Wir müssen wissen, wir werden wissen.
(We must know, we will know.)
— David Hilbert (1862–1943)

5 Homotopy

In Lecture 4, we learned about an algebraic method for describing and classifying structures. In this lecture, we look at using algebra to find combinatorial descriptions of topological spaces. We begin by looking at an equivalence relation called *homotopy* that gives a classification of spaces that is coarser that homeomorphism, but respects the finer classification. That is, two spaces that have the same topological type *must* have the same homotopy type, but the reverse does not necessarily hold. This property should remind you of our definition of a topological invariant. We then continue by looking at a powerful method for understanding topological spaces by forming algebraic images of them using *functors*. One functor is the *fundamental group*, the first group description of a space we will see. Unfortunately, this group is hard to compute and may not give us a finite description. It does, however, give us a method for proving that both the homeomorphism problem and the homotopy problem (detecting whether two spaces are homotopic) are undecidable.

5.1 Homotopy

We defined topological type using homeomorphisms. Often, we observe a qualitative similarity in shape that does not need the full power of a homeomorphism. Rather, we map a space onto a subset of the space that identity map characterizes its connectivity.

Definition 5.1 (deformation retraction) A *deformation retraction* of a space $\mathbb{X}$ onto a subspace $\mathbb{A}$ is a family of maps $f_t : \mathbb{X} \to \mathbb{X}, t \in [0, 1]$ such that f_0 is the identity map, $f_1(\mathbb{X}) = \mathbb{A}$, and $f_t|\mathbb{A}$ is the identity map for all t. The family should be continuous, in the sense that the associated map $\mathbb{X} \times [0, 1] \to \mathbb{X}$, $(x, t) \mapsto f_t(x)$ is continuous.

In other words, starting from the original space $\mathbb{X}$ at time 0, we continuously deform the space until it becomes the subspace $\mathbb{A}$ at time 1. We do this without ever moving the subspace $\mathbb{A}$ in the process. In Figure 1, the space $\mathbb{X}$ is a fat letter 'A', and its subspace $\mathbb{A}$ is a thin letter 'A'. We retract the fat letter onto the thin letter continuously to get a deformation retraction. Note that the two spaces areconnected similarly but are of different dimension. We may continue this retraction until we get the cycle on the right. Once we get the cycle, we are stuck. We cannot go further and retract the space into a single point.

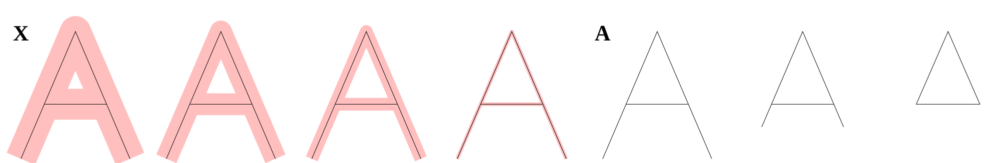


Figure 1. The deformation retraction of a fat letter 'A' onto a thin one, and finally to a cycle.

A deformation retraction is a special case of a homotopy, where the requirement of the final space being a subspace is relaxed.

Definition 5.2 (homotopy) A *homotopy* is a family of maps $f_t : \mathbb{X} \to \mathbb{Y}, t \in [0, 1]$, such that the associated map $F : \mathbb{X} \times [0, 1] \to \mathbb{Y}$ given by $F(x, t) = f_t(x)$ is continuous. Then, $f_0, f_1 : \mathbb{X} \to \mathbb{Y}$ are *homotopic* via the homotopy f_t. We denote this as $f_0 \simeq f_1$.

Suppose we have a retraction f_t as in Definition 5.1. If we let $i : \mathbb{A} \to \mathbb{X}$ to be the inclusion map, we have $f_1 \circ i \simeq 1_{\mathbb{A}}$ and $i \circ f_1 \simeq 1_{\mathbb{X}}$. This allows us to classify $\mathbb{X}$ and its subspace $\mathbb{A}$ as having the same connectivity using the maps f_1, i. This is just a special case of homotopy equivalence.

