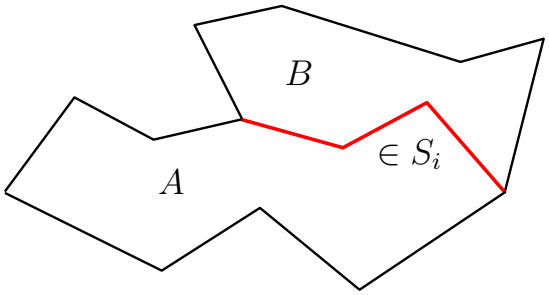
  - By construction, all segments in S_i are $\ll R_i$ and $\gg R_{i-1}$





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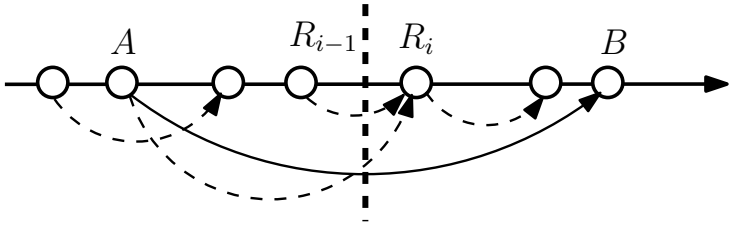
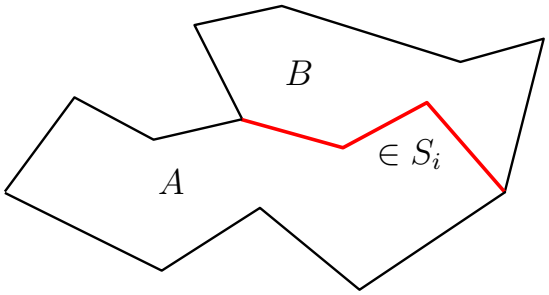
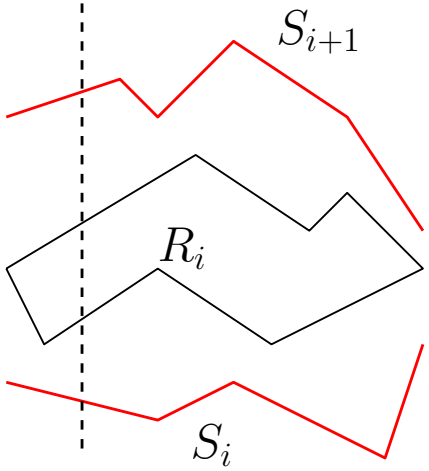




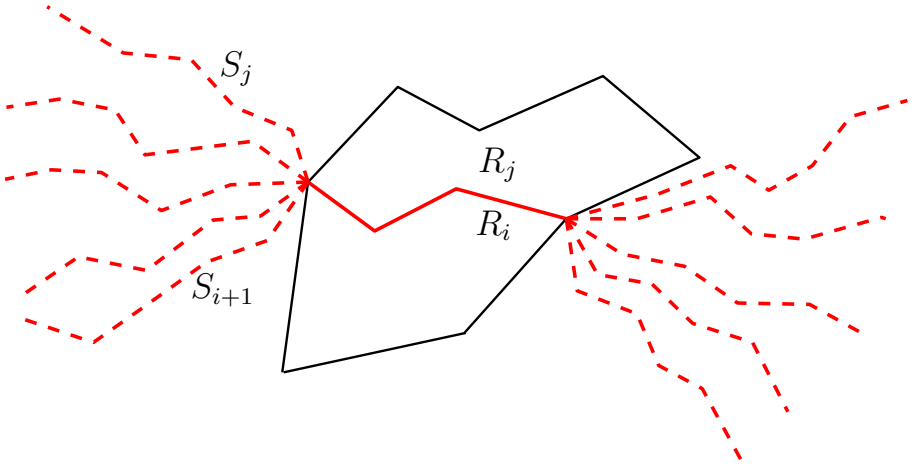
- Sufficient: no two **consecutive** S_i and S_{i+1} are equal



Properties of Separators

- **Consequence**:
  $R_0 \ll S_1 \ll R_1 \ll S_2 \ll \cdots \ll R_{n-2} \ll S_{n-1} \ll R_{n-1}$

