CS 448Z: Physically Based Animation & Sound

Acoustic Waves & Radiation

Doug James
Stanford University

Acoustic Waves

Diffraction is important!

- Speed of sound, $c = 343 \text{ m/s} = \lambda f$
- Audible frequencies: 16 Hz — 20 kHz
- Audible wavelengths: 21 m — 17 mm

Diffraction Shaders [Stam 99]

Linear acoustics

Consider tiny acoustic fluctuations in key quantities about mean values, e.g., in air:

- **Pressure,** p: About p_0=1 atmosphere.
- **Density,** ρ: About ρ_0 air density.
- **Velocity,** $\mathbf{v}$: Consider acoustic particle velocity $\mathbf{v}$ fluctuations about background flow. Usually assume stationary air speed is zero (c.f. Doppler effects)

Sound Intensity

The intensity of a sound pressure p in air is usually measured on a decibel scale by the quantity

$$20 \times \log_{10}\left(\frac{|p|}{p_{\text{ref}}}\right),$$

where the reference pressure $p_{\text{ref}} = 2 \times 10^{-5}$ N/m^2. Thus, $p = p_0 \equiv 1$ atmosphere ($= 10^5$ N/m^2) is equivalent to 194 dB. A very loud sound $\sim$120 dB corresponds to

$$\frac{p}{p_0} \approx \frac{2 \times 10^{-5}}{10^5} \times 10^{\left(\frac{120}{20}\right)} = 2 \times 10^{-4} \ll 1.$$

Similarly, for a 'deafening' sound of 160 dB, $p/p_0 \sim 0.02$. This corresponds to a pressure of about 0.3 lbs/in^2 and is loud enough for nonlinear effects to begin to be important.

[Howe, Section 1.3]

Acoustic Particle Velocity

$$\text{acoustic particle velocity} \approx \frac{\text{acoustic pressure}}{\text{mean density} \times \text{speed of sound}}.$$

In air the speed of sound is about 340 m/sec. Thus, at 120 dB $v \sim 5$ cm/sec; at 160 dB $v \sim 5$ m/sec.

[Howe, Section 1.3]

Wave equation follows from…

(see [Howe 2003], or Bridson's “fluid” course notes)

- Linearized continuity eqn (mass conservation)

$$\frac{1}{\rho}\frac{\partial \tilde{\rho}}{\partial t} + \nabla \cdot \mathbf{v} = 0$$

Acoustic particle velocity

- Linearized momentum eqn (“ma=f”)

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p$$

Acoustic pressure

- Energy eqn (relates acoustic pressure and density at constant entropy)

$$p = \left(\frac{\partial p}{\partial \rho}\right)_0 \tilde{\rho} = c^2 \tilde{\rho}$$

Acoustic density fluctuation

$$\tilde{\rho} = \frac{1}{c^2} p \xrightarrow{①} \frac{1}{\rho c^2} \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{v} = 0$$

$$\frac{\partial}{\partial t}$$

$$\frac{1}{\rho c^2} \frac{\partial^2 p}{\partial t^2} + \nabla \cdot \underbrace{\frac{\partial}{\partial t} \mathbf{v}}_{} = 0$$

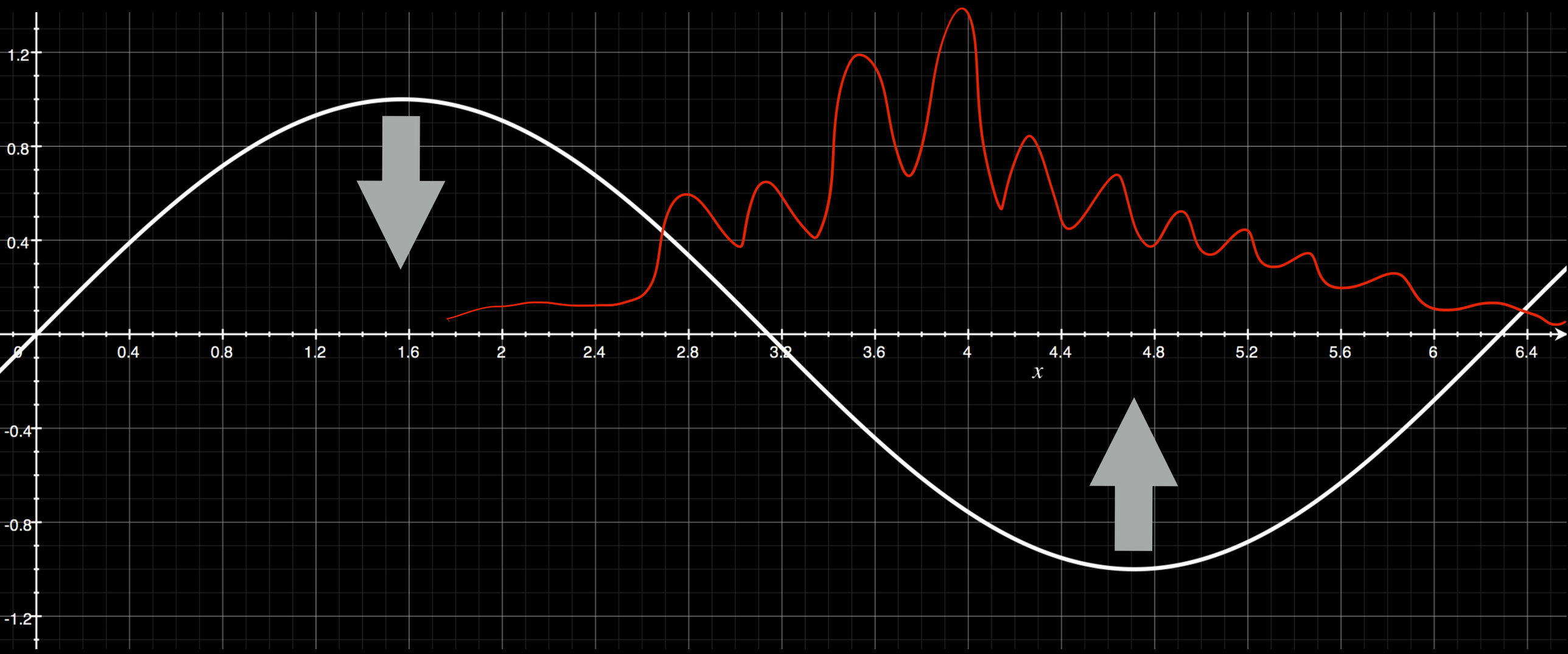
mom. eqn. $\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho} \nabla p$

$$\frac{1}{\cancel{\rho_0} c^2} \frac{\partial^2 p}{\partial t^2} + \underbrace{\nabla \cdot \left(-\frac{1}{\cancel{\rho_0}} \nabla p \right)}_{-\nabla \cdot \nabla p = -\nabla^2 p} = 0$$

Acoustic Waves

1D Wave Equation

$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2}$$



Traveling wave solutions of the form:

$$p(x,t) = f(x \pm ct)$$

3D Wave Equation

Acoustic pressure, $p(\boldsymbol{x}, t)$, satisfies

$$\boxed{\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0}$$ (*Wave Equation*)

where

$$\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}$$

is the Laplacian.

Radiation boundary condition (1)

Need to impose (vibrating) solid boundaries.