Definition 5.3 (homotopy equivalence) A map $f : \mathbb{X} \to \mathbb{Y}$ is called a *homotopy equivalence* if there is a map $g : \mathbb{Y} \to \mathbb{X}$, such that $f \circ g \simeq 1_{\mathbb{Y}}$ and $g \circ f \simeq 1_{\mathbb{X}}$. Then, $\mathbb{X}$ and $\mathbb{Y}$ are *homotopy equivalent* and have the same *homotopy type*. This fact is denoted as $\mathbb{X} \simeq \mathbb{Y}$.

The difference between two spaces being homeomorphic versus being homotopic lies in what the compositions of the two functions f, g are: in the former, we require the compositions to be *equivalent* to the identity maps. In the latter, they only need to be *homotopic* them.

$$\begin{array}{lll} \text{Homeomorphism:} & g \circ f = 1_{\mathbb{X}} & f \circ g = 1_{\mathbb{Y}} \\ \text{Homotopy:} & g \circ f \simeq 1_{\mathbb{X}} & f \circ g \simeq 1_{\mathbb{Y}} \end{array}$$

In Lecture 2, we saw an equivalence class based on homeomorphisms. Homotopy is also an equivalence relation, but it does not have the differentiating power of homeomorphisms: two spaces with different topological type could have the same homotopy type. The simplest type of spaces have the homotopy type of a point.

Definition 5.4 (contractible) A space with the homotopy type of a point is called *contractible*.

Figure 1 shows a non-contracible space that is homotopy equivalent to a circle. As a weaker invariant, homotopy is still quite useful, as homeomorphic spaces are homotopic.

Theorem 5.1 $\mathbb{X} \approx \mathbb{Y} \Rightarrow \mathbb{X} \simeq \mathbb{Y}$.

Again, we may utilize the theorem by using its contrapositive: if two spaces are not homotopic, they are not homeomorphic. If the spaces turn out to be homotopic, however, we gain no information about their topological types.

5.2 Categories and Functors

A powerful technique for studying topological spaces is to form and study algebraic images of them. This idea forms the crux of algebraic topology. Usually, these "images" are groups, but richer structures also arise, although we do not have time to discuss them in this class. Our hope is that in the process of forming these images, we retain enough detail to accurately reconstruct the shapes of spaces. As we are interested in understanding how spaces are structurally related, we also want maps between spaces to be converted into maps between the images. The mechanism we use for forming these images are *functors*. To use functors, we need a concept called categories, which may be viewed as an abstraction of abstractions.

Definition 5.5 (category) A *category* $\mathcal{C}$ consists of:

1. a collection $\mathrm{Ob}(\mathcal{C})$ of *objects*,
2. sets $\mathrm{Mor}(X, Y)$ of *morphisms* for each pair $X, Y \in \mathrm{Ob}(\mathcal{C})$, including a distinguished identity morphism $1 = 1_X \in \mathrm{Mor}(X, X)$ for each X.
3. a composition of morphisms function $\circ : \mathrm{Mor}(X, Y) \times \mathrm{Mor}(Y, Z) \to \mathrm{Mor}(X, Z)$ for each triple $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, satisfying $f \circ 1 = 1 \circ f = f$, and $(f \circ g) \circ h = f \circ (g \circ h)$.

We have already seen a few examples of categories, as listed in Table 1. A functor relates two categories *and* their morphisms. We are very familiar with maps of spaces and their images. What is new here is that the functor also maps maps, as shown in Figure 2.

category	morphisms
sets	arbitrary functions
groups	homomorphisms
topological spaces	continuous maps
topological spaces	homotopy classes of maps

Table 1. Some categories and their morphisms.

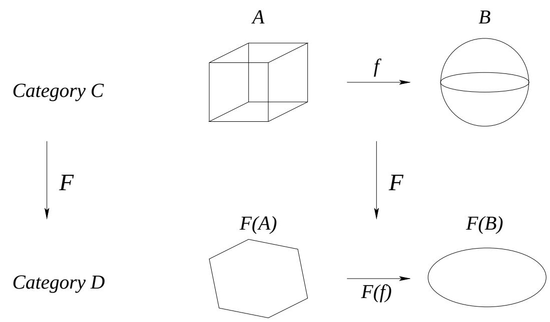


Figure 2. A functor F creates images $F(A), F(B)$ of not only the objects A, B in a category, but also of maps between the objects, such as $F(f)$.

