Klein bottle for rent – inquire within.
— Anonymous

2 Surface Topology

Last lecture, we spent a considerable amount of effort defining *manifolds*. We like manifolds because they are locally Euclidean. So, even though it is hard for us to reason about them globally, we know what to do in small neighborhoods. It turns out that this ability is all we really need. This is rather fortunate, because we suddenly have spaces with more interesting structure than the Euclidean spaces to study.

Recall that topology, like Euclidean geometry, is a study of the properties of spaces that remain invariant (do not change) under a fixed set of transformations. In topology, we expand the transformations that are allowed from rigid motions (Euclidean geometry) to *homeomorphisms*: bijective bi-continuous maps. In this lecture, we ask whether we may classify manifolds under this set of transformations, and we see that such a classification is possible for two-dimensional manifolds or *surfaces*.

2.1 Topological Type

To begin with, we should indicate what we mean by a *classification*. This notion has a nice mathematical definition, which you may have seen in high school.

Definition 2.1 (partition) A *partition of a set* is a decomposition of the set into subsets (*cells*) such that every element of the set is in one and only one of the subsets.

We wish to partition the set of manifolds according to their connectivity. We are forced to look at different partitioning schemes in our search for one which is computationally feasible. Each scheme depends on an equivalence relation.

Definition 2.2 (equivalence) Let S be a nonempty set and let $\sim$ be a relation between elements of S that satisfies the following properties for all $a, b, c \in S$:

1. (Reflexive) $a \sim a$.
2. (Symmetric) If $a \sim b$, then $b \sim a$.
3. (Transitive) If $a \sim b$ and $b \sim c$, $a \sim c$.

Then, the relation $\sim$ is an *equivalence relation* on S.

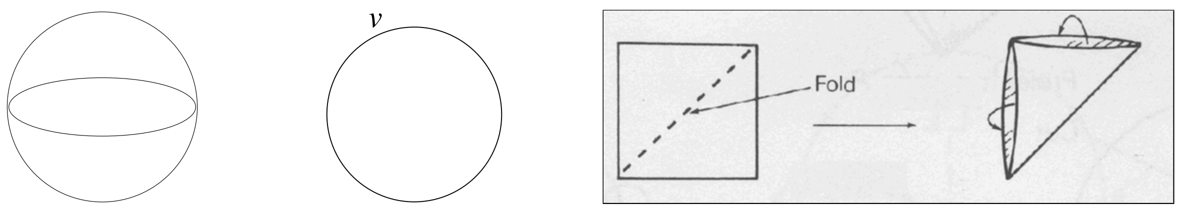
It is clear from the definition of homeomorphism that it is an equivalence relation. The following theorem allows us to derive a partition from an equivalence relation.

Theorem 2.1 *Let S be a nonempty set and let $\sim$ be an equivalence relation on S. Then, $\sim$ yields a natural partition of S, where $\bar{a} = \{x \in S \mid x \sim a\}$. $\bar{a}$ represents the subset to which a belongs to. Each cell $\bar{a}$ is an* equivalence class.

As homeomorphism is an equivalence relation, we may use it to partition manifolds by this theorem. If two manifolds are placed in the same subset, they are connected the same way, and we say that they have the same *topological type*. One of the fundamental questions in topology is whether this partition is computable. In this lecture, we focus on the solution to this problem in two dimensions.

2.2 Basic 2-Manifolds

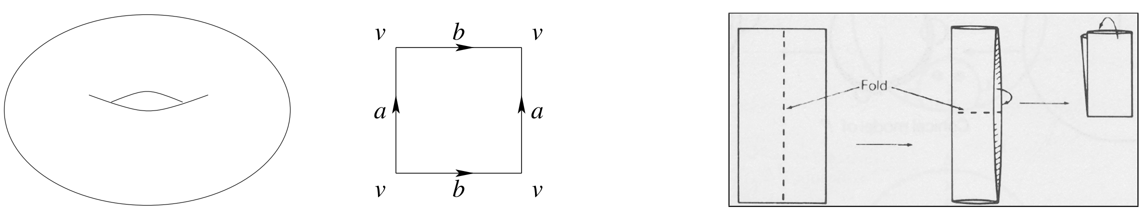
Before classifying 2-manifolds, however, it would be nice to meet a few of them. In this section, we look at a few basic 2-manifolds.



(a) $\{x \in \mathbb{R}^3 \mid |x| = 1\}$ (b) Identify boundary to v (c) Instructions for a flat sphere

Figure 1. The sphere $\mathbb{S}^2$

The sphere. Topologically, the sphere $\mathbb{S}^2$ is the simplest surface. We are most comfortable with the implicit surface definition in Figure 1 (a), that defines the unit sphere as a subspace of $\mathbb{R}^3$. The sphere may be defined, however, using a diagram (b), which asks us to make the entire boundary of a disc to a single point. This process is called *identification*: this means that all the points in the boundary should be treated as if they were the same point. The identification here gives us a topological sphere. We may also make a sphere out of paper. Paper has no curvature, so it has flat geometry, and we get a *flat* sphere (c). The abstract sphere defined by the diagram (b), along with the flat sphere, highlight the difference between the sphere as a topological concept, and a sphere as a geometric entity. It is important for you to consider the difference carefully. We only care about connectivity in topology, and what is connected *like* the geometric sphere *is* a sphere, no matter its geometry.