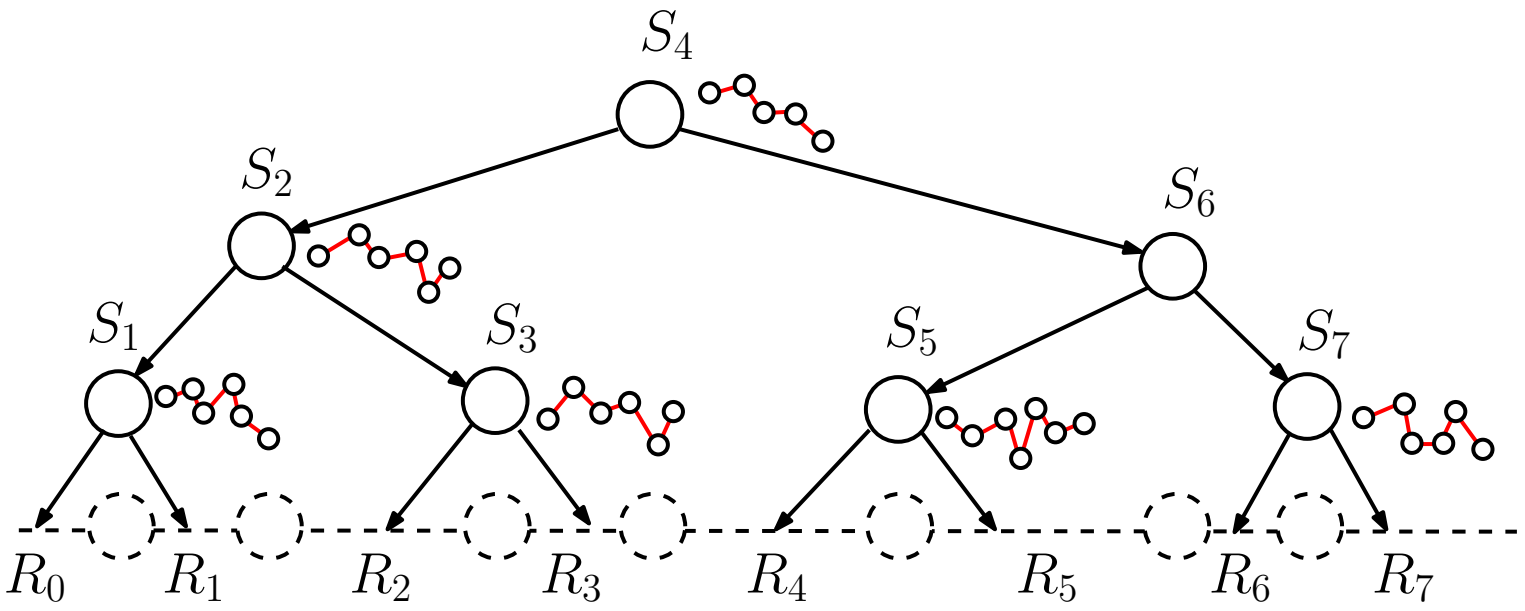
- **Consequence**: If e is in the common boundary of R_i and R_j, the separators that contain it are **exactly** $S_i, S_i, \ldots, S_{j-1}$



- Computing separators?

Point Location Data Structure

- **Binary search tree on the ordered set of separators**
  - Binary search on y-coordinate
- Within each node, **binary search tree on the segments of the associated separator**
  - Segments ordered by x-coordinate
- **Two** geometric primitives
  - Query point **above/below** a segment (signed distance computation)
  - Query point **left/right** of a subdivision vertex (segment endpoint) (x-coordinate comparison)



