Sensor Placement for Isotropic Source Localization

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Abstract. This paper proposes a sensor placement method for three-dimensional point source localization using multiple sensors. An observation model of multiple sensors and one source of unknown position is introduced. Given that model, the minimum number of sensors needed to localize the source is determined, and an optimal sensor placement is derived. The placement is optimal in the sense that it minimizes the effect of the measurement error on the localization error bound. The proposed placement is also shown to simplify the computational complexity. The theoretical results are experimentally evaluated through simulation. The simulation results reveal that the proposed placement significantly reduces the relative localization error.

1. Introduction and Related Work

Although source localization is a very active area of research, the literature seems to be rather scarce on sensor placement in the context of point source localization using multiple point sensors. Hence, in this paper we introduce a sensor placement technique for a three-dimensional source localization problem.

Source localization using distributed sensor networks has been an active area of research for many years [1, 2]. The application domain of source localization_includes intrusion detection in a surveillance system, contamination source detection in space shuttles, fault detection, and object identification and tracking in military combats. Special attention has been paid to wide-band and acoustic sources, using maximum-likelihood estimators. For example, the authors of [5] derive the maximum-likelihood location estimator based on the Cramér–Rao bound for wideband sources in the near field of the sensor array.

On the other hand, it seems that relatively less work has been done on sensor placement for point source localization using multiple sensors (in comparison with the amount of work done for source localization). The most relevant published works are presented in [3] and [6]. The author of [3] studied the problem of placing sensors for minimizing the variance of passive position estimates. A simple method was developed for optimally placing the sensors subject to constraints on their positions. The problem of placing sensors constrained to a line segment (originally studied in [4]) was used as an example. In [6], the authors studied three different methods for identifying the location of an impulsive source via point sensor measurements for systems described by partial differential equations (PDE). The authors analyze the minimum number of sensors and “appropriate” sensor locations for each method based on the PDE model.

In this paper, we propose a sensor placement method for the source localization problem driven by the goal of minimizing the localization error bound in a linear-algebraic framework. Further, the minimum number of sensors needed to localize the source is determined. To evaluate the proposed method, we performed a set of simulation experiments. The experimental results show that the proposed placement greatly reduces the localization error.

The paper is organized as follows: Section 2 introduces the observation model and states the problem addressed in this paper. A solution model and a derivation of the proposed sensor placement are given in Section 3. The proposed placement is evaluated in Section 4. Finally, Section 5 concludes the paper and discusses the current and future work.

2. Observation Model and Problem Statement

The observation model and the problem statement are introduced in Section 2.1 and Section 2.2 respectively.

2.1 Observation Model

The observation model includes one isotropic radiation point sensor whose intensity is known but its position vector $\mathbf{p}_s$ is unknown. Since the source is isotropic, the energy flows equally in all directions out of the source. Hence, the intensity of the source observed at a distance d from the source is inversely proportional to d^2. This is known as the *inverse square law* [9]. This model applies to a wide variety of sources including light, magnetic, and electric charge sources.

The model includes a set of N sensors $\{s_i | 0 \le i \le N-1\}$, where the position $\mathbf{p}_i$ of each sensor s_i is known. Based on the discussion given above, the signal $\hat{u}_i$ picked from the sensor s_i is given by

$$\hat{u}_i = u_i + e_i = \frac{k}{\|\mathbf{p}_i - \mathbf{p}_s\|^2} + e_i \tag{1}$$

where u_i is the error free measurement, *k* is a positive real constant, "$\| . \|$" is the norm operator, and e_i is the measurement error, due to calibration error, noise, etc. The source signal is assumed to propagate at a large enough speed so that we can ignore propagation delays.

2.2 Problem Statement

The end goal of a general source localization problem is to find the position of the source, given the measurements of the sensors. The main objectives to be addressed are (1) to derive a sensor placement configuration that will minimize the effect of the error component e_i of the sensor measurement on the localization error, and (2) to minimize the number of the sensors required to localize the source. Mathematically, this may be stated as follows.

$$\underset{\mathbf{p}_i}{\text{Min}} \left[\frac{\|\Delta\mathbf{p}_s\|}{\|\mathbf{p}_s\|}, N \right] \tag{2}$$

where $\Delta\mathbf{p}_s$ is the localization error vector.

3. Error-Bound Driven Sensor Placement

This section presents proposed approach. Section 3.1 derives the solution model for the problem and, as a byproduct, answers the question of "what is the minimum number of sensors needed to find the source location?" Section 3.2 proposes a sensor placement configuration for minimizing the effect of the measurement error on the relative error of the source localization.