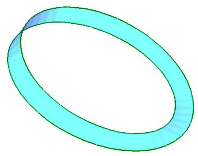


(a) Donut surface (b) Diagram (c) Instructions for a flat torus

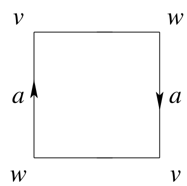
Figure 2. The torus $\mathbb{T}^2$

The torus. The torus is familiar to us as the surface of a bagel or a donut, as shown in Figure 2 (a). We may describe a torus as a subspace of $\mathbb{R}^3$ geometrically. For example, a *torus of revolution* is created when we sweep a circle around the z-axis: $T(u, v) = ((1 + \cos u) \cos v, (1 + \cos u) \sin v, \sin(u))$. The torus may also be described via a diagram (b), in which the edges are glued according to their direction of their arrows. Finally, we can build a flat torus (c) easily.

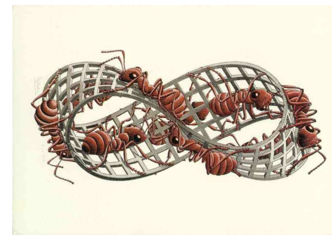
The Möbius strip. Figure 3 (a) shows an embedded Möbius strip: a 2-manifold with boundary. It is easy to construct one by gluing one end of a strip of paper to the other end with a single twist, as shown in the diagram (b). This manifold is not *orientable*. The notes for last lecture included a definition of orientability for smooth manifolds in an appendix. We will see another formal definition of orientability in the next lecture. For now, orientability means that the surface has two sides. M. C. Escher establishes that the Möbius strip is one-sided by marching ants on the strip (c). Note that the boundary of the Möbius strip is a single cycle. This cycle corresponds to the two unglued edges in the diagram (b) which we may now glue with or without a twist.



(a) Embedded

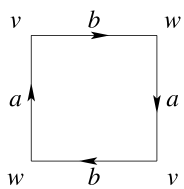


(b) Diagram

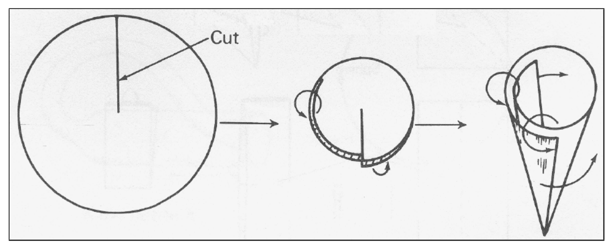


(c) Escher's *Möbius Strip II* (on its side)

Figure 3. The Möbius strip is a non-orientable manifold with boundary.



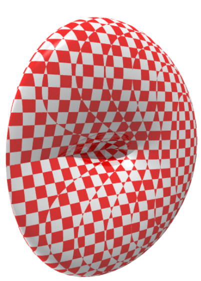
(a) Diagram



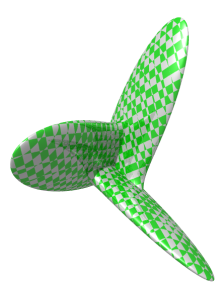
(b) Instructions for a flat $\mathbb{R}\mathrm{P}^2$

Figure 4. The projective plane $\mathbb{R}\mathrm{P}^2$

The projective plane. If we put non-matching arrows on the remaining two edges of the Möbius diagram as in Figure 4, we get the projective plane $\mathbb{R}\mathrm{P}^2$. This action corresponds to gluing the boundary of a disk to the boundary of the Möbius strip. This manifold has this name because of its association with *projective geometry* used in art and computer graphics for representing what we see on a flat canvas. For example, we know that railway lines never intersect, as they are parallel. When we look at them in real life, however, we see that they come together at the horizon, or at "infinity". They also intersect at horizon behind us. We would like any two lines to intersect at most once, so we identify the two intersecting points as the same point. Imagine the boundary of the diagram in (a) is the horizon. The arrows on the diagram identify a point and its reflected image around the origin. These points are called *anti-podal* points. This manifold is non-orientable as it contains a Möbius strip. It cannot be embedded in $\mathbb{R}^3$, so we have to be content with immersions. Figure 5 shows three immersions of the projective plane, all of which self-intersect. These immersions are famous as they contain interesting geometry in addition to their shared topology. To make an paper model, we have to cut the paper to allow for the self-intersection.