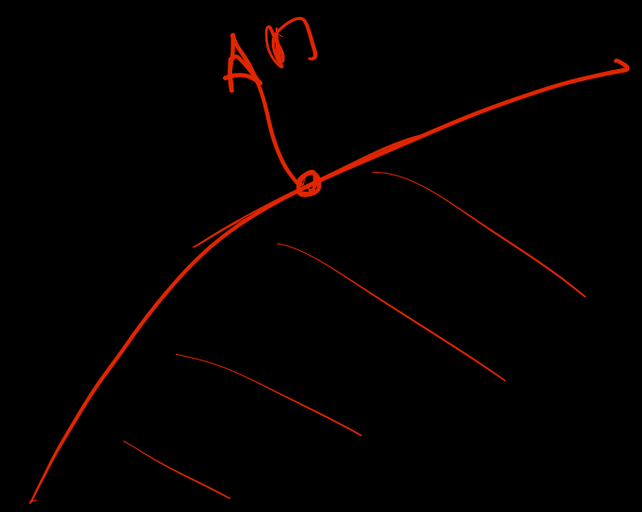
From the linearized momentum equation (F=0),

$\frac{\partial}{\partial n} \equiv (n \cdot \nabla)$

$a_n \equiv n \cdot \frac{\partial v}{\partial t} = \frac{\partial v_n}{\partial t}$

$$\boxed{\rho \frac{\partial \boldsymbol{v}}{\partial t} = -\nabla p}$$

$\cdot\, n$



the normal component yields

$$\boxed{\frac{\partial p}{\partial n} = -\rho\, a_n(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \Gamma}$$ *(Neumann BC)*

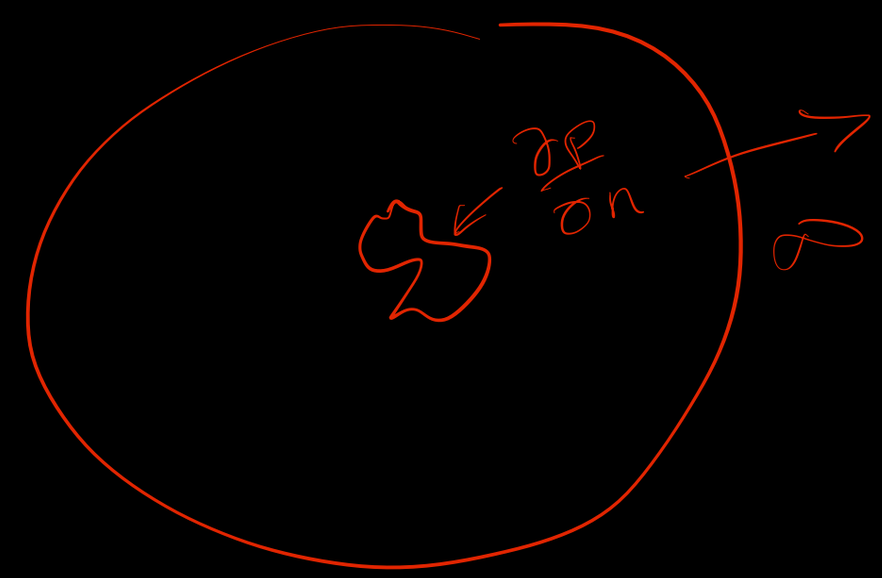
where $a_n(\boldsymbol{x}, t)$ is the surface acceleration,

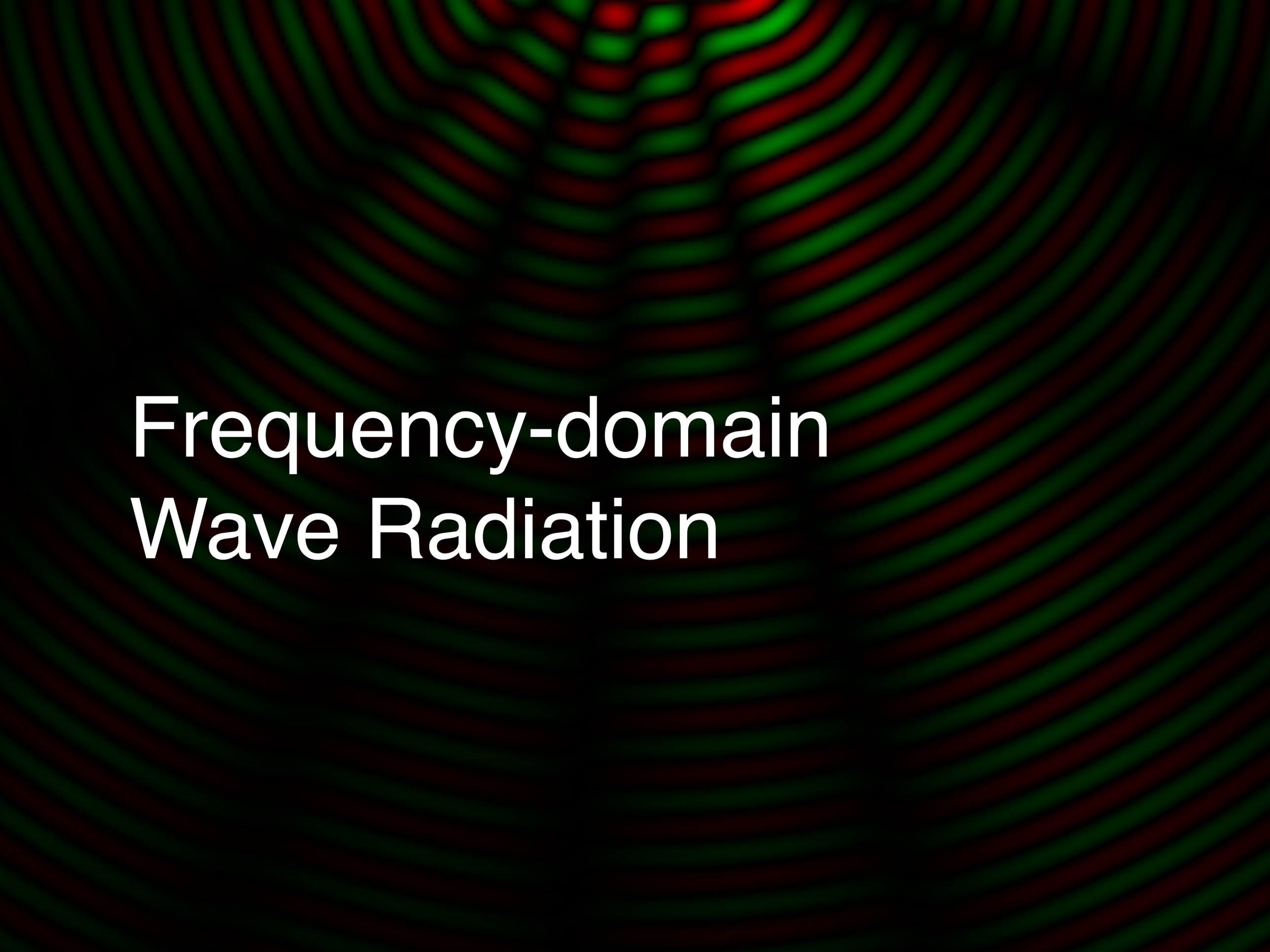
and $\rho = 1.2041\ kg/m^3$ is air density at STP.

Radiation boundary condition (2)

Need condition at infinity.

- So-called *Sommerfeld* radiation condition.
- Solutions must decay at infinity and correspond to out-going waves.
- Avoids nonphysical solutions, such as infinite sources at infinity.