3.1 Solution Model

In the case of error-free measurement, Equation (1) can be rewritten

$$\|\mathbf{p}_i - \mathbf{p}_s\|^2 = \frac{k}{u_i}; \quad e_i = 0 \tag{3}$$

This means that if the sensor signal is error-free, we can determine that the source is on a spherical locus (or a circle in the case of two-dimensional localization) whose center is the sensor location $\mathbf{p}_i$ and whose radius is k/u_i. Equation (3) can be expanded as follows

$$\|\mathbf{p}_i\|^2 + \|\mathbf{p}_s\|^2 - 2\mathbf{p}_i \cdot \mathbf{p}_s = \frac{k}{u_i} \tag{4}$$

where "." indicates scalar product. A similar equation can be obtained for each sensor. As a result, to find the location of the source, we need to solve multiple sphere equations. To make the equations linear, consider the sphere equation obtained for the sensor s_m

$$\|\mathbf{p}_m\|^2 + \|\mathbf{p}_s\|^2 - 2\mathbf{p}_m \cdot \mathbf{p}_s = \frac{k}{u_m} \tag{5}$$

Subtracting (5) from (4) and rearranging, we obtain the following linear equation, which represents the common chord of the two spheres described by (4) and (5)

$$-2(\mathbf{p}_i - \mathbf{p}_m)\cdot \mathbf{p}_s = k\left(\frac{1}{u_i} - \frac{1}{u_m}\right) - \left(\|\mathbf{p}_i\|^2 - \|\mathbf{p}_m\|^2\right) \tag{6}$$

To locate a source in three dimensions, we need at least three linear equations of the form (6). To obtain three linearly-independent equations of the form (6), we need a set of four sensors $\{s_i | 0 \le i \le 3\}$.

Without loss of generality, we will choose to place s_0 at the origin of the system of coordinate. The three linearly-independent equations can then be obtained by repeating Equation (6) three times. m will be fixed to zero, while i will vary from 1 to 3. Thus, the system of linear equations can be written as $\mathbf{A}\mathbf{p}_s = \mathbf{b}$ where

$$\mathbf{A} = -2.\begin{bmatrix}\mathbf{p}_1^T \\ \mathbf{p}_2^T \\ \mathbf{p}_3^T\end{bmatrix} = -2\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \tag{7}$$

$$,\mathbf{b} = \begin{bmatrix} k\left(\frac{1}{u_1} - \frac{1}{u_0}\right) - \|\mathbf{p}_1\|^2 \\ k\left(\frac{1}{u_2} - \frac{1}{u_0}\right) - \|\mathbf{p}_2\|^2 \\ k\left(\frac{1}{u_3} - \frac{1}{u_0}\right) - \|\mathbf{p}_3\|^2 \end{bmatrix}$$

The matrix $\mathbf{A}$ is a 3×3 coefficient matrix expressed in terms of the sensor positions. $\mathbf{b}$ is the vector of constants for the system of equations.

3.2 Sensor Placement

The system of linear equations derived in the last section can be solved as $\mathbf{p}_s = \mathbf{A}^{-1}\mathbf{b}$. Note that this is the error-free solution. Now, let us introduce the error into the computation.

The measurement error only affects the vector $\mathbf{b}$, while $\mathbf{A}$ remains unchanged because its elements depend only on the sensor positions, and hence it is always fixed for a given sensor placement. After introducing the error, the system of linear equations will become

$$\mathbf{A}(\mathbf{p}_s + \Delta\mathbf{p}_s) = \mathbf{b} + \Delta\mathbf{b} \tag{8}$$

where $\Delta\mathbf{p}_s$ is the localization error and $\Delta\mathbf{b}$ is the error that occurs in the constant vector due to the measurements error. We assume that the error e_i is bounded, otherwise arbitrary, i.e. $|e_i| \le \varepsilon$, where ε is a small positive real number. Furthermore, we assume that $|\varepsilon/u_i| << 1$. This can be achieved by imposing the following constraint which is derived by combining $|\varepsilon/u_i| << 1$ and (1).

$$\frac{\varepsilon\left\|\mathbf{p}_i-\mathbf{p}_s\right\|^2}{k} << 1 \Rightarrow \left\|\mathbf{p}_i-\mathbf{p}_s\right\| << \sqrt{\frac{k}{\varepsilon}} \qquad \textbf{(9)}$$

