Acoustic Waves & Radiation

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Acoustic Waves

Diffraction is important!

- Speed of sound, \( c = 343 \text{ m/s} = \lambda f \)
- Audible frequencies: 16 Hz — 20 kHz
- Audible wavelengths: 21 m — 17 mm

Diffraction Shaders [Stam 99]
Consider tiny acoustic fluctuations in key quantities about mean values, e.g., in air:

- **Pressure**, $p$: About $p_0 = 1$ atmosphere.
- **Density**, $\rho$: About $\rho_0$ air density.
- **Velocity**, $v$: Consider acoustic particle velocity $v$ fluctuations about background flow. Usually assume stationary air speed is zero (c.f. Doppler effects).
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Sound Intensity

The intensity of a sound pressure $p$ in air is usually measured on a decibel scale by the quantity

$$20 \times \log_{10} \left( \frac{|p|}{p_{\text{ref}}} \right),$$

where the reference pressure $p_{\text{ref}} = 2 \times 10^{-5}$ N/m$^2$. Thus, $p = p_0 \equiv 1$ atmosphere ($=10^5$ N/m$^2$) is equivalent to 194 dB. A very loud sound $\sim 120$ dB corresponds to

$$\frac{p}{p_0} \approx \frac{2 \times 10^{-5}}{10^5} \times 10^{\left( \frac{120}{20} \right)} = 2 \times 10^{-4} \ll 1.$$ 

Similarly, for a ‘deafening’ sound of 160 dB, $p/p_0 \sim 0.02$. This corresponds to a pressure of about 0.3 lbs/in$^2$ and is loud enough for nonlinear effects to begin to be important.
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Acoustic Particle Velocity

\[
\text{acoustic particle velocity} \approx \frac{\text{acoustic pressure}}{\text{mean density} \times \text{speed of sound}}.
\]

In air the speed of sound is about 340 m/sec. Thus, at 120 dB \( v \sim 5 \text{ cm/sec} \); at 160 dB \( v \sim 5 \text{ m/sec} \).
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Wave equation follows from...
(see [Howe 2003], or Bridson's "fluid" course notes)

- **Linearized continuity eqn** (mass conservation)
  \[
  \frac{1}{\rho} \frac{\partial \tilde{\rho}}{\partial t} + \nabla \cdot \mathbf{v} = 0
  \]

- **Linearized momentum eqn** ("ma=f")
  \[
  \rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla \tilde{p}
  \]

- **Energy eqn** (relates acoustic pressure and density at constant entropy)
  \[
  p = \left( \frac{\partial p}{\partial \rho} \right)_{0} \tilde{\rho} = c^2 \tilde{\rho}
  \]

Acoustic particle velocity
Acoustic pressure
Acoustic density fluctuation
\[ j^{\nu} = \frac{1}{c^2} p \quad \Rightarrow \quad \frac{1}{p c^2} \frac{\partial p}{\partial t} + \nabla \cdot v = 0 \]

\[ \frac{\partial j^{\nu}}{\partial t} = \frac{p^2}{2} + \nabla \cdot \left( \frac{\nabla \cdot v}{p} \right) = 0 \]

\[ \frac{\partial v}{\partial t} = -\frac{1}{\rho} \frac{\nabla \rho}{\rho} \]

\[ -\nabla \cdot \nabla p = -\nabla^2 p \]
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1D Wave Equation

\[ \frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2} \]

Traveling wave solutions of the form:

\[ p(x, t) = f(x \pm ct) \]
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3D Wave Equation

Acoustic pressure, $p(x, t)$, satisfies

$$\nabla^2 p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0$$

(Wave Equation)

where

$$\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}$$

is the Laplacian.
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Radiation boundary condition (1)

Need to impose (vibrating) solid boundaries.

From the linearized momentum equation (F=0),

\[
\frac{\partial v}{\partial t} = -\nabla p
\]

the normal component yields

\[
\frac{\partial p}{\partial n} = -\rho \, a_n(x, t), \quad x \in \Gamma
\]

(Neumann BC)

where \( a_n(x, t) \) is the surface acceleration, and \( \rho = 1.2041 \, kg/m^3 \) is air density at STP.
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Radiation boundary condition (2)

Need condition at infinity.