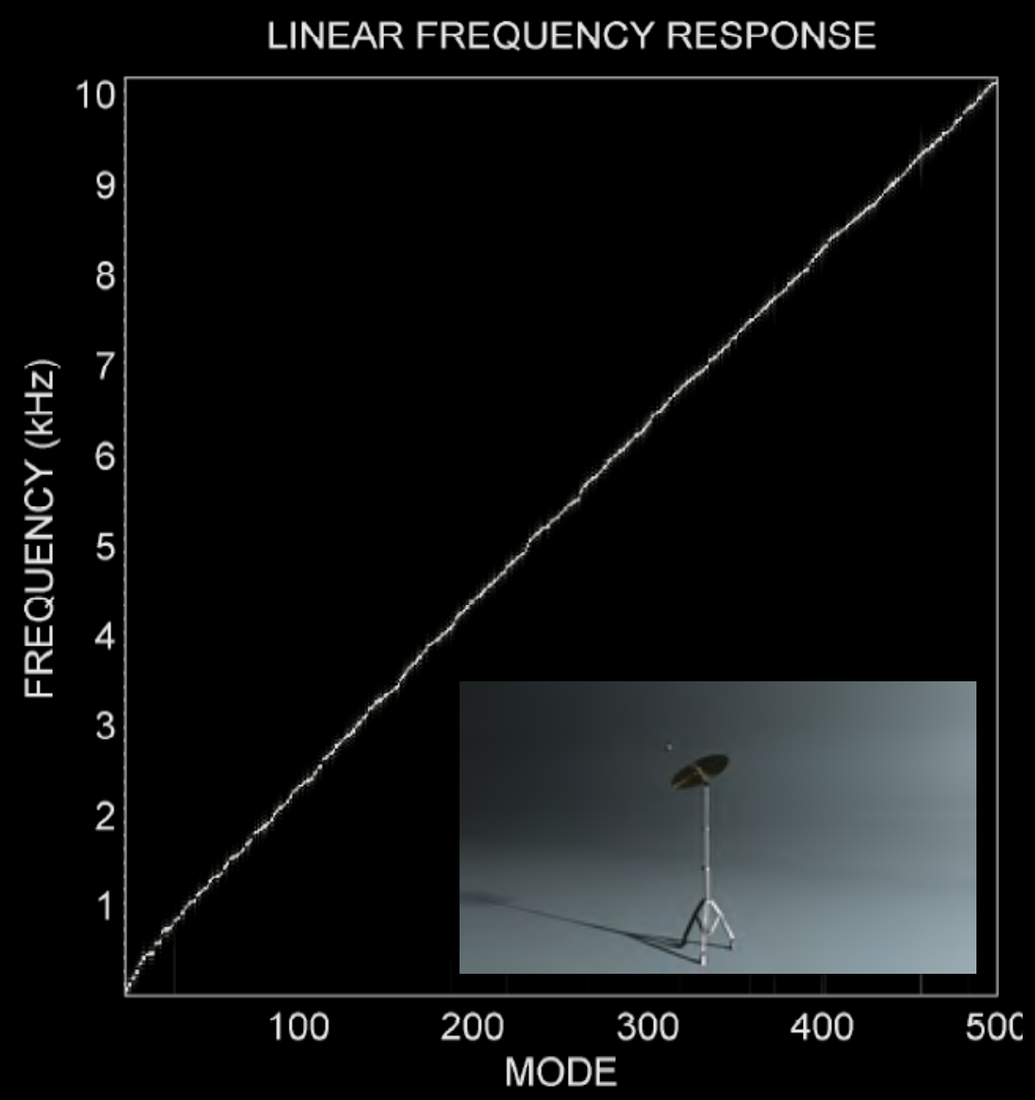




Frequency-domain Wave Radiation

Frequency-domain Wave Radiation

Motivation: Individual mode vibrations are localized in frequency domain



Impulse response of a cymbal impact modeled using a linear modal vibration model.

Math Flashback

Using complex numbers to represent oscillations

By linearity of the wave equation & BCs, sufficient to consider a unit amplitude harmonic vibration at natural frequency ω,

$$e^{+i\omega t} = \cos(\omega t) + i\,\sin(\omega t)$$

Different phases introduced by $e^{+i\phi}$ factors since

$$e^{+i\omega t}e^{i\phi} = \cos(\omega t + \phi) + i\,\sin(\omega t + \phi)$$

Can think of $q(t)$ as an amplitude and a unit complex part.

Harmonic pressure oscillations

Consider harmonic pressure solutions of the form

$$p(\boldsymbol{x}, t) = p(\boldsymbol{x}) e^{+i\omega t}$$

where $p(\boldsymbol{x})$ is a complex-valued pressure field—will be the *acoustic transfer function*.

Using Euler's formula, we can write $p(\boldsymbol{x}) = |p| e^{i\phi}$ so that

$$\begin{aligned} p(\boldsymbol{x}, t) &= p(\boldsymbol{x})\, e^{+i\omega t} = |p(\boldsymbol{x})|\, e^{+i(\omega t + \phi)} \\ &= |p(\boldsymbol{x})| \cos(\omega t + \phi) + i\, |p(\boldsymbol{x})| \sin(\omega t + \phi). \end{aligned}$$

The Helmholtz equation

(the frequency-domain wave equation)

Substituting $p(\boldsymbol{x}, t) = p(\boldsymbol{x}) e^{+iwt}$ into the wave equation yields

$$0 = \nabla^2 p(\boldsymbol{x})\, e^{+i\omega t} - \frac{1}{c^2}\, p(\boldsymbol{x}) \left(\frac{\partial^2}{\partial t^2}\, e^{+i\omega t} \right)$$

$$= \left(\nabla^2 p(\boldsymbol{x}) - \frac{\omega^2}{c^2} p(\boldsymbol{x}) \right) e^{+i\omega t}$$

Never zero

and so

$$\boxed{\nabla^2 p(\boldsymbol{x}) + k^2\, p(\boldsymbol{x}) = 0} \quad (\textit{Helmholtz eqn})$$

where k is the wavenumber,

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

Radiation boundary conditions

Recall time-domain Neumann BC:

$$\boxed{\frac{\partial p}{\partial n} = -\rho\, a_n(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \Gamma}$$

What is frequency-domain BC?

What is $a_n(\boldsymbol{x}, t)$?

Radiation boundary conditions

Need harmonic surface acceleration $a_n(\boldsymbol{x}, t)$ in terms of eigenmode displacement, $\bar{\mathbf{u}}(\boldsymbol{x})$:

$$\mathbf{u}(\boldsymbol{x}, t) = \bar{\mathbf{u}}(\boldsymbol{x})\, e^{+i\omega t} \qquad \Longrightarrow \qquad u_n = \bar{u}_n e^{+i\omega t}$$

$$\boldsymbol{v}(\boldsymbol{x}, t) = \frac{d\mathbf{u}}{dt}(\boldsymbol{x}, t) = i\omega\, \bar{\mathbf{u}}(\boldsymbol{x})\, e^{+i\omega t} \qquad \Longrightarrow \qquad v_n = i\omega \bar{u}_n e^{+i\omega t}$$

$$\boldsymbol{a}(\boldsymbol{x}, t) = \frac{d^2\mathbf{u}}{dt^2}(\boldsymbol{x}, t) = (i\omega)^2\, \bar{\mathbf{u}}(\boldsymbol{x})\, e^{+i\omega t} \qquad \Longrightarrow \qquad a_n = -\omega^2 \bar{u}_n e^{+i\omega t}$$

$$\boxed{a_n = -\omega^2 \bar{u}_n e^{+i\omega t}}$$

Frequency-domain Wave Radiation

Radiation boundary conditions

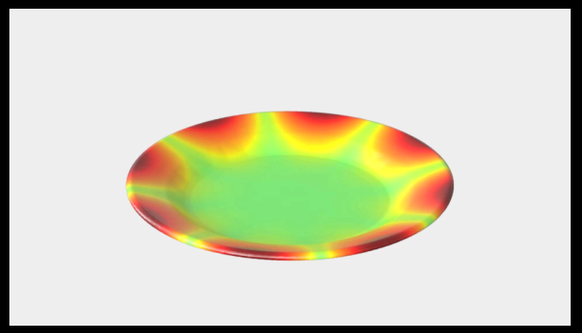
The Neumann boundary condition becomes

$$\frac{\partial p(\boldsymbol{x}, t)}{\partial n} = -\rho\, a_n(\boldsymbol{x}, t)$$

Never zero

$$\frac{\partial p(\boldsymbol{x})}{\partial n} \cancel{e^{+i\omega t}} = \rho\, \omega^2\, \bar{u}_n\, \cancel{e^{+i\omega t}}$$

$$\boxed{\frac{\partial p(\boldsymbol{x})}{\partial n} = \rho\, \omega^2\, \bar{u}_n(\boldsymbol{x})}$$



(+Sommerfeld radiation BC: Out-going decaying waves.)

Frequency-domain Wave Radiation

Acoustic Transfer BVP

Given mode data $\bar{u}_n(\boldsymbol{x})$ and frequency ω.