Definition 5.6 (functor) A *(covariant) functor* F from a category $\mathcal{C}$ to a category $\mathcal{D}$ assigns to each object $X \in \mathcal{C}$ an object $F(X) \in \mathcal{D}$, and to each morphism $f \in \mathrm{Mor}(X, Y)$, a morphism $F(f) \in \mathrm{Mor}(F(X), F(Y))$, such that $F(1) = 1$ and $F(f \circ g) = F(f) \circ F(g)$.

The concepts of functors and categories help our intuition in unifying the various techniques we will employ in understanding topological spaces. We will not, however, need these concepts in any deeper fashion.

5.3 Fundamental Group

One of the simplest and most important functors in algebraic topology is the fundamental group. This is the first functor we will examine. It captures the topology of loops on a surface in form of a group. To form this group, we need a set and a binary operation that has the appropriate properties. Our set will be composed of loops on a surface.

Definition 5.7 (path, loop) A *path* in $\mathbb{X}$ is a continuous map $f : [0, 1] \to \mathbb{X}$. A *loop* is a path f with $f(0) = f(1)$, i.e. a loop starts and ends at the same *base-point*. The equivalence class of a path f under the equivalence relation of homotopy is $[f]$.

For example, Figure 3 shows two homotopic paths. You may imagine the path as a string that is somehow stuck on a

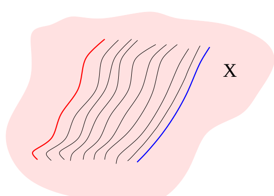


Figure 3. The path on the left is homotopic to the path on the right. The image of the path under the homotopy is shown for some instances.

surface and may only be moved on that surface. You then smoothly deform the piece of string to go from one path to a homotopic path. If we have two paths such that the first path ends at the origin of the second path, we may form the product of the paths.

Definition 5.8 (product path) Given two paths $f, g : [0, 1] \to \mathbb{X}$, the *product path* $f \cdot g$ is a path which traverses f and then g.

The speed of traversal is doubled in order for $f \cdot g$ to be traversed in unit time. Clearly, this product operation respects homotopy classes. Furthermore, if we restrict ourselves to loops, the operation is closed and associative. The identity loop is the *trivial loop* that never moves from the point. The inverse of a loop is simply the loop traversed backwards. Therefore, homotopic loops along with the product path binary operation form a group.

Definition 5.9 (fundamental group) The *fundamental group* $\pi_1(\mathbb{X}, x_0)$ of $\mathbb{X}$ and x_0 has the homotopy classes of loops in $\mathbb{X}$ based at x_0 as its elements, and $[f][g] = [f \cdot g]$ as its binary operation.

Example 5.1 ($\pi_1(\mathbb{T}^2)$) Figure 4 shows three loops on a torus. The loops on the right are homotopic to each other,

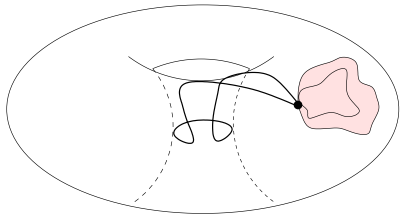


Figure 4. The thick loop goes around the neck of the torus and is not homotopic to the other two loops, which are homotopic through the highlighted surface.

and may be deformed to the base-point through the highlighted surface. The thick loop, however, goes around the neck of the torus and may not be deformed to the base-point, as it does not bound any surface around the neck. As a torus is connected, the base-point may be moved around, so we can omit it from our notation. The thick loop is one of the generators of $\pi_1(\mathbb{T}^2)$. The other generator goes around the width of the torus. The two generators are not homotopic, and $\pi_1(\mathbb{T}^2) \cong \mathbb{Z} \times \mathbb{Z}$, although this result is not immediate.

Note that what distinguishes the generators of the fundamental group in our example is that there is no disk whose boundary is that loop. That is, if we can find any disk that *bounds* our loop, we may then retract the loop via a deformation retraction to the base-point. All bounding loops are equivalent to the trivial loop and contract to the basepoint.