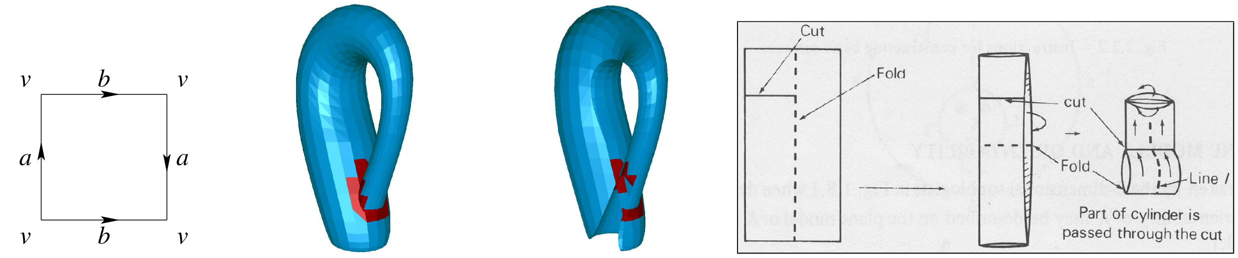
(a) Cross cap



(b) Boy's Surface

(c) Steiner's Roman Surface

Figure 5. Models of the projective plane $\mathbb{R}\mathrm{P}^2$



(a) Diagram

(b) An immersion

(c) Cut in half (a Möbius strip)

(d) Instructions for a flat $\mathbb{K}^2$

Figure 6. The Klein bottle $\mathbb{K}^2$

The Klein bottle. If we glue the free edges of the Möbius strip in the same direction, we get the Klein bottle $\mathbb{K}^2$, as shown in Figure 6 (a). The Klein bottle is therefore equivalent to gluing two Möbius strips to each other along their boundary. Like the projective plane, it is a closed non-orientable surface. It is not embeddable in $\mathbb{R}^3$, and its immersions self-intersect (b, c) with the intersecting triangle colored in red. Once again, we need to cut paper in order to make a flat model (d).

2.3 Connected Sum

We may use the surfaces we just defined to form larger manifolds. To do this, we form connected sums.

Definition 2.3 (connected sum) The *connected sum* of two n-manifolds $\mathbb{M}_1, \mathbb{M}_2$ is

$$\mathbb{M}_1 \# \mathbb{M}_2 \quad = \quad \mathbb{M}_1 - \mathring{D}_1^n \bigcup_{\partial \mathring{D}_1^n = \partial \mathring{D}_2^n} \mathbb{M}_2 - \mathring{D}_2^n,$$

where D_1^n, D_2^n are n-dimensional closed disks in $\mathbb{M}_1, \mathbb{M}_2$, respectively.

In other words, we cut out two disks and glue the manifolds together along the boundary of those disks using a homeomorphism. In Figure 7, for example, we connect two tori to form a sum with two *handles*.

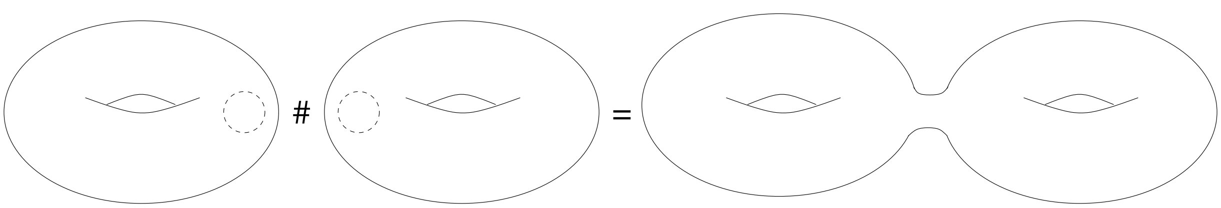


Figure 7. The connected sum of two tori is a genus 2 torus.

2.4 The Classification Theorem

We are now able to state a result that gives a complete classification of compact 2-manifolds.

Theorem 2.2 (classification of compact 2-manifolds) *Every closed compact surface is homeomorphic to a sphere, the connected sum of tori, or connected sum of projective planes.*

We will see in the next lecture that this classification is easily computable. In the remainder of this lecture, we will look at Conway's ZIP proof [2] of this theorem. The paper is provided on the website as the notes for the rest of the lecture.

The theorem answers the homeomorphism question for compact manifolds in two dimensions. After learning about groups, we will see that this question is undecidable for dimensions four and higher. This problem is still open in three dimensions, and *three-manifold topology* is an active area of research. For a very accessible overview, see Weeks [3].

Acknowledgments

The instruction for making flat 2-manifolds are from Firby and Gardiner [1]. I rendered the models of projective plane in Figure 5 in POV-Ray using descriptions by Tore Nordstrand. Figure 6 (b, c) are from [4].

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[1] FIRBY, P. A., AND GARDINER, C. F. *Surface Topology*. Ellis Horwood Limited, Chichester, England, 1982.

[2] FRANCIS, G. K., AND WEEKS, J. R. Conway's ZIP proof. *American Math. Monthly* **106** (May 1999).

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[4] ZOMORODIAN, A., AND EDELSBRUNNER, H. Fast Software for Box Intersection. In *Proc. 16th Ann. ACM Sympos. Comput. Geom.* (2000), pp. 129–138.