- So-called *Sommerfeld radiation condition*.
- Solutions must decay at infinity and correspond to out-going waves.
- Avoids nonphysical solutions, such as infinite sources at infinity.
Frequency-domain Wave Radiation
Frequency-domain Wave Radiation

Motivation: Individual mode vibrations are localized in frequency domain

Impulse response of a cymbal impact modeled using a linear modal vibration model.
Using complex numbers to represent oscillations

By linearity of the wave equation & BCs, sufficient to consider a unit amplitude harmonic vibration at natural frequency $\omega$,

$$e^{+i\omega t} = \cos(\omega t) + i \sin(\omega t)$$

Different phases introduced by $e^{+i\phi}$ factors since

$$e^{+i\omega t}e^{i\phi} = \cos(\omega t + \phi) + i \sin(\omega t + \phi)$$

Can think of $q(t)$ as an amplitude and a unit complex part.
Frequency-domain Wave Radiation

Harmonic pressure oscillations

Consider harmonic pressure solutions of the form

$$p(x, t) = p(x) e^{i\omega t}$$

where $p(x)$ is a complex-valued pressure field—will be the acoustic transfer function.

Using Euler’s formula, we can write $p(x) = |p| e^{i\phi}$ so that

$$p(x, t) = p(x) e^{i\omega t} = |p(x)| e^{i(\omega t + \phi)}$$

$$= |p(x)| \cos(\omega t + \phi) + i |p(x)| \sin(\omega t + \phi).$$
Frequency-domain Wave Radiation

The Helmholtz equation
(the frequency-domain wave equation)

Substituting $p(x, t) = p(x)e^{+i\omega t}$ into the wave equation yields

$$0 = \nabla^2 p(x) e^{+i\omega t} - \frac{1}{c^2} p(x) \left( \frac{\partial^2}{\partial t^2} e^{+i\omega t} \right)$$

$$= \left( \nabla^2 p(x) - \frac{\omega^2}{c^2} p(x) \right) e^{+i\omega t}$$

and so

$$\nabla^2 p(x) + k^2 p(x) = 0 \quad (\text{Helmholtz eqn})$$

where $k$ is the wavenumber,

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

Never zero
Frequency-domain Wave Radiation

Radiation boundary conditions

Recall time-domain Neumann BC:

$$\frac{\partial p}{\partial n} = -\rho a_n(x, t), \quad x \in \Gamma$$

What is frequency-domain BC?

What is $a_n(x, t)$?
Frequency-domain Wave Radiation

Radiation boundary conditions

Need harmonic surface acceleration \( a_n(x, t) \) in terms of eigenmode displacement, \( \bar{u}(x) \):

\[
\begin{align*}
u(x, t) &= \frac{du}{dt}(x, t) = i\omega \bar{u}(x) e^{+i\omega t} \quad &\implies\quad v_n &= i\omega \bar{u}_n e^{+i\omega t} \\\na(x, t) &= \frac{d^2u}{dt^2}(x, t) = (i\omega)^2 \bar{u}(x) e^{+i\omega t} \quad &\implies\quad a_n &= -\omega^2 \bar{u}_n e^{+i\omega t}
\end{align*}
\]

\[
a_n = -\omega^2 \bar{u}_n e^{+i\omega t}
\]
Frequency-domain Wave Radiation

Radiation boundary conditions

The Neumann boundary condition becomes

\[
\frac{\partial p(x, t)}{\partial n} = -\rho \ a_n(x, t)
\]

\[
\frac{\partial p(x)}{\partial n} e^{+i\omega t} = \rho \omega^2 \ \vec{u}_n \ e^{+i\omega t}
\]

\[
\frac{\partial p(x)}{\partial n} = \rho \omega^2 \ \vec{u}_n(x)
\]

(+Sommerfeld radiation BC: Out-going decaying waves.)
Frequency-domain Wave Radiation

Acoustic Transfer BVP

Given mode data $\bar{u}_n(x)$ and frequency $\omega$.

