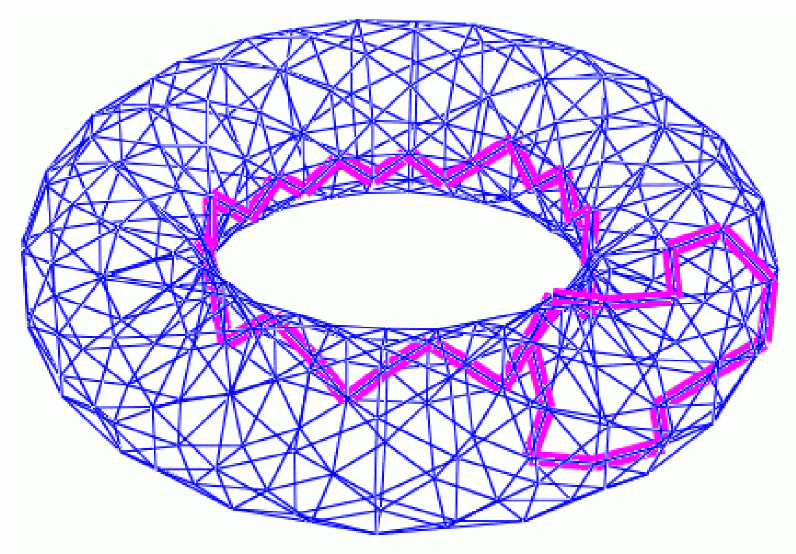
INTRODUCTION TO COMPUTATIONAL TOPOLOGY



Afra Zomorodian

CS 468 – Lecture 1

1-14-4

ORGANIZATION

- Wednesdays, 12:30-2 PM, in Gates 392
- Lectures
- Papers
- Projects

WHY ORGANIZED?

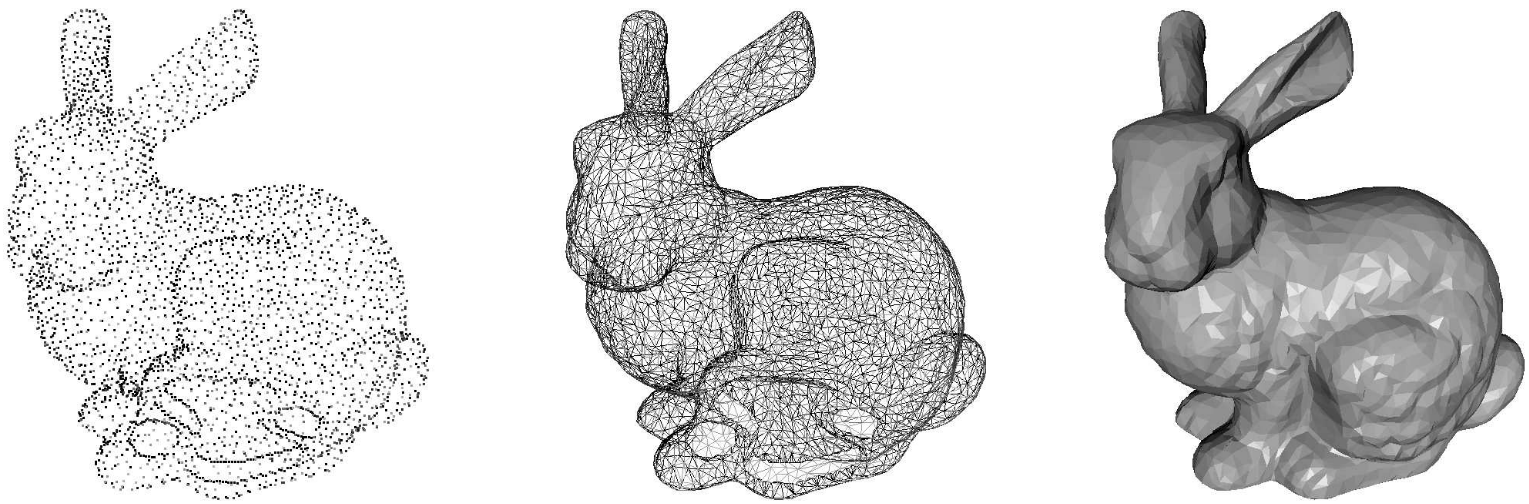
- Topological problems arise in computer science
- We don't know topology
- Topology is
  - large
  - unfamiliar
  - axiomatic (therefore unintuitive)
  - cryptic
- Goal: present background for computer scientists
- So, you can read papers!