The physical implication of this constraint is that the less the measurement error bound, the further the source can be from the sensors without introducing a significant localization error. With this in mind, it can be shown (using the binomial expansion with negative powers) that the norm of the error vector $\mathbf{\Delta b}$ is bounded according to the following inequality.

$$\left\|\mathbf{\Delta b}\right\| \le \varepsilon \left\| \begin{bmatrix} \frac{1}{u_1^{\;2}}+\frac{1}{u_0^{\;2}} \\ \frac{1}{u_1^{\;2}}+\frac{1}{u_0^{\;2}} \\ \frac{1}{u_1^{\;2}}+\frac{1}{u_0^{\;2}} \end{bmatrix} \right\| \qquad \textbf{(10)}$$

Now, the localization error bound can be expressed as follows [7]

$$\frac{\left\|\mathbf{\Delta p}_s\right\|}{\left\|\mathbf{p}_s\right\|} \le \kappa(\mathbf{A})\frac{\left\|\mathbf{\Delta b}\right\|}{\left\|\mathbf{b}\right\|} \qquad \textbf{(11)}$$

where $\kappa(\mathbf{A})$ is the condition number of $\mathbf{A}$, which is the ratio between the largest singular value to the smallest singular value of $\mathbf{A}$ in the singular value decomposition (SVD) of $\mathbf{A}$, given by

$$\mathbf{A} = \mathbf{U}\,\mathbf{\Sigma}\,\mathbf{V}^T \qquad \textbf{(12)}$$

where $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices (note that $\mathbf{A}$ is a real matrix) and $\mathbf{\Sigma}$ is a diagonal matrix whose elements are the singular values of $\mathbf{A}$ [8].

From the definition of the condition number, it is clear that the "best-conditioned" matrix will have a condition number of 1. This is achieved by having equal singular values, i.e. $\mathbf{\Sigma} = \alpha\,\mathbf{I}$, where $\mathbf{I}$ is the identity matrix, and α is a real number that is equal to all the singular values of $\mathbf{A}$. Thus, having $\mathbf{\Sigma} = \alpha\,\mathbf{I}$ corresponds to the optimal placement of the sensors in the sense of minimizing the condition number of $\mathbf{A}$, and hence minimizing the localization error.

If we choose $\mathbf{U}$ and $\mathbf{V}$ to be any arbitrary orthogonal matrices, the resulting placement is a rotated version of the placement shown in Figure 1. If we choose $\mathbf{U}$ and $\mathbf{V}$ so that each of them is equal to the identity matrix, as a special case, this will lead to the placement shown in Fig. 1. In this case, we will have $\mathbf{A} = \alpha\,\mathbf{I}$. Not only does this placement minimize the localization error bound by minimizing the condition number of $\mathbf{A}$, but it also eliminates the need to invert the matrix $\mathbf{A}$, which in turn simplifies the computations and eliminates the effect of the round off error due to the inversion process. We will call this placement the "best-conditioned aligned pyramid" (BCAP). The placement is identified as a pyramid since the sensors are on the vertices of a 4-vertex symmetric pyramid whose base is the triangle $s_1\,s_2\,s_3$, and whose top is s_0. We describe the pyramid as aligned to indicate that its edges are aligned with the coordinate axes.

The solution to the system of equations in this case is given as follows. As mentioned above no matrix inversion is required.

$$\hat{\mathbf{p}}_s = \begin{bmatrix} \frac{k}{\alpha}\left(\frac{1}{\hat{u}_1} - \frac{1}{\hat{u}_0}\right) - \alpha \\ \frac{k}{\alpha}\left(\frac{1}{\hat{u}_2} - \frac{1}{\hat{u}_0}\right) - \alpha \\ \frac{k}{\alpha}\left(\frac{1}{\hat{u}_3} - \frac{1}{\hat{u}_0}\right) - \alpha \end{bmatrix} \tag{13}$$

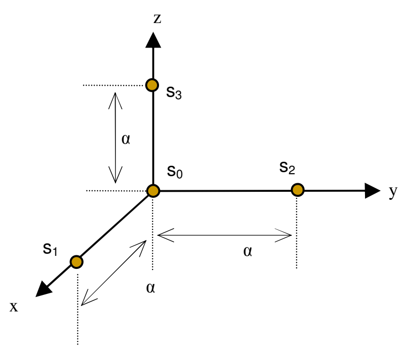

Fig. 1. Sensor placement that minimizes the coefficient matrix condition number.