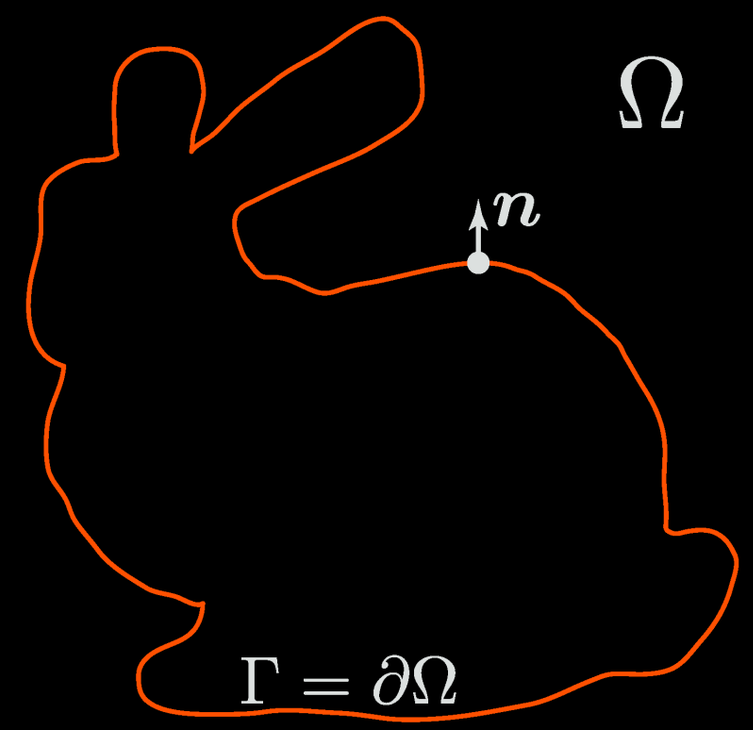
Solve exterior Helmholtz eqn

$$(\nabla^2 + k^2)p(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \Omega$$

subject to BC:

$$\frac{\partial p(\boldsymbol{x})}{\partial n} = \rho\,\omega^2\,\bar{u}_n(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma$$

(+Sommerfeld radiation BC.)



Frequency-domain Wave Radiation

Visualizing solutions

Can visualize solutions by looking at real part of

$$p(\boldsymbol{x})e^{+i\omega t}$$

or the AT amplitude

$$|p(\boldsymbol{x})|$$

MODE 01

MODE 02

MODE 03

MODE 04


MODE 05

MODE 06

MODE 07

MODE 08


MODE 09


MODE 10


MODE 11


MODE 12


MODE 13


MODE 14


MODE 15


MODE 16


MODE 17


MODE 18


MODE 19


MODE 20


MODE 21


MODE 22


MODE 23


MODE 24


MODE 25


MODE 26


MODE 27


MODE 28


MODE 29


MODE 30


MODE 31


MODE 32


MODE 33


MODE 34


MODE 35

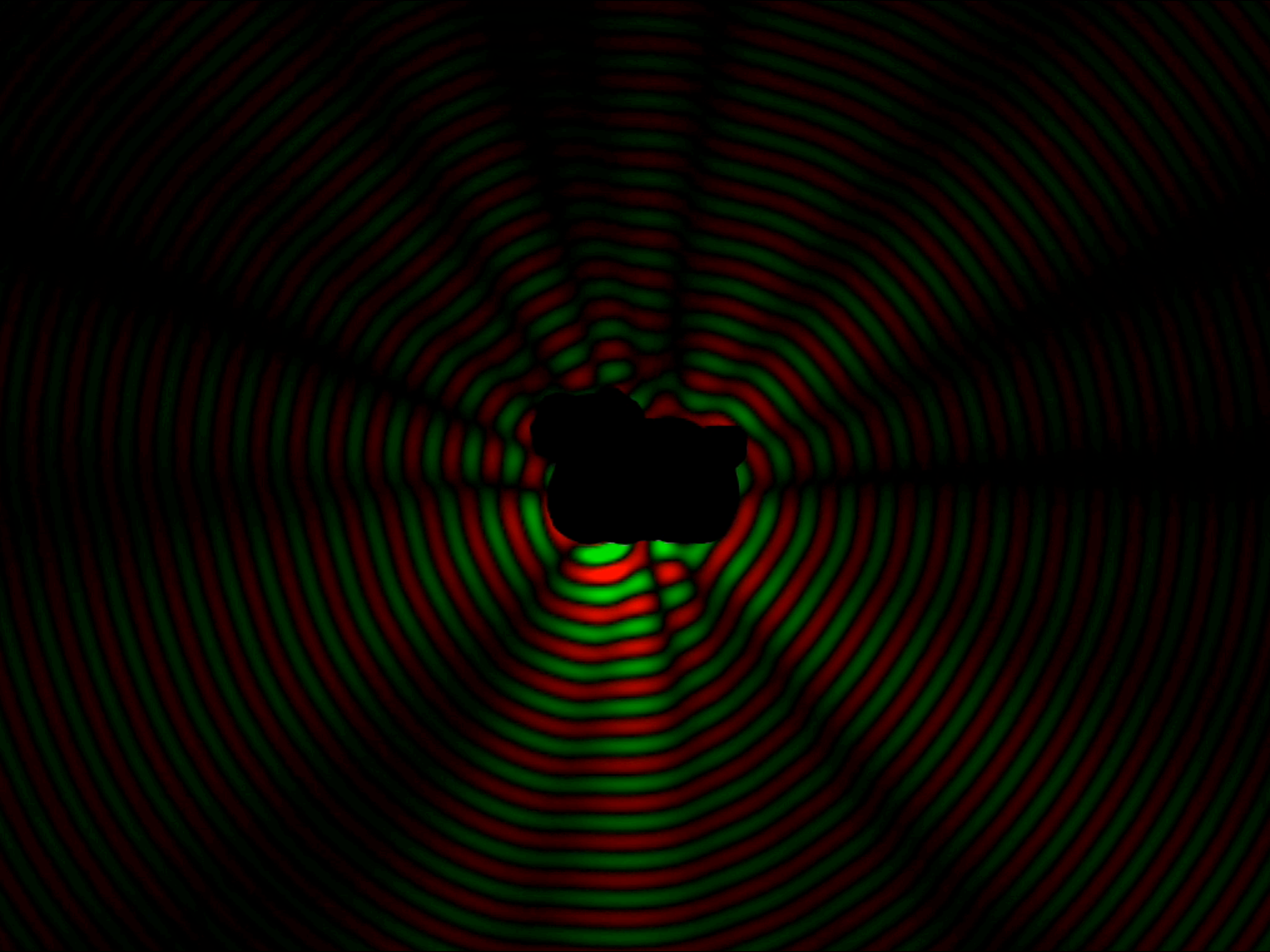

MODE 36


MODE 37


MODE 38


MODE 39


MODE 40



Including Source Terms

3D Wave Equation (without sources)

Acoustic pressure, $p(\boldsymbol{x}, t)$, satisfies

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0 \quad \text{(\textit{Wave Equation})}$$

subject to boundary conditions, e.g., vibrations.

What about RHS source terms?

Re-deriving the wave equation

(see [Howe 2003])

- Linearized continuity eqn (mass conservation)

$$\frac{1}{\rho_0}\frac{\partial \tilde{\rho}}{\partial t} + \nabla \cdot \mathbf{v} = \underbrace{q}_{\text{Volume source}}(\mathbf{x}, t)$$

- Linearized momentum eqn ("ma=f")

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} + \nabla p = \underbrace{\mathbf{F}}_{\text{Force density}}(\mathbf{x}, t)$$

- Energy eqn (relates acoustic pressure and density at constant entropy)

$$p = \left(\frac{\partial p}{\partial \rho}\right)_0 \underbrace{\tilde{\rho}}_{\text{Acoustic density fluctuation}} = c^2 \, \tilde{\rho}$$

Acoustic Waves

3D Wave Equation (with sources)

Acoustic pressure, $p(\boldsymbol{x}, t)$, satisfies

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) p = \rho \frac{\partial q}{\partial t} - \nabla \cdot \mathbf{F}$$

subject to boundary conditions, e.g., vibrations.

- Can be hard to model $\mathbf{F}$.

Solve for p. (what about $\tilde{\rho}$ and $\mathbf{v}$?)

Velocity Potential

A useful trick

Velocity Potential, φ

Scalar velocity potential, $\varphi(\mathbf{x}, t) \in \mathbb{R}$ satisfies

$$\mathbf{v} = \nabla \varphi$$

Exists when $\mathbf{F} = 0$, due to momentum eqn:

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p$$

$\Longrightarrow$ changes in $\mathbf{v}$ are gradient of something.