Solve exterior Helmholtz eqn

\[
(\nabla^2 + k^2) p(x) = 0, \quad x \in \Omega
\]

subject to BC:

\[
\frac{\partial p(x)}{\partial n} = \rho \omega^2 \bar{u}_n(x), \quad x \in \Gamma
\]

(+Sommerfeld radiation BC.)
Frequency-domain Wave Radiation

Visualizing solutions

Can visualize solutions by looking at real part of

\[ p(x) e^{+i\omega t} \]

or the AT amplitude

\[ |p(x)| \]
MODE 10
MODE 18
MODE 19
MODE 20
MODE 35
Including Source Terms
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3D Wave Equation (without sources)

Acoustic pressure, \( p(x, t) \), satisfies

\[
\nabla^2 p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0 \quad (\text{Wave Equation})
\]

subject to boundary conditions, e.g., vibrations.

What about RHS source terms?
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Re-deriving the wave equation
(see [Howe 2003])

• Linearized continuity eqn (mass conservation)
\[
\frac{1}{\rho_0} \frac{\partial \tilde{\rho}}{\partial t} + \nabla \cdot \mathbf{v} = q(x, t)
\]

• Linearized momentum eqn ("ma=f")
\[
\rho_0 \frac{\partial \mathbf{v}}{\partial t} + \nabla p = \mathbf{F}(x, t)
\]

• Energy eqn (relates acoustic pressure and density at constant entropy)
\[
p = \left( \frac{\partial p}{\partial \rho} \right)_0 \tilde{\rho} = c^2 \tilde{\rho}
\]
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3D Wave Equation (with sources)

Acoustic pressure, $p(x, t)$, satisfies

$$\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) p = \rho \frac{\partial q}{\partial t} - \nabla \cdot F$$

subject to boundary conditions, e.g., vibrations.

- Can be hard to model $F$.

Solve for $p$. (what about $\tilde{p}$ and $\mathbf{v}$?)
Velocity Potential

A useful trick
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Velocity Potential, $\varphi$

Scalar velocity potential, $\varphi(x, t) \in \mathbb{R}$ satisfies

$$\mathbf{v} = \nabla \varphi$$

Exists when $\mathbf{F} = 0$, due to momentum eqn:

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p$$

$\implies$ changes in $\mathbf{v}$ are gradient of something.

Useful trick! Knowing $\varphi$ gives you both $\mathbf{v}$ and $p$:

$$\mathbf{v} = \nabla \varphi \quad \text{and} \quad p = -\rho \frac{\partial \varphi}{\partial t}$$
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3D Wave Equation using $\varphi(x, t)$

Wave equation with $F = 0$, i.e., only volume srcs:

$$
\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) p = \rho \frac{\partial q}{\partial t}
$$

Substitute $p = -\rho \frac{\partial \varphi}{\partial t}$ and remove one $\frac{\partial}{\partial t}$:

$$
\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \varphi = -q(x, t)
$$
Limiting case of an Incompressible Fluid
Acoustic Waves

**Incompressible limit \((c \to \infty)\)**

Since the speed of sound is related to small pressure and density fluctuations:

\[
c^2 \approx \frac{\delta p}{\delta \rho} \quad \delta \rho \to 0 \to \infty
\]

In this limit, the wave equation (using \(\varphi\)) simplifies to

\[
\nabla^2 \varphi = q(x, t)
\]

where the volume source is \(q = \nabla \cdot \mathbf{v}\)
Incompressible example (Howe 1.4.1)

Pulsating Sphere (radius $a$)

Fig. 1.4.1.
Incompressible example (Howe 1.4.1)

Pulsating Sphere (radius $a$)

\[
\begin{align*}
\nabla^2 \varphi &= 0, & r > a, \\
\frac{\partial \varphi}{\partial r} &= v_n(t), & r = a
\end{align*}
\]

where $r = |\mathbf{x}|$.