What is Topology?

- Not how things look (geometry)
- But how they are connected
- Classifications
- Invariants
  1. transform space in a fixed way
  2. observe what stays the same
- Erlanger Programm (Felix Klein)
- Intrinsic vs. Extrinsic

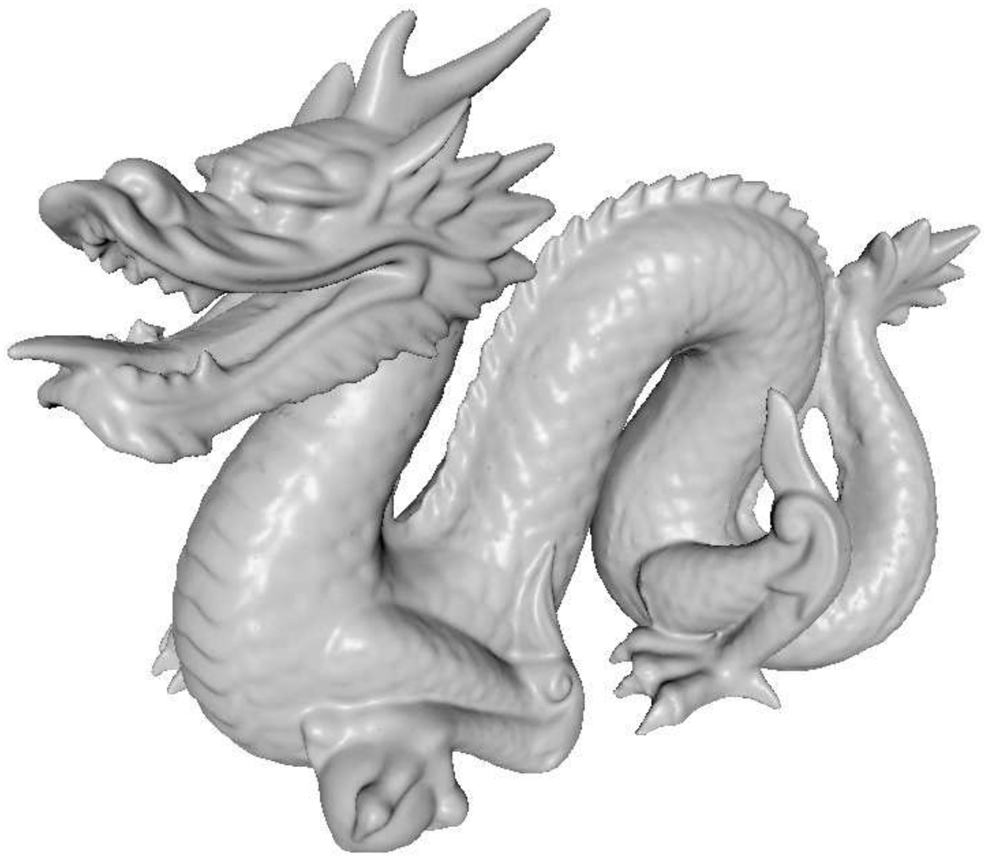
MOTIVATION

GRAPHICS: SURFACE RECONSTRUCTION



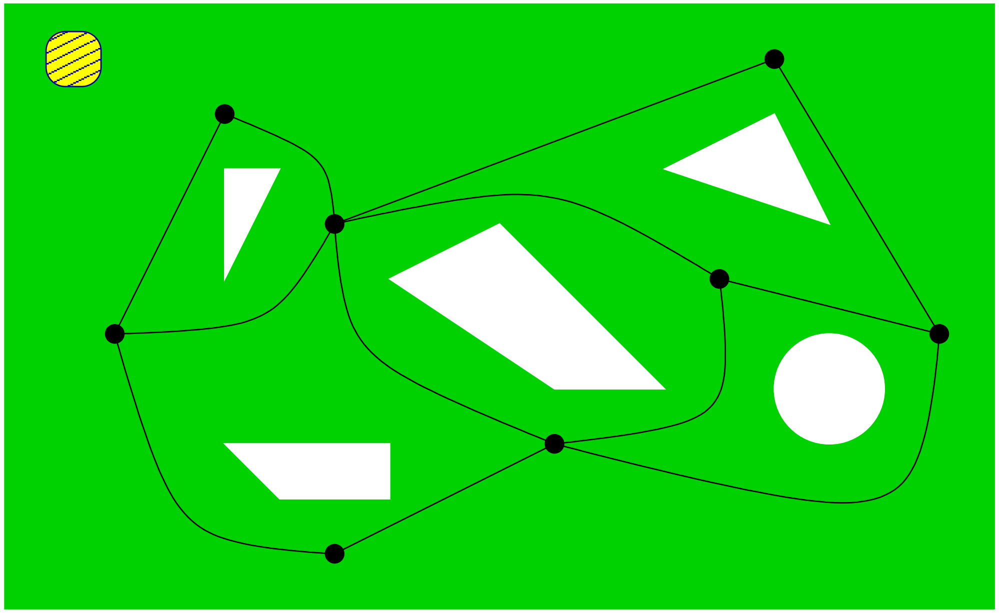