Useful trick! Knowing φ gives you *both* $\mathbf{v}$ and p:

$$\mathbf{v} = \nabla \varphi \quad \text{and} \quad p = -\rho \frac{\partial \varphi}{\partial t}$$

3D Wave Equation using $\varphi(\mathbf{x}, t)$

Wave equation with $\mathbf{F} = 0$, i.e., only volume srcs:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) p = \rho \frac{\partial q}{\partial t}$$

Substitute $p = -\rho \dfrac{\partial \varphi}{\partial t}$ and remove one $\dfrac{\partial}{\partial t}$:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) \varphi = -q(\mathbf{x}, t)$$

Limiting case of an Incompressible Fluid

Incompressible limit ($c \to \infty$)

Since the speed of sound is related to small pressure and density fluctuations:

$$c^2 \approx \frac{\delta p}{\delta \rho} \xrightarrow{\delta\rho \to 0} \infty$$

In this limit, the wave equation (using φ) simplifies to

$$\nabla^2 \varphi = q(\mathbf{x}, t)$$

where the volume source is $q = \nabla \cdot \mathbf{v}$

Incompressible example (Howe 1.4.1)

Pulsating Sphere (radius a)

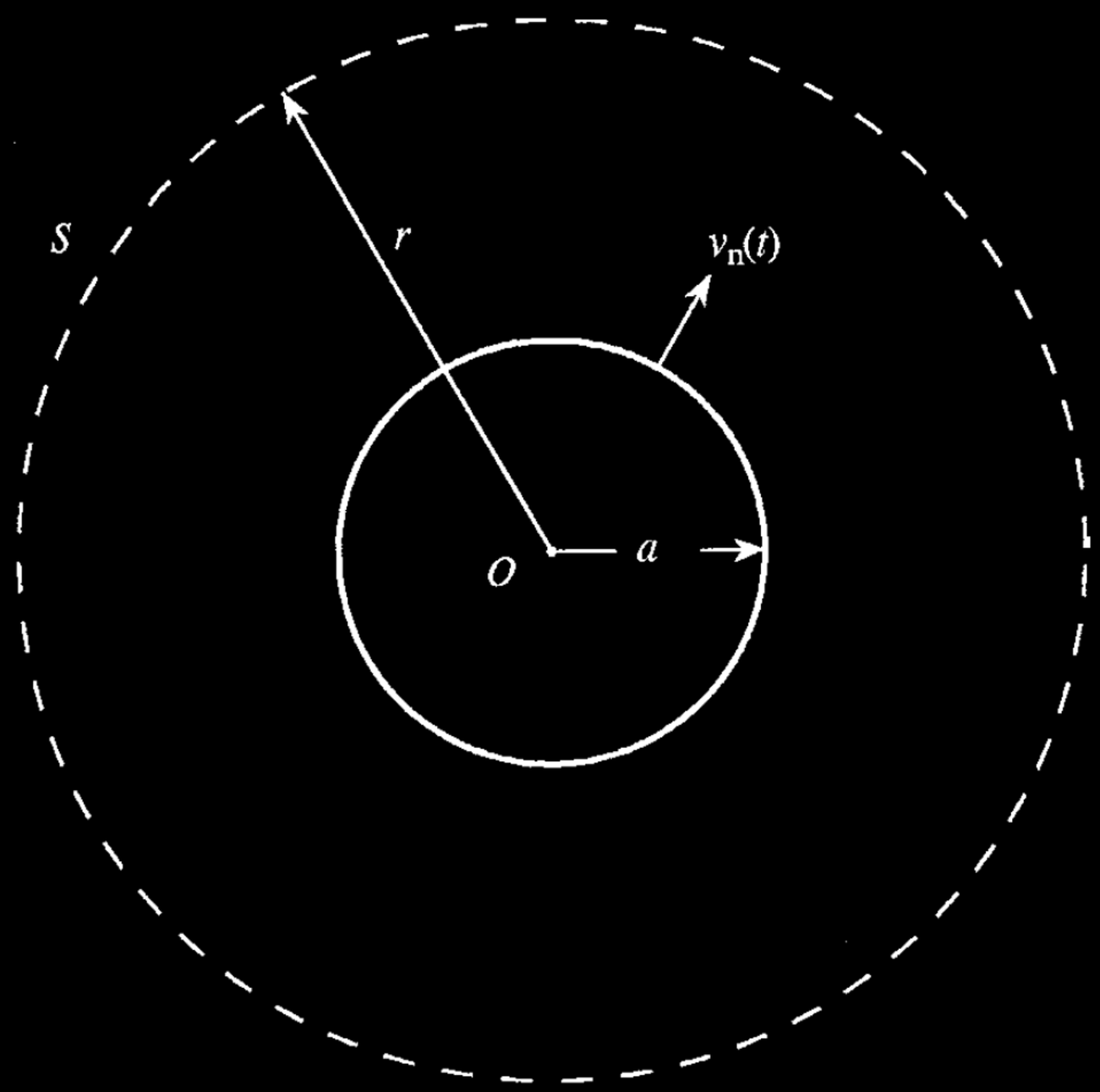


Fig. 1.4.1.

Incompressible example (Howe 1.4.1)

Pulsating Sphere (radius a)

$$\left.\begin{aligned} \nabla^2\varphi &= 0, & r &> a, \\ \partial\varphi/\partial r &= v_{\mathrm{n}}(t), & r &= a \end{aligned}\right\} \quad \text{where } r = |\mathbf{x}|.$$

The motion is obviously radially symmetric, so that

$$\nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\varphi = 0, \quad r > a.$$

Hence,

$$\varphi = \frac{A}{r} + B,$$

Incompressible example (Howe 1.4.1)

Pulsating Sphere (radius a)

Applying the BC ($\partial\varphi/\partial r = v_{\mathrm{n}}(t), \quad r = a$):

$$\varphi = -\frac{a^2 v_{\mathrm{n}}(t)}{r}, \quad r > a.$$

$$p = -\rho_0 \frac{\partial\varphi}{\partial t} = \rho_0 \frac{a^2}{r} \frac{dv_{\mathrm{n}}}{dt}(t)$$

Incompressible example (Howe 1.4.1)

Pulsating Sphere (radius a)

Volume flux (or "volume velocity") of source:

For any time t, the volume flux $q(t)$ of fluid is the same across any closed surface enclosing the sphere. Evaluating it for any sphere S of radius $r > a$, as shown in Fig. 1.4.1, we find

$$q(t) = \oint_S \nabla\varphi \cdot d\mathbf{S} = 4\pi a^2 v_{\mathrm{n}}(t),$$

and we may also write

$$\varphi = \frac{-q(t)}{4\pi r}, \quad r > a.$$

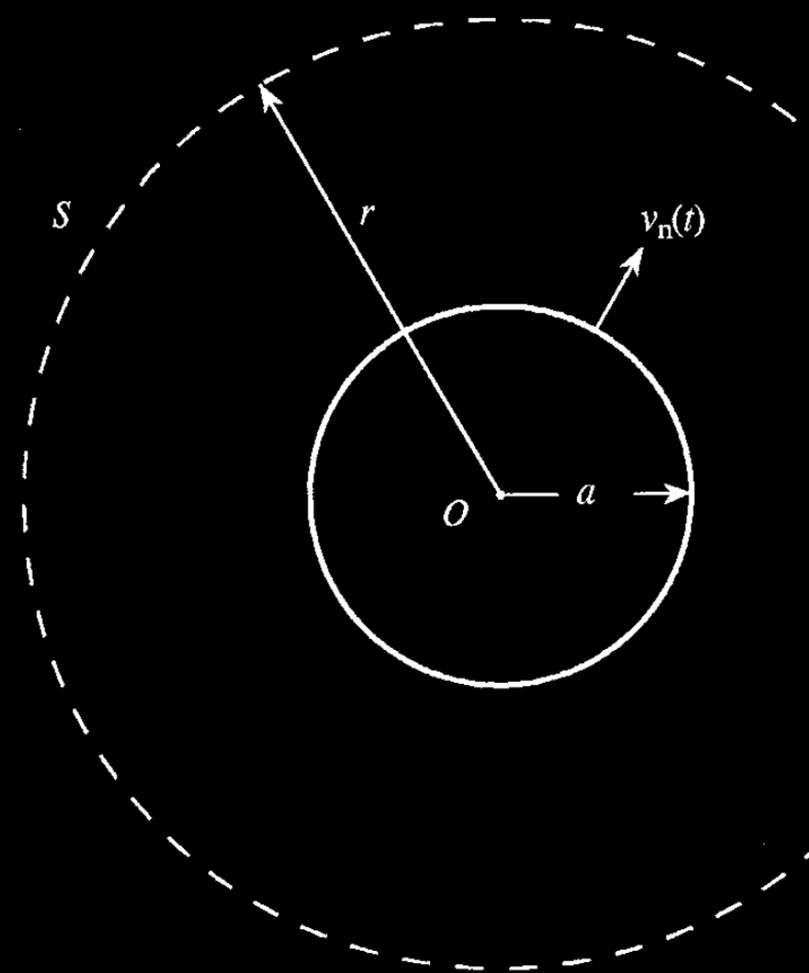
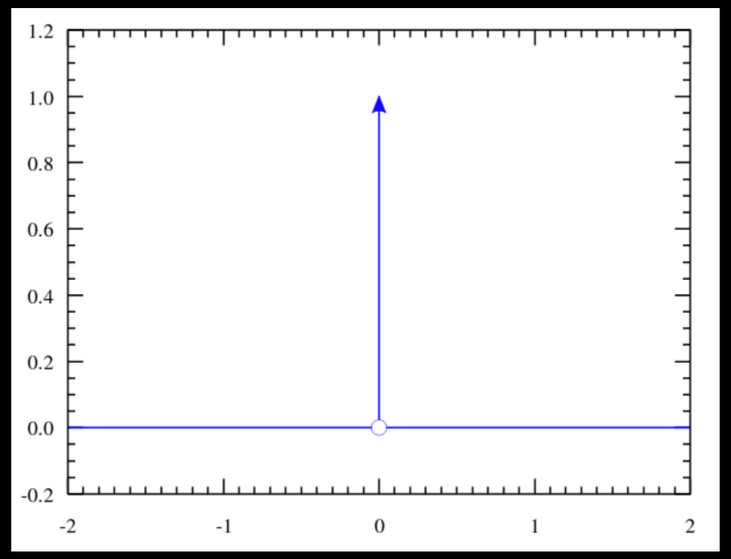