4. Experimental Evaluation

To verify the theoretical results obtained in the previous section, we conducted a set of simulation experiments where the source was moving slowly along a helical path as shown in Fig. 2. The experiments were performed for three different sensor placements. The first placement is the BCAP introduced in the previous section with $\alpha = 100$. The second placement corresponds to a matrix **A** that is "badly-conditioned", with a condition number of 10^6. The third placement results in a coefficient matrix with a condition number of 10^3. Fig. 3 depicts the second placement used in the experiments.

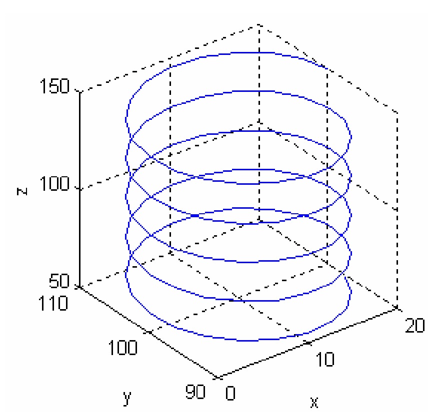

Fig. 2. The source moves along a helical path.

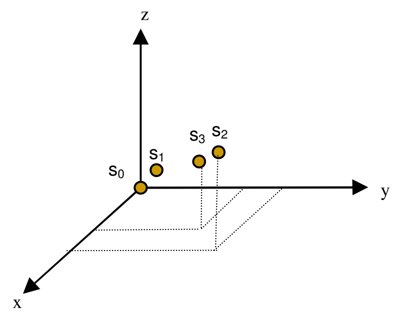

Fig. 3. Sensor configuration that leads to a badly-conditioned matrix: s_0 is at $(0, 0, 0)^T$, s_1 is at $(1.55, 1.87, 2.61)^T$, and s_2 is at $(36.31, 43.08, 57.92)^T$, s_3 is at $(26.47, 31.39, 42.11)^T$

To conduct the simulation, the path was sampled into a number sample points. At each sample point, the location of the source was estimated according to the solution model derived in the previous section. The simulation was conducted for different measurement relative error bounds $|\varepsilon/u_i|$. For each bound and for each placement the absolute value of the relative root mean square (RMS) error was computed along the path. Fig. 4 shows a plot for the results for the different placement on a log scale. Thus, each point on the x-axis corresponds to a complete experiment for the source moving along the path with a measurement relative error bound equal to the value on the x-axis at that point. The proposed BCAP placement consistently results in less RMS relative localization errors.

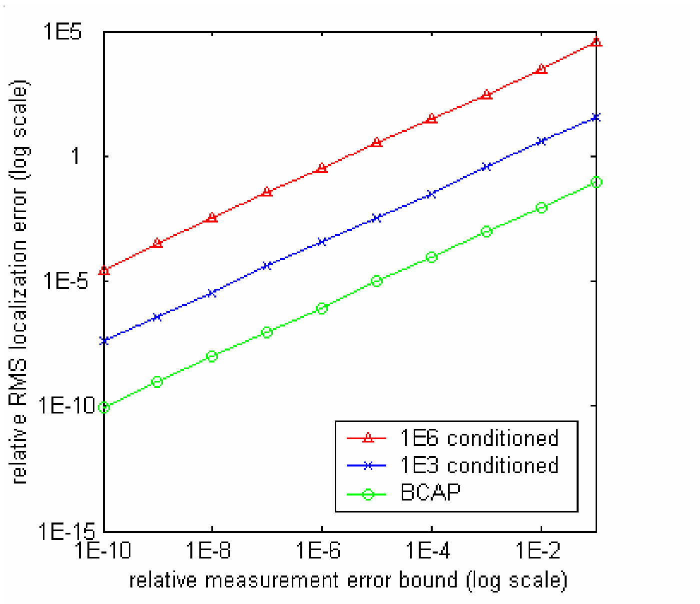

Fig. 4. A plot of the path RMS error vs. the relative measurement error

Fig. 5 shows an example of the estimated path using the BCAP placement versus the placement that corresponds to the condition number of 10^3 for the cases where the bound of the relative measurement error is 1% and 0.1%. The case of the "1e3 conditioned" placement with 1% error results in so huge localization error that the actual path and the estimated path cannot be shown on the same scale (hence, will not be shown in the figure), whereas the corresponding BCAP case shows that the estimated path is still close to the actual path (Fig. 5a). In the case of 0.1% relative error bound, the estimated path using BCAP follows the actual path to a great precision (Fig. 5c), while the localization error of the "1e3 conditioned" is so high (Fig. 5b) that the estimated path is useless in the sense that it does not provide any good estimation of the actual path. The placement that corresponds to a condition number of 10^6 showed a poorer performance in these particular cases.