MOTIVATION

GRAPHICS: TUNNELS



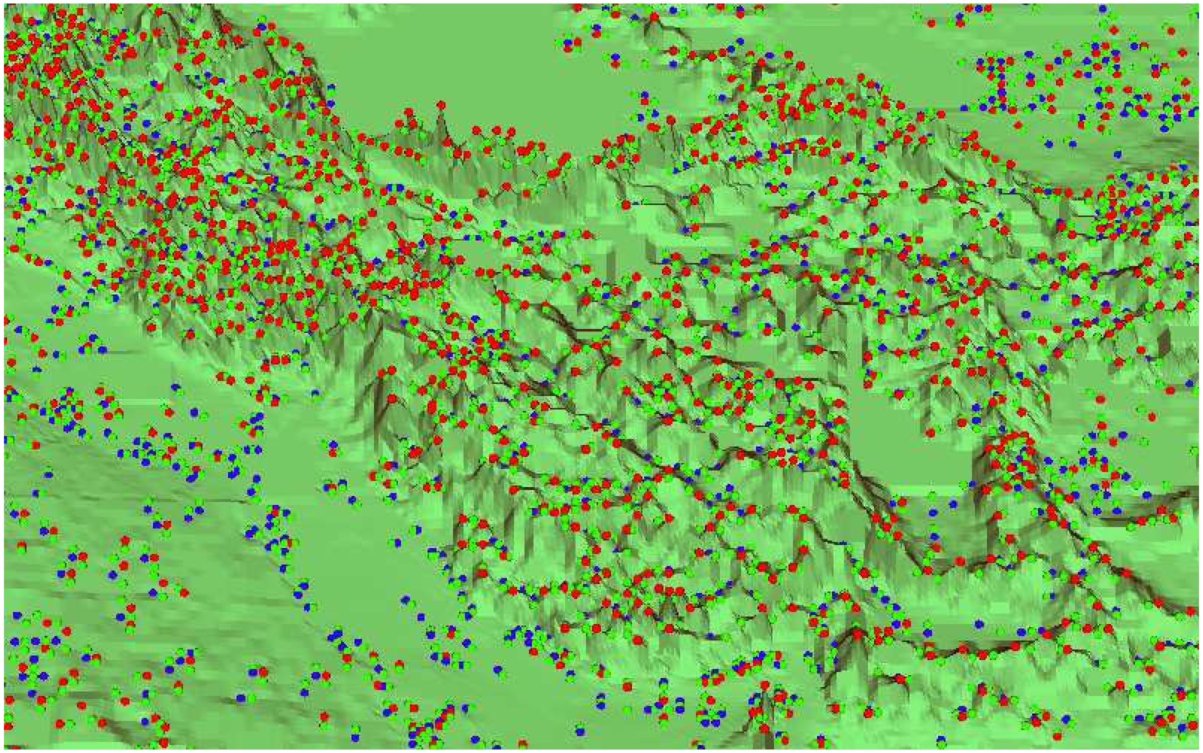
Motivation

Robotics: Configuration Space



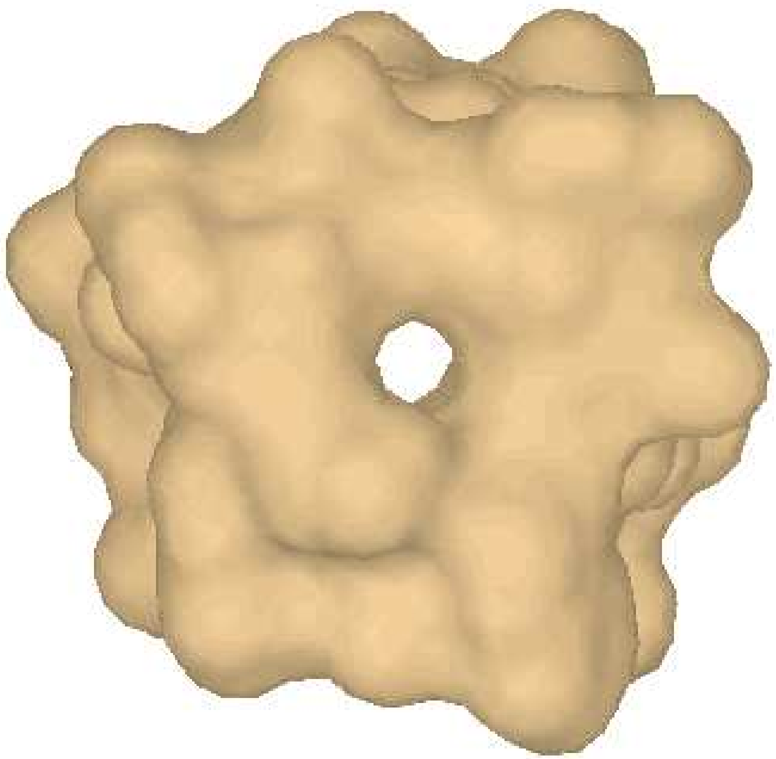
MOTIVATION

GEOGRAPHY: TERRAINS



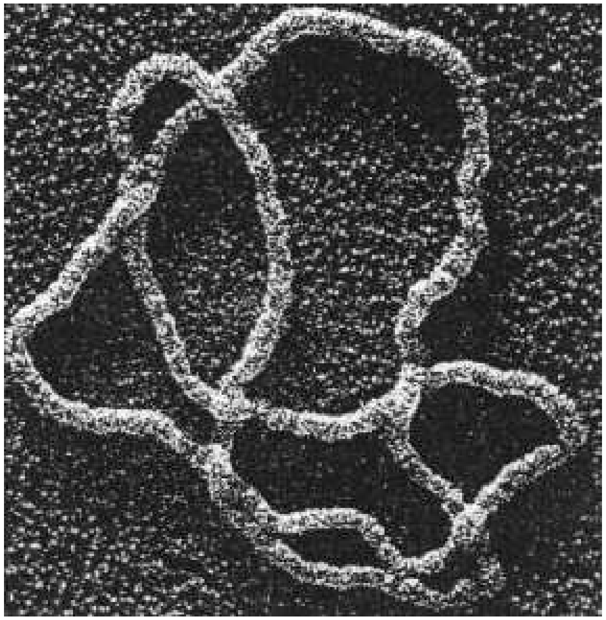
MOTIVATION

BIOLOGY: STRUCTURE

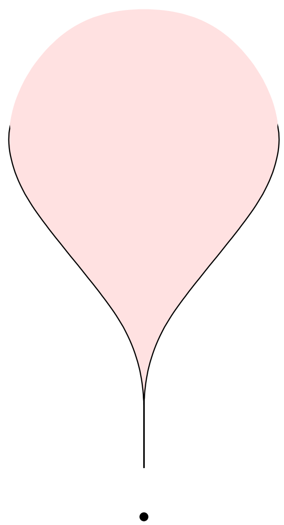


MOTIVATION

CHEMISTRY: KNOTS & LINKS



POINT SET TOPOLOGY



Afra Zomorodian

CS 468 – Lecture 1

1-14-4

MOTIVATION

- Connectivity
- Neighborhoods
- $f : D \subseteq \mathbb{R} \to \mathbb{R}$
- ϵ-δ definition: $\lim_{x \to x_0} f(x) = y_0$ iff for all $\epsilon > 0$, $\exists \delta > 0$ such that if $x \in D$ and $|x - x_0| < \delta$, then $|f(x) - y_0| < \epsilon$.
- Mapping of open intervals
- Metric