Fig. 1.4.1.

Math Background

Dirac Delta Function

https://en.wikipedia.org/wiki/Dirac_delta_function



The Dirac delta can be loosely thought of as a function on the real line which is zero everywhere except at the origin, where it is infinite,

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

and which is also constrained to satisfy the identity

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

Useful property: $$\int_{-\infty}^{\infty} f(t)\delta(t-T)\, dt = f(T)$$

Incompressible case (Howe 1.4.2)

Point Source

A volume point source of strength $q(t)$ at the origin satisfies a related problem:

$$\nabla^2 \varphi = q(t)\delta(\mathbf{x})$$

where $\delta(\mathbf{x}) = \delta(x_1)\delta(x_2)\delta(x_3)$.

Gives the solution:

$$\varphi(\mathbf{x}, t) = -\frac{q(t)}{4\pi r}$$

(agrees with pulsating sphere for $r > a$)

Sound Produced by an Impulsive Point Source

Wave equation (Howe 1.5)

Impulsive Point Source

Consider an impulsive ("pop") event at $t = 0$, $\mathbf{x} = 0$:

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\varphi = \delta(\mathbf{x})\delta(t)$$

Has two types of solutions (where $r = \|\mathbf{x}\|$):

$$\varphi(\mathbf{x}, t) = \frac{1}{4\pi r}\,\delta\left(t - \frac{r}{c}\right) \qquad \text{(outgoing)}$$

and

$$\varphi(\mathbf{x}, t) = \frac{1}{4\pi r}\,\delta\left(t + \frac{r}{c}\right) \qquad \text{(incoming)}$$

Wave equation (Howe 1.5)

Impulsive Point Source

We only consider the *causal* solution

$$\varphi(\mathbf{x}, t) = \frac{1}{4\pi r}\,\delta\left(t - \frac{r}{c}\right)$$

Represents a spherical pulse that is only nonzero about $r = ct > 0$.

Wave equation (Howe 1.6)

Free-space Green's Function

Physical solution due to "pop" event at $t = \tau$, $\mathbf{x} = \mathbf{y}$:

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) G = \delta(\mathbf{x} - \mathbf{y})\delta(t - \tau),$$

Gives the solution:

$$G(\mathbf{x}, \mathbf{y}, t - \tau) = \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|}\,\delta\left(t - \tau - \frac{|\mathbf{x} - \mathbf{y}|}{c_0}\right)$$

This represents an impulsive, spherically symmetric wave expanding from the source at $\mathbf{y}$ at the speed of sound. The wave amplitude decreases inversely with distance $|\mathbf{x} - \mathbf{y}|$ from the source point $\mathbf{y}$.

Wave equation (Howe 1.6)

Free-space Green's Function

Gives general solution to wave equation (φ or p form)

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) p = \mathcal{F}(\mathbf{x}, t)$$

using linear superposition of sources

$$\mathcal{F}(\mathbf{x}, t) = \iint_{-\infty}^{\infty} \mathcal{F}(\mathbf{y}, \tau)\, \underbrace{\delta(\mathbf{x} - \mathbf{y})\, \delta(t - \tau)}_{\text{Same source as Green's function}}\, d^3\mathbf{y}\, d\tau$$

Wave equation (Howe 1.6)

Free-space Green's Function

Source

$$\delta(\mathbf{x}-\mathbf{y})\,\delta(t-\tau)$$

$$\mathscr{F}(\mathbf{y},\tau)\,\delta(\mathbf{x}-\mathbf{y})\,\delta(t-\tau)$$

$$\iint_{-\infty}^{\infty} \mathscr{F}(\mathbf{y},\tau)\,\delta(\mathbf{x}-\mathbf{y})\,\delta(t-\tau)\,d^3\mathbf{y}\,d\tau$$

$$= \mathscr{F}(\mathbf{x},t)$$

Causal Solution

$$G(\mathbf{x},\mathbf{y},t-\tau)$$

$$\mathscr{F}(\mathbf{y},\tau)\,G(\mathbf{x},\mathbf{y},t-\tau)$$

$$\iint_{-\infty}^{\infty} \mathscr{F}(\mathbf{y},\tau)\,G(\mathbf{x},\mathbf{y},t-\tau)\,d^3\mathbf{y}\,d\tau$$

$$= \frac{1}{4\pi}\iint_{-\infty}^{\infty} \frac{\mathscr{F}(\mathbf{y},\tau)}{\|\mathbf{x}-\mathbf{y}\|}\,\delta\left(t-\tau-\frac{\|\mathbf{x}-\mathbf{y}\|}{c}\right)\,d^3\mathbf{y}\,d\tau$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\mathscr{F}\left(\mathbf{y},\frac{\|\mathbf{x}-\mathbf{y}\|}{c}\right)}{\|\mathbf{x}-\mathbf{y}\|}\,d^3\mathbf{y}$$

Wave equation (Howe 1.6)

Free-space Green's Function

So, the general solution to wave equation

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) p = \mathcal{F}(\mathbf{x}, t)$$

is given by

$$p(\mathbf{x}, t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\mathcal{F}\left(\mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right)}{|\mathbf{x}-\mathbf{y}|}\, d^3\mathbf{y}$$

Wave equation (Howe 1.6)

Approximating the solution

$$p(\mathbf{x}, t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\mathcal{F}\left(\mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right)}{|\mathbf{x} - \mathbf{y}|} d^3\mathbf{y}$$

Difficult to evaluate integral in general, hence:

- Special solutions (monopoles, dipoles, etc.)
- Far-field approximations (Fraunhofer)
- Numerical integration

Major limitation:

- No boundaries (no reflections, no surface radiation)
  - Green's function extension for acoustically compact shapes
  - Need general numerical solvers (FEM, FDM, BEM, etc.)

Point Sources

Time-Domain Monopoles, Dipoles, ...

RECALL

Incompressible case (Howe 1.4.2)

Point Source

A volume point source of strength $q(t)$ at the origin satisfies a related problem:

$$\nabla^2 \varphi = q(t)\delta(\mathbf{x})$$

where $\delta(\mathbf{x}) = \delta(x_1)\delta(x_2)\delta(x_3)$.