Definition 5.10 (boundaries) A loop on manifold $\mathbb{M}$ that is the boundary of a disk is a *boundary*. Otherwise, the loop is *non-bounding*.

The fundamental group is, in fact, one in a series of *homotopy groups* $\pi_n(\mathbb{X})$ for a space $\mathbb{X}$. The higher-dimensional homotopy groups extend the notion of a loop to n-dimensional cycles and capture the homotopy classes of these cycles. Once again, homeomorphic spaces have the same homotopy groups. Equivalent homotopy groups, however, do not imply the same topological type. We may still use these groups to differentiate between spaces using the contrapositive statement. We do not, however, on the following grounds:

1. The definition of the fundamental group is inherently non-combinatorial, as it depends on smooth maps and the topology of the space.
2. The higher-dimensional homotopy groups are very complicated and hard to compute. In particular, they are not directly computable from a cell decomposition of a space, such as a simplicial decomposition.
3. Even if we were able to compute the homotopy groups, we may get an infinite description of a space: only a finite number of homotopy groups may be non-trivial for an n-dimensional space. Infinite descriptions are certainly not viable for computational purposes.

5.4 Markov's Proof

The definition of the fundamental group enables us to give a quick sketch of Markov's proof of the undecidability of the homeomorphism problem in dimensions greater than 4. In 1912, Dehn proposed the following problem: given two finitely presented groups, decide whether or not they are isomorphic. In 1955, Adyan showed that for any fixed group, Dehn's problem is undecidable. Markov knew that homeomorphic manifolds have the same fundamental group. So, he described a procedure for building a manifold whose fundamental group was related to a given finitely presented group. In particular, its fundamental group would not be the trivial group unless the manifold itself was a sphere. In this fashion, Markov reduced the homeomorphism problem to the isomorphism of groups, proving its undecidability.

Suppose we have a presentation of a group $G : (a_1, \ldots, a_n : r_1, \ldots, r_m)$ with n generators and m relations. Markov maps each generator to an equivalence class of homotopic loops in a 4-manifold. To do so, he attaches n handles to $\mathbb{B}^4$, the four dimensional closed ball, as shown in Figure 5. This base manifold $\mathbb{M}$ is equivalent to the connected sum of n four-dimensional tori. The fundamental group of this manifold, then, is generated by n

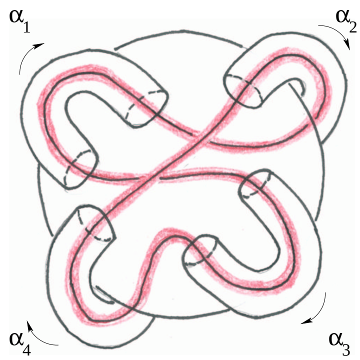


Figure 5. A four dimensional closed ball $\mathbb{B}^4$ with four handles, corresponding to generators α_1 through α_4 with the indicated directions. The loop corresponds to loop $\alpha_1^{-1}\alpha_3\alpha_4\alpha_2$.

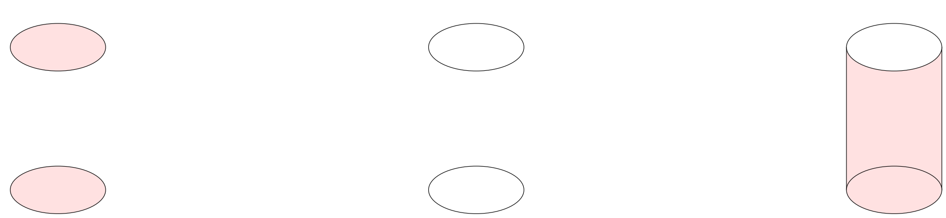
generators, each of whom is represented by one of the handles. We name each handle, with one of the two directions, as a generator. The inverse of each generator is when we travel in the opposite direction in each handle.

Having constructed a manifold with the appropriate generators, Markov next considers the relations. Each relation states $r_i = 1$, that is, the word r_i is equivalent to the identity element. Markov maps the relation r_i into an equivalence class of homotopic loops in $\mathbb{M}$, as shown for the loop $\alpha_1^{-1}\alpha_3\alpha_4\alpha_2$ in Figure 5. Any loop C_i associated to r_i in $\mathbb{M}$ should be bounding and equivalent to the trivial loop. To establish this, we begin by taking a tubular N_i neighborhood of C_i. We make sure these neighborhoods do not intersect each other. We carve N_i out of $\mathbb{M}$ to get $\mathbb{M}'$, leaving an tunnel that represents the relation r_i.