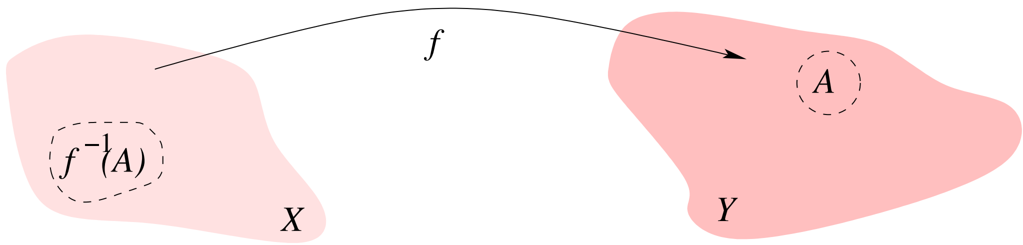
SETS OF POINTS

- A set is a well-defined collection of objects, such that
  1. elements $a \in S$.
  2. one empty set $\emptyset$.
  3. description: $\{x \mid \mathrm{P}(x)\}$ or $\{1, 2, 3\}$
  4. well-defined if $a \in S$ or $a \notin S$
- Point

Topological Spaces

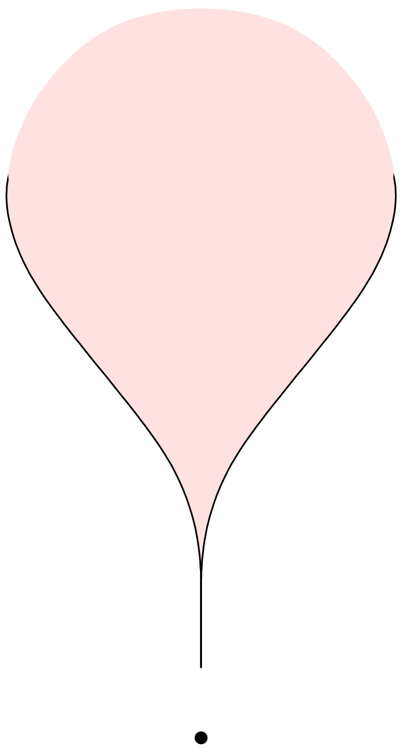
- Given set X
- Take $T \subseteq 2^X$ such that:
  1. If $S_1, S_2 \in T$, then $S_1 \cap S_2 \in T$.
  2. If $\{S_J \mid j \in J\} \subseteq T$, then $\cup_{j \in J} S_j \in T$.
  3. $\emptyset, X \in T$.
- T is a topology on set X
- $S \in T$ is an open set.
- Complement of S is closed.
- All possibilities
- finite intersections, infinite unions
- The pair (X, T) topological space $\mathbb{X}$

CONTINUITY



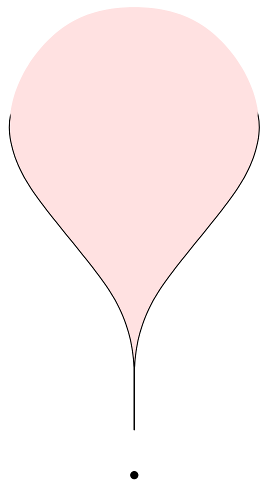
- $f : \mathbb{X} \rightarrow \mathbb{Y}$
- Open set A in $\mathbb{Y}$
- Suppose $f^{-1}(A)$ is open in $\mathbb{X}$.
- f is continuous
- f is a map