Gives the solution:

$$\varphi(\mathbf{x}, t) = -\frac{q(t)}{4\pi r}$$

(agrees with pulsating sphere for $r > a$)

Compressible case (Howe 1.7)

Time-Domain Monopole Solution

A volume point source of strength $q(t)$ at the origin satisfies a related problem:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\varphi = -q(t)\delta(\mathbf{x})$$

Gives the solution:

Time delay

$$\varphi(\mathbf{x}, t) = -\frac{q\left(t - \frac{r}{c}\right)}{4\pi r}$$

(same as incompressible case when $c \to \infty$)

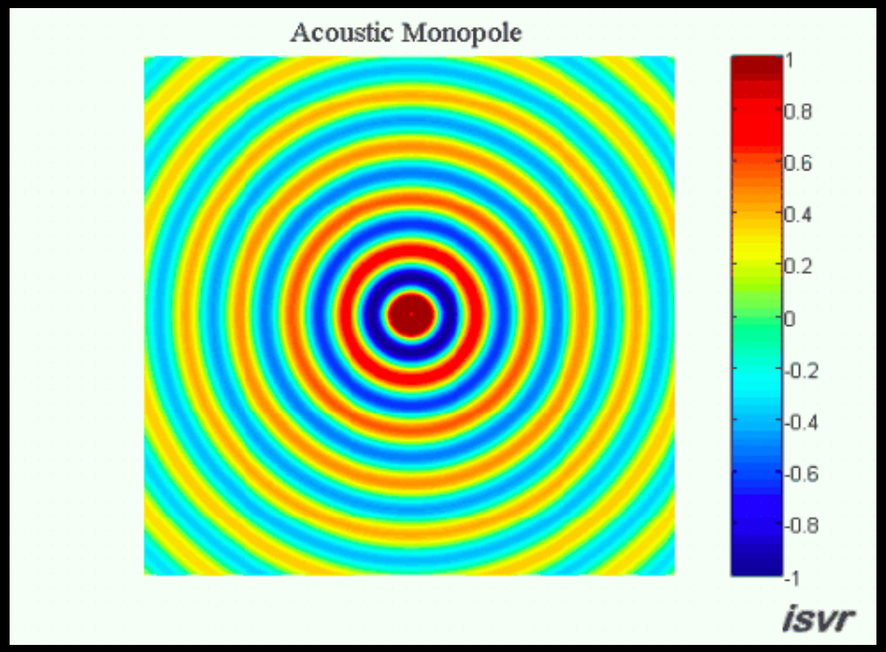
Compressible case (Howe 1.7)

Time-Domain Monopole Solution

Visualization of the solution:

$$\varphi(\mathbf{x}, t) = -\frac{q\left(t - \frac{r}{c}\right)}{4\pi r}$$

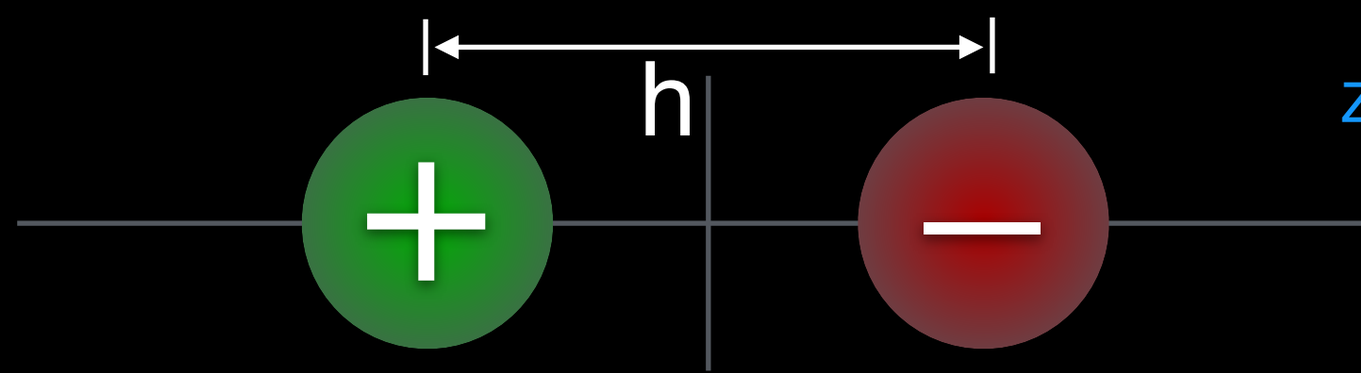
for $q(t) = sin(t)$.



https://blog.soton.ac.uk/soundwaves/wave-basics/point-sources-inverse-square-law/

Compressible case (Howe 1.7)

Dipole as the limit of two monopoles



Zero net volume change.

Consider two equal & opposite monopole sources:

$$\frac{1}{h}\, q(t)\, \delta(x - \frac{h}{2})\, \delta(y)\, \delta(z) - \frac{1}{h}\, q(t)\, \delta(x + \frac{h}{2})\, \delta(y)\, \delta(z)$$

$$= -q(t)\, \delta'(x)\, \delta(y)\, \delta(z)$$

$$= -\frac{\partial}{\partial x}\Big(q(t)\, \delta(x)\, \delta(y)\, \delta(z)\Big)$$

$$= -\frac{\partial}{\partial x}\Big(q(t)\, \delta(\mathbf{x})\Big) = \text{Gradient of monopole source}$$

Solution?

Useful Trick

Differentiate to get other solutions

Given solution $\varphi(\mathbf{x}, t)$ for some RHS $b(\mathbf{x}, t)$,

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\varphi = b$$

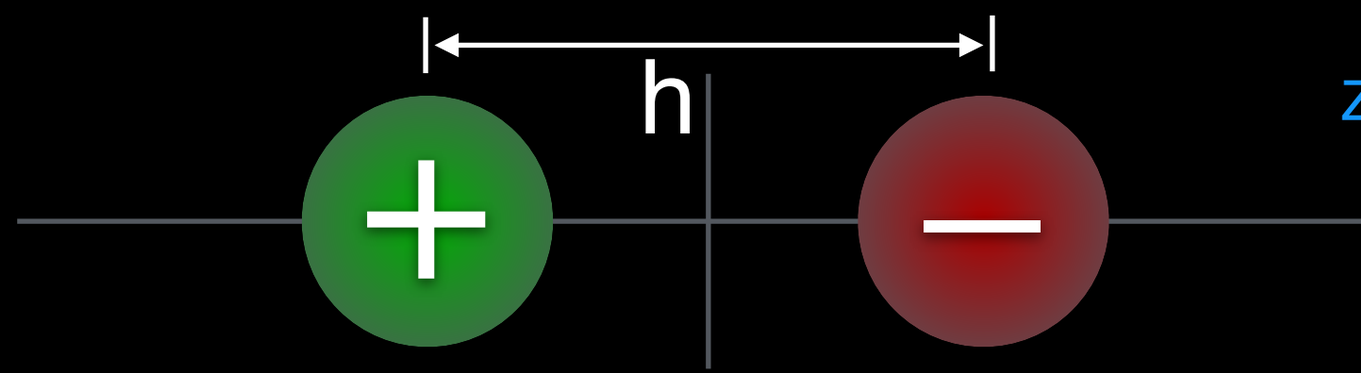
then $\mathscr{D}\varphi$ is the wave solution to RHS $\mathscr{D}b$ since

$$\mathscr{D}\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\varphi = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathscr{D}\varphi = \mathscr{D}b$$

for a constant-coefficient differential operator $\mathscr{D}$, e.g., a directional derivative $(\hat{\mathbf{n}} \cdot \nabla)$.

Compressible case (Howe 1.7)

Dipole as the limit of two monopoles



Zero net volume change.

$$RHS = -\frac{\partial}{\partial x}\left(q(t)\,\delta(\mathbf{x})\right) = \text{Gradient of monopole source}$$

so solution must be monopole gradient:

$$\varphi(\mathbf{x}, t) = \frac{\partial}{\partial x}\varphi_{monopole}(\mathbf{x}, t) = \frac{\partial}{\partial x}\left[-\frac{q\left(t - \frac{r}{c}\right)}{4\pi r}\right]$$

Dipole example (Howe 1.7.3)

Vibrating Sphere

Small sphere radius: a.

Vibrating along $\hat{\mathbf{x}}$.