To turn C_i into the trivial loop, we need to "sew in" an appropriate disk whose boundary is the loop, turning the loop into a boundary. Each loop C_i is homeomorphic to $\mathbb{S}^1$ by definition. When creating the neighborhoods N_i, we place a copy of $\mathbb{B}^3$ at every point of C_i. This action corresponds to getting the product of the two spaces.

Definition 5.11 (products of manifolds) The *product* of two topological spaces consists of the Cartesian product of their sets, along with the *product topology* that consists of the Cartesian product of their open sets.

Figure 6 displays three product spaces with the middle space being the boundarhy of the other two spaces. This means that we may glue the two spaces on the sides along their common boundaries. We follow this procedure to glue a disk along the loop C_i. According to the definition, our tubular neighborhood is $N_i \approx \mathbb{S}^1 \times \mathbb{B}^3$. Consequently, its boundary is $\partial N_i \approx \mathbb{S}^1 \times \mathbb{S}^2$, with the closed ball contributing the boundary. We now use a trick we used in creating connected sums of 2-manifolds, as shown in Figure 6 in lower dimensions. That is, we find another space whose boundary is homeomorphic to ∂N_i. We have $\partial N_i \approx \mathbb{S}^1 \times \mathbb{S}^2 \approx \partial(\mathbb{B}^2 \times \mathbb{S}^2)$. So, we glue the boundary of $\mathbb{B}^2 \times \mathbb{S}^2$ to the boundary left by N_i to get $\mathbb{M}_1$. Within $\mathbb{M}_1$, the loop corresponding to relation r_i is retractable, because we just gave it a disk through which it can contract to a point. So, by performing a *Dehn surgery*, we have killed r_i. But we have also killed several other relations, too. For example, in Figure 5, we have also killed $\alpha_3\alpha_4\alpha_2\alpha_1^{-1}$. This is equivalent to adding relations to the finitely presented group. We perform this surgery on the other relations, arriving at $\mathbb{M}_m$, a topological space whose fundamental group has the relations of the presented group G as well as some others.



(a) $\mathbb{S}^0 \times \mathbb{B}^2$ (b) $\mathbb{S}^0 \times \mathbb{S}^1$ (c) $\mathbb{B}^1 \times \mathbb{S}^1$

Figure 6. The two circles in (b) constitute the boundary of both disks in (a) and the cylinder in (c). This fact allows us to construct connected sums of 2-manifolds: we carved out two disks (a) and connected a handle (c) on the boundary (b).

But now, we are done. By Adyan's result, the isomorphism problem for any fixed group is undecidable. In particular, we may pick the trivial group, the fundamental group of the sphere. Given a group presentation, we build a manifold $\mathbb{M}_m$ according to Markov's directions. This manifold has a fundamental group equivalent to the presented group with some additional relations. But the presented group is isomoprhic to the trivial group, the additional relations do not change anything. Therefore, if we could decide whether $\mathbb{M}_m$ is homeomorphic to $\mathbb{S}^4$, we could decide whether the group is the trivial group. As the latter problem is undecidable, so is the former problem.

The same proof works if we go back and replace all occurrences of "homeomorphism" by "homotopy", making the latter classification undecidable. It also works for higher dimensional manifolds. Markov eventually extended his undecidability proof to any "interesting property", although this result is known as *Rice's Theorem*, as it was independently proven and published by Rice in the West.

Acknowledgments

I borrow heavily from Hatcher [4] for the treatment of homotopy and the fundamental group. An English translation of Markov's result [5] is available off my web site. He worked during the golden age of Soviet mathematics at the Steklov Institute. Matiyasevich [6] and Adyan [2] discuss the Markov and Novikov schools of mathematics, respectively. Adyan's result [1] is only available in Russian, but one may substitute Rabin's independent proof [7]. For a history of undecidability theory, see Davis [3].

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