The motion is obviously radially symmetric, so that

\[
\nabla^2 \varphi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) \varphi = 0, \quad r > a.
\]

Hence,

\[
\varphi = \frac{A}{r} + B,
\]
Incompressible example (Howe 1.4.1)

Pulsating Sphere (radius $a$)

Applying the BC $(\partial \varphi / \partial r = v_n(t), \quad r = a)$:

$$\varphi = -\frac{a^2 v_n(t)}{r}, \quad r > a.$$  

$$p = -\rho_0 \frac{\partial \varphi}{\partial t} = \rho_0 \frac{a^2}{r} \frac{dv_n}{dt}(t)$$
Incompressible example (Howe 1.4.1)

Pulsating Sphere (radius $a$)

Volume flux (or "volume velocity") of source:

For any time $t$, the volume flux $q(t)$ of fluid is the same across any closed surface enclosing the sphere. Evaluating it for any sphere $S$ of radius $r > a$, as shown in Fig. 1.4.1, we find

$$q(t) = \int_S \nabla \varphi \cdot dS = 4\pi a^2 v_n(t),$$

and we may also write

$$\varphi = \frac{-q(t)}{4\pi r}, \quad r > a.$$
Math Background

Dirac Delta Function

https://en.wikipedia.org/wiki/Dirac_delta_function

The Dirac delta can be loosely thought of as a function on the real line which is zero everywhere except at the origin, where it is infinite,

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

and which is also constrained to satisfy the identity

$$\int_{-\infty}^{\infty} \delta(x) \, dx = 1.$$ 

Useful property:

$$\int_{-\infty}^{\infty} f(t) \delta(t - T) \, dt = f(T)$$
Incompressible case (Howe 1.4.2)

Point Source

A volume point source of strength $q(t)$ at the origin satisfies a related problem:

$$\nabla^2 \varphi = q(t)\delta(x)$$

where $\delta(x) = \delta(x_1)\delta(x_2)\delta(x_3)$.

Gives the solution:

$$\varphi(x, t) = -\frac{q(t)}{4\pi r}$$

(agrees with pulsating sphere for $r > a$)
Sound Produced by an Impulsive Point Source
Wave equation (Howe 1.5)

**Impulsive Point Source**

Consider an impulsive ("pop") event at $t = 0, \mathbf{x} = 0$:

$$\left( \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \varphi = \delta(\mathbf{x})\delta(t)$$

Has two types of solutions (where $r = \|\mathbf{x}\|$):

$$\varphi(\mathbf{x}, t) = \frac{1}{4\pi r} \delta \left( t - \frac{r}{c} \right)$$  \hspace{1cm} \text{(outgoing)}$$

and

$$\varphi(\mathbf{x}, t) = \frac{1}{4\pi r} \delta \left( t + \frac{r}{c} \right)$$  \hspace{1cm} \text{(incoming)}$$
Wave equation (Howe 1.5)

**Impulsive Point Source**

We only consider the *causal* solution

\[
\varphi(x, t) = \frac{1}{4\pi r} \delta \left( t - \frac{r}{c} \right)
\]

Represents a spherical pulse that is only nonzero about \( r = ct > 0 \).
Wave equation (Howe 1.6)

Free-space Green's Function

Physical solution due to "pop" event at $t = \tau$, $x = y$:

$$\left( \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) G = \delta(x - y)\delta(t - \tau),$$

Gives the solution:

$$G(x, y, t - \tau) = \frac{1}{4\pi|x - y|} \delta \left( t - \tau - \frac{|x - y|}{c_0} \right)$$

This represents an impulsive, spherically symmetric wave expanding from the source at $y$ at the speed of sound. The wave amplitude decreases inversely with distance $|x - y|$ from the source point $y$. 
Free-space Green's Function

Gives general solution to wave equation (\(\varphi\) or \(p\) form)

\[
\left( \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) p = \mathcal{F}(\mathbf{x}, t)
\]

using linear superposition of sources

\[
\mathcal{F}(\mathbf{x}, t) = \iiint_{-\infty}^{\infty} \mathcal{F}(\mathbf{y}, \tau) \delta(\mathbf{x} - \mathbf{y}) \delta(t - \tau) \, d^3y \, d\tau
\]

Same source as Green's function
Wave equation (Howe 1.6)