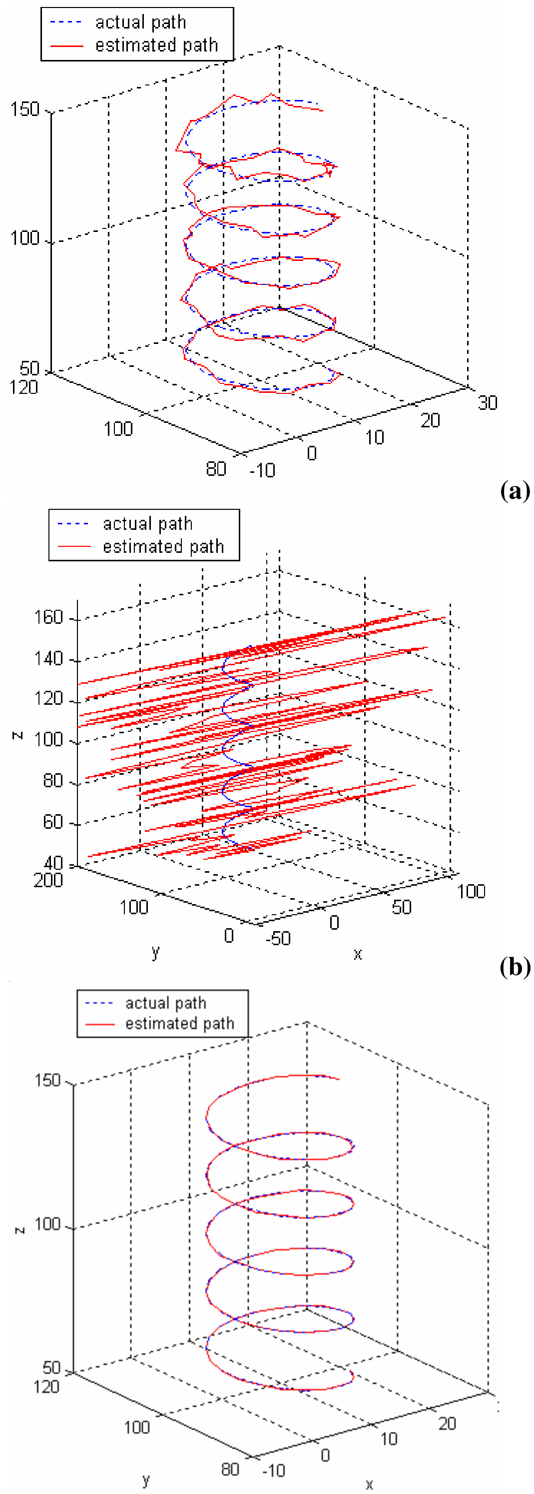

Fig. 5. The estimated path shown vs. the actual path for (a) BCAP and relative measurement error of 1%, (b) a badly-conditioned placement and relative measurement error of 0.1%, and (c) BCAP and relative measurement error of 0.1%.

5. Conclusion and Future Work

In this paper we present a three-dimensional observation model for a source localization problem. Based on the model, we imposed two questions: (1) “What is the minimum number of sensors needed to find the location of the sensor?” and (2) “What is the optimal sensor placement that minimizes the localization error?” We formulated the problem in a linear-algebraic framework, which directly answered the first question. Then, we used the formulation to derive an error bound for the localization error, based on which, we derived a sensor placement configuration that minimizes the error bound of the localization error and eliminated the round-off error due to inversion. To verify the results we obtained, we conducted a set of simulations. The experimental results revealed that the proposed placement outperformed other placements in terms of the localization error.

Thus, the contribution of this paper is (1) formulating the source localization problem for the given model in a linear-algebraic framework for the goal of obtaining an optimal sensor placement, and (2) the proposed sensor placement (BCAP) that is optimal in the sense that it minimizes the effect of the measurement error on the localization error. Our current and future work aim at extending the problem in many directions by relaxing many of the assumptions made while developing the proposed solution. For example, we are interested in studying the problem for multiple sources, unknown source signals, etc. We are also interested in the problem where we can use smart sensors for fusing information and reconfiguration.

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