Basic Point Location Algorithm

- **Input**: Point p

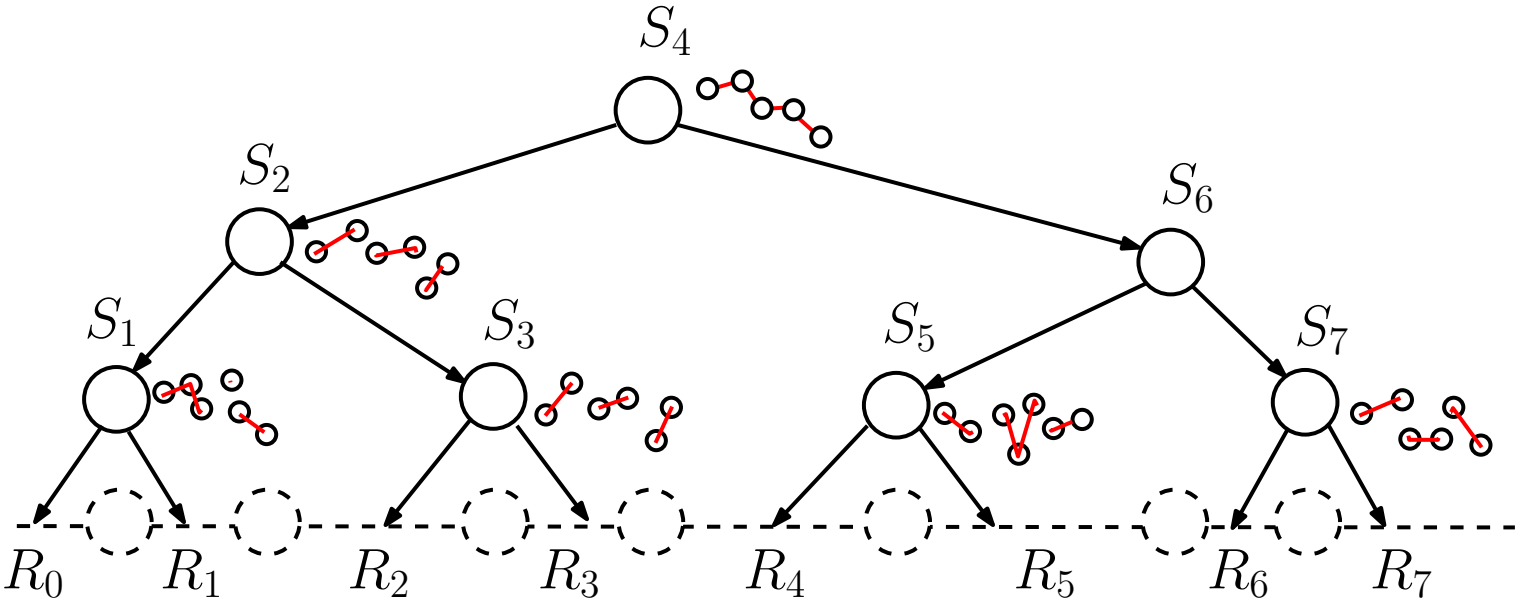
```
i ← 0; j ← n − 1; k ← root;
while (i < j) {
    if (i < k ≤ j) {
        find e ∈ S_k such that e ≪ p or e ≫ p
        if p ∈ e return e
        if (p ≫ e) i ← index of the region above e
        if (p ≪ e) j ← index of the region below e
    }
    else if (k > j) k ← left child of k
    else if (k <= i) k ← right child of k
}
```

Analysis of the Basic Algorithm

- **Space**: $S(m) = O(m^2)$
  - "Outer" search: number of faces $O(n) = O(m)$
  - "Inner" search: number of segments in a separator $O(m)$
- **Query time**: $Q(m) = O(\log^2 m)$
  - Two nested binary searches
- Still don't care about the preprocessing time

Improving the Space Requirement

- Avoid storing entire separators



- **Space**: $S(m) = O(m)$
- **Preprocessing time**: $P(m) = O(m)$
  - $O(1)$-time computation of lca

$$\text{lca}(i, j) = j \ \& \ \sim (\text{msb}(i \hat{} j) - 1)$$

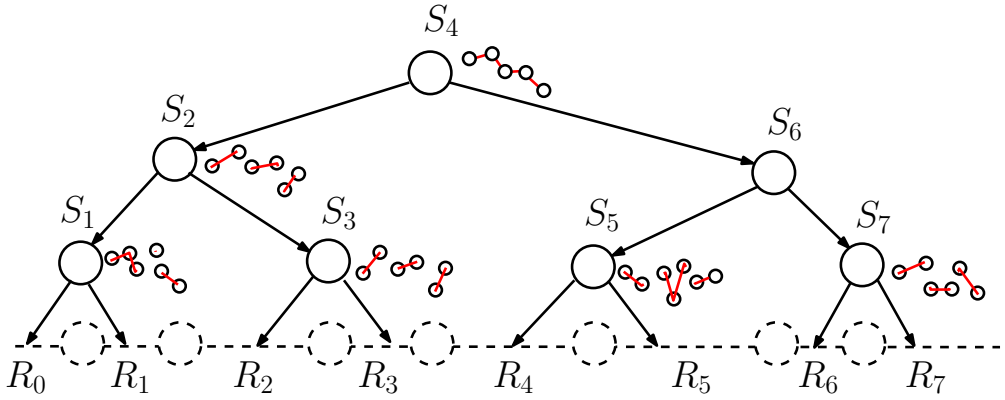
msb(x): obtained by zeroing all but the most significant bit of x
(precompute in a table)

^: bitwise XOR $\sim$: bitwise NOT &: bitwise AND

Improving the Time Requirement

- The idea of **fractional cascading**
- Replace **independent** binary searches by **correlated** ones

- Start from the bottom level of the "outer" tree
- Propagate a **fixed fraction** of the breakpoints to the parent
- Nodes in the next level repeat



- Only perform the "inner" binary search **once**, pay $O(\log m)$
- For each level of the "outer" search, do only $O(1)$ additional work

- **Query time** $Q(m) = O(\log m + \log m)$