Free-space Green's Function

Source

$$\int \int_\infty^{\infty} \int _{-\infty}^{\infty} \mathcal{F}(y, \tau) \delta(x - y) \delta(t - \tau) \ d^3 y \ d\tau$$

$$\delta(x - y) \delta(t - \tau)$$

$$\mathcal{F}(x, t)$$

Causal Solution

$$\int \int_\infty^{\infty} \int _{-\infty}^{\infty} \mathcal{F}(y, \tau) G(x, y, t - \tau) \ d^3 y \ d\tau$$

$$G(x, y, t - \tau)$$

$$\frac{1}{4\pi} \int \int_\infty^{\infty} \int _{-\infty}^{\infty} \mathcal{F}(y, \tau) \delta\left(t - \tau - \frac{\|x - y\|}{c}\right) \ d^3 y \ d\tau$$

$$\frac{1}{4\pi} \int \int^{\infty}_{-\infty} \mathcal{F}\left(y, \frac{\|x - y\|}{c}\right) \ d^3 y$$

$$\mathcal{F}(y, \tau) G(x, y, t - \tau)$$
Free-space Green's Function

So, the general solution to wave equation

\[
\left( \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) p = \mathcal{F}(x, t)
\]

is given by

\[
p(x, t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\mathcal{F}(y, t - \frac{|x-y|}{c_0})}{|x-y|} \, d^3y
\]
Wave equation (Howe 1.6)

Approximating the solution

\[ p(x, t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\mathcal{F}(y, t - \frac{|x-y|}{c_0})}{|x - y|} \, d^3y \]

Difficult to evaluate integral in general, hence:

- Special solutions (monopoles, dipoles, etc.)
- Far-field approximations (Fraunhofer)
- Numerical integration

Major limitation:

- No boundaries (no reflections, no surface radiation)
- Green's function extension for acoustically compact shapes
- Need general numerical solvers (FEM, FDM, BEM, etc.)
Point Sources
Time-Domain Monopoles, Dipoles, ...
Incompressible case (Howe 1.4.2)

Point Source

A volume point source of strength $q(t)$ at the origin satisfies a related problem:

$$\nabla^2 \varphi = q(t)\delta(x)$$

where $\delta(x) = \delta(x_1)\delta(x_2)\delta(x_3)$.

Gives the solution:

$$\varphi(x, t) = -\frac{q(t)}{4\pi r}$$

(agree with pulsating sphere for $r > a$)
Compressible case (Howe 1.7)

**Time-Domain Monopole Solution**

A volume point source of strength $q(t)$ at the origin satisfies a related problem:

$$\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \varphi = -q(t)\delta(x)$$

Gives the solution:

$$\varphi(x, t) = -\frac{q \left( t - \frac{r}{c} \right)}{4\pi r}$$

(same as incompressible case when $c \to \infty$)
Compressible case (Howe 1.7)

Time-Domain Monopole Solution

Visualization of the solution:

\[ \varphi(x, t) = -\frac{q \left( t - \frac{r}{c} \right)}{4\pi r} \]

for \( q(t) = \sin(t) \).

https://blog.soton.ac.uk/soundwaves/wave-basics/point-sources-inverse-square-law/
Compressible case (Howe 1.7)

**Dipole as the limit of two monopoles**

Consider two equal & opposite monopole sources:

\[
\frac{1}{h} q(t) \delta(x - \frac{h}{2}) \delta(y) \delta(z) - \frac{1}{h} q(t) \delta(x + \frac{h}{2}) \delta(y) \delta(z)
\]

\[
= - q(t) \delta'(x) \delta(y) \delta(z)
\]

\[
= - \frac{\partial}{\partial x} \left( q(t) \delta(x) \delta(y) \delta(z) \right)
\]

\[
= - \frac{\partial}{\partial x} \left( q(t) \delta(x) \right) = \text{Gradient of monopole source}
\]

Zero net volume change.
Useful Trick

Differentiate to get other solutions

Given solution $\varphi(x, t)$ for some RHS $b(x, t)$,

$$\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \varphi = b$$

then $\mathcal{D}\varphi$ is the wave solution to RHS $\mathcal{D}b$ since

$$\mathcal{D} \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \varphi = \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \mathcal{D}\varphi = \mathcal{D}b$$

for a constant-coefficient differential operator $\mathcal{D}$, e.g., a directional derivative $(\mathbf{n} \cdot \nabla)$. 
Compressible case (Howe 1.7)