Speed: $\dot{x}(t) = U(t)$

Equivalent dipole strength:

$$q(t) = 2\pi a^3 U(t)$$

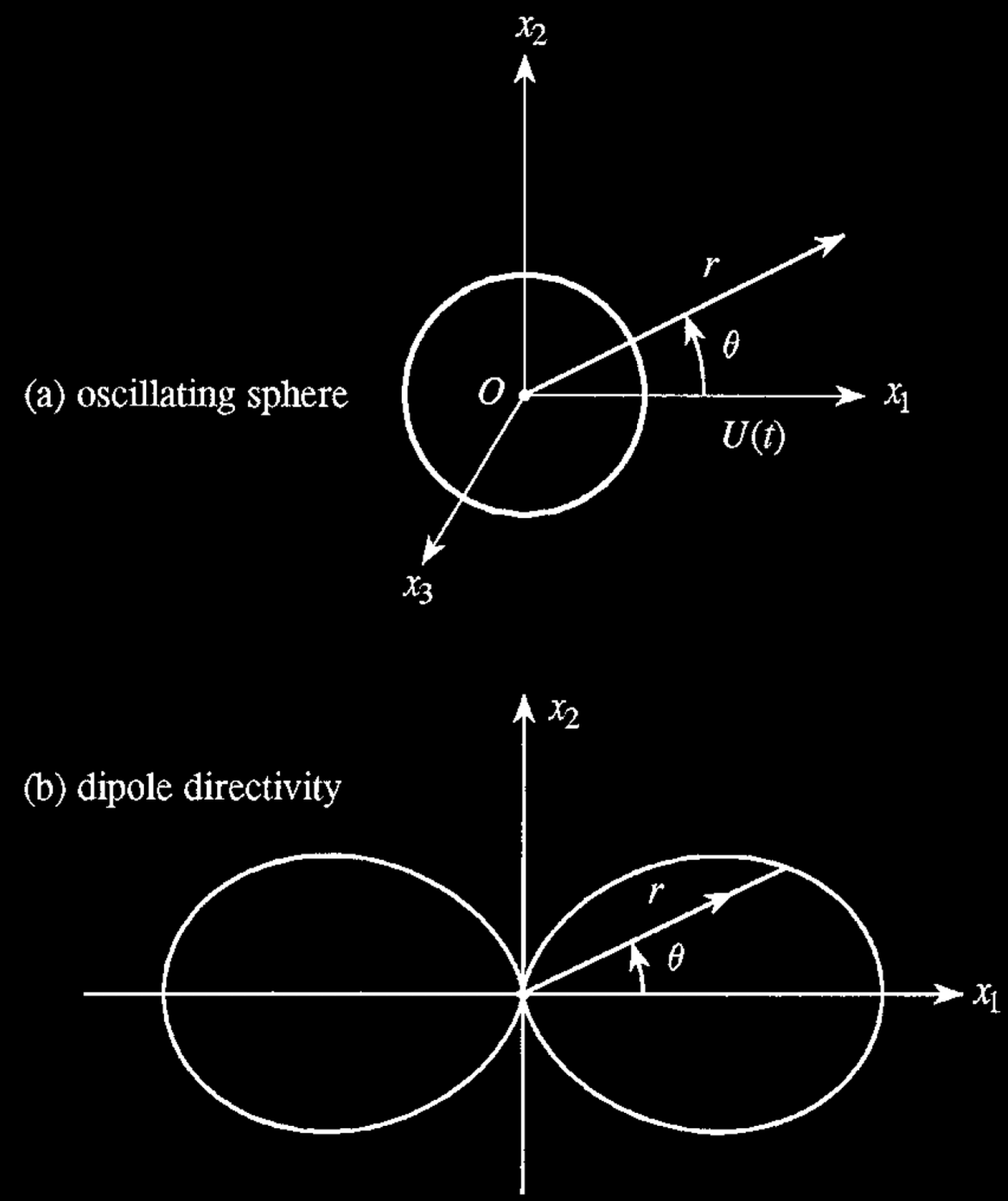
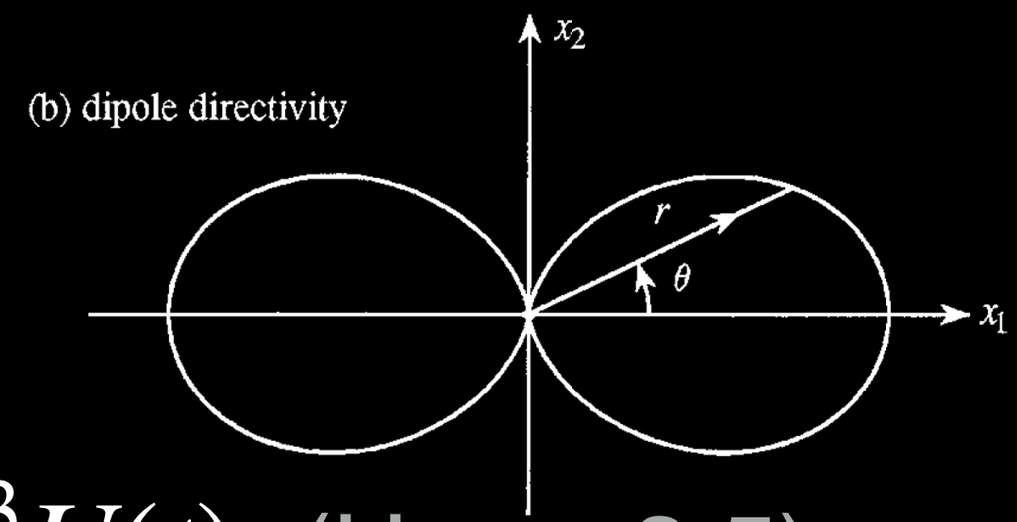


Fig. 1.7.1.

Dipole example (Howe 1.7.3)

Vibrating Sphere



Equivalent dipole: $q(t) = 2\pi a^3 U(t)$ (Howe 3.5)

Solution (using derivative trick):

$$\varphi(\mathbf{x}, t) = \frac{\partial}{\partial x_1}\left(\frac{2\pi a^3 U\left(t - \frac{|\mathbf{x}|}{c_0}\right)}{4\pi |\mathbf{x}|}\right)$$

Since $r = \|\mathbf{x}\|$ and $x_1 = r\cos\theta$, it follows that

$$\varphi = \underbrace{-\frac{a^3\cos\theta}{2r^2}U\left(t - \frac{r}{c_0}\right)}_{\text{near field}} \underbrace{- \frac{a^3\cos\theta}{2c_0 r}\frac{\partial U}{\partial t}\left(t - \frac{r}{c_0}\right)}_{\text{far field}}$$

Dipole example (Howe 1.7.3)

Vibrating Sphere

$$\varphi = -\frac{a^3 \cos\theta}{2r^2} U\left(t - \frac{r}{c_0}\right) - \frac{a^3 \cos\theta}{2c_0 r} \frac{\partial U}{\partial t}\left(t - \frac{r}{c_0}\right)$$

near field

far field

The near-field term is dominant at sufficiently small distances r from the origin such that

$$\frac{1}{r} \gg \frac{1}{c_0}\frac{1}{U}\frac{\partial U}{\partial t} \sim \frac{f}{c_0},$$

where f is the characteristic frequency of the oscillations of the sphere. But, sound of frequency f travels a distance

$$c_0/f = \lambda \equiv \text{one acoustic wavelength}$$

in one period of oscillation $1/f$. Hence, the near-field term is dominant when

$$r \ll \lambda.$$

Dipole example (Howe 1.7.3)

Vibrating Sphere

$$\varphi = -\frac{a^3 \cos\theta}{2r^2} U\left(t - \frac{r}{c_0}\right) - \frac{a^3 \cos\theta}{2c_0 r} \frac{\partial U}{\partial t}\left(t - \frac{r}{c_0}\right)$$

near field

far field

Far-field radiation ($r \gg \lambda$):

- involves time-derivative of $q(t)$.
- disappears as $c \to \infty$ (incompressible limit).

Acoustic compactness: Dipole model implicitly assumes that $a \ll \lambda$, i.e., the sphere is *acoustically compact:* characteristic dimensions are smaller than wavelengths occurring.

Dipole example (Howe 1.7.3)

Vibrating Sphere

$$\varphi = -\frac{a^3 \cos\theta}{2r^2} U\left(t - \frac{r}{c_0}\right) - \frac{a^3 \cos\theta}{2c_0 r} \frac{\partial U}{\partial t}\left(t - \frac{r}{c_0}\right)$$

near field

far field

Far-field pressure solution:

$$p(\mathrm{x}, t) = -\rho \frac{\partial \varphi}{\partial t}$$

$$\approx \frac{\rho a^3 \cos\theta}{2cr} \frac{\partial^2 U}{\partial t^2}\left(t - \frac{r}{c}\right), \quad r = \|\mathbf{x}\| \to \infty\,.$$

So, we hear the "jerk" of the sphere ;)