**Dipole as the limit of two monopoles**

\[
RHS = - \frac{\partial}{\partial x} \left( q(t) \delta(x) \right) = \text{Gradient of monopole source}
\]

so solution must be monopole gradient:

\[
\varphi(x, t) = \frac{\partial}{\partial x} \varphi_{\text{monopole}}(x, t) = \frac{\partial}{\partial x} \left[ -\frac{q \left( t - \frac{r}{c} \right)}{4\pi r} \right]
\]

Zero net volume change.
Vibrating Sphere

Small sphere radius: \( a \).
Vibrating along \( \hat{x} \).
Speed: \( \dot{x}(t) = U(t) \).
Equivalent dipole strength:

\[
q(t) = 2\pi a^3 U(t)
\]
Dipole example (Howe 1.7.3)

**Vibrating Sphere**

Equivalent dipole: \( q(t) = 2\pi a^3 U(t) \) (Howe 3.5)

Solution (using derivative trick):

\[
\varphi(x, t) = \frac{\partial}{\partial x_1} \left( \frac{2\pi a^3 U(t - \frac{|x|}{c_0})}{4\pi |x|} \right)
\]

Since \( r = ||x|| \) and \( x_1 = r \cos \theta \), it follows that

\[
\varphi = -\frac{a^3 \cos \theta}{2r^2} U \left( t - \frac{r}{c_0} \right) - \frac{a^3 \cos \theta}{2c_0 r} \frac{\partial U}{\partial t} \left( t - \frac{r}{c_0} \right)
\]

near field

far field
Dipole example (Howe 1.7.3)

Vibrating Sphere

\[
\varphi = - \frac{a^3 \cos \theta}{2r^2} U \left( t - \frac{r}{c_0} \right) - \frac{a^3 \cos \theta}{2c_0 r} \frac{\partial U}{\partial t} \left( t - \frac{r}{c_0} \right)
\]

near field

far field

The near-field term is dominant at sufficiently small distances \( r \) from the origin such that

\[
\frac{1}{r} \gg \frac{1}{c_0} \frac{1}{U} \frac{\partial U}{\partial t} \sim \frac{f}{c_0},
\]

where \( f \) is the characteristic frequency of the oscillations of the sphere. But, sound of frequency \( f \) travels a distance

\[
c_0/f = \lambda \equiv \text{one acoustic wavelength}
\]

in one period of oscillation \( 1/f \). Hence, the near-field term is dominant when

\[
r \ll \lambda.
\]
Vibrating Sphere

\[
\varphi = -\frac{a^3 \cos \theta}{2r^2} U \left( t - \frac{r}{c_0} \right) - \frac{a^3 \cos \theta}{2c_0 r} \frac{\partial U}{\partial t} \left( t - \frac{r}{c_0} \right)
\]

Near field (\( r \gg \lambda \))

Far field (\( r \gg \lambda \)):

- involves time-derivative of \( q(t) \).
- disappears as \( c \to \infty \) (incompressible limit).

Acoustic compactness: Dipole model implicitly assumes that \( a \ll \lambda \), i.e., the sphere is **acoustically compact**: characteristic dimensions are smaller than wavelengths occurring.
Dipole example (Howe 1.7.3)

Vibrating Sphere

\[ \varphi = -\frac{a^3 \cos \theta}{2r^2} U \left( t - \frac{r}{c_0} \right) - \frac{a^3 \cos \theta}{2c_0 r} \frac{\partial U}{\partial t} \left( t - \frac{r}{c_0} \right) \]

Far-field pressure solution:

\[ p(x, t) = -\rho \frac{\partial \varphi}{\partial t} \approx \frac{\rho a^3 \cos \theta}{2cr} \frac{\partial^2 U}{\partial t^2} \left( t - \frac{r}{c} \right), \quad r = \|x\| \to \infty. \]

So, we hear the "jerk" of the sphere ;)}