SET PROPERTIES

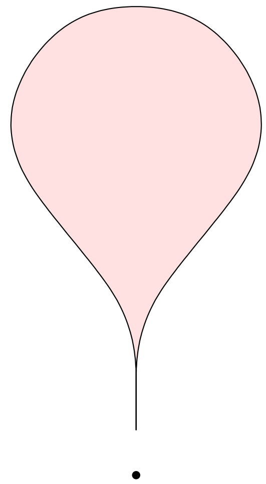


$A \subseteq \mathbb{X}$

CLOSURE



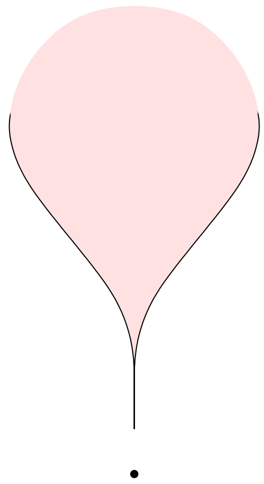
(a) $A \subseteq \mathbb{X}$



(b) $\overline{A}$

The closure $\overline{A}$ of A is the intersection of all closed sets containing A.

INTERIOR



(a) $A \subseteq \mathbb{X}$

(b) $\mathring{A}$

The interior $\mathring{A}$ of A is the union of all open sets contained in A.

BOUNDARY

(a) $A \subseteq \mathbb{X}$ (b) ∂A

The **boundary** ∂A **of** A is $\partial A = \overline{A} - \mathring{A}$.

NEIGHBORHOODS

- $\mathbb{X} = (X, T)$, a topological space.
- $x \in X$
- $A \in T$ such that $x \in \mathring{A}$ is a neighborhood of x
- Suppose we have a collection B of neighborhoods of x
- Every neighborhood of x contains a neighborhood in B
- We call B a basis of neighborhoods at $x \in X$

Subspaces

- $\mathbb{X} = (X, T)$, a topological space.
- $A \subseteq X$, a subset
- $T_A = \{S \cap A \mid S \in T\}$
- T_A is the relative or induced topology
- $\mathbb{A} = (A, T_A)$ is a topological space, a subspace of $\mathbb{X}$.
- Not the only topology possible (as we will see)

METRIC

- A metric or distance function $d : X \times X \to \mathbb{R}$ is a function that satisfies:
  1. For all $x, y \in X$, $d(x, y) \geq 0$ (positivity).
  2. If $d(x, y) = 0$, then $x = y$ (non-degeneracy).
  3. For all $x, y \in X$, $d(x, y) = d(y, x)$ (symmetry).
  4. For all $x, y, z \in X$, $d(x, y) + d(y, z) \geq d(x, z)$ (the triangle inequality).
- Euclidean metric: $d(x, y) = \sqrt{\sum_{i=1}^{n} (u_i(x) - u_i(y))^2}$

METRIC SPACES

- The open ball $B(x, r)$ with center x and radius $r > 0$ with respect to metric d is defined to be $B(x, r) = \{y \mid d(x, y) < r\}$.
- A set X with a metric function d is called a metric space.
- Endowed with metric topology of d, where the set of open balls defined using d serve as basis neighborhoods.
- The Cartesian product of n copies of $\mathbb{R}$ along with the Euclidean metric is the n-dimensional Euclidean space $\mathbb{R}^n$.
- Circle Example

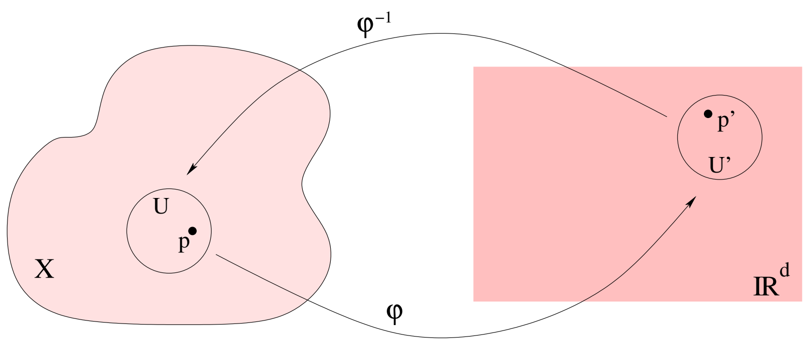
INTUITION

- We *love* Euclidean spaces
- Spaces that look like them? Manifolds!
- How about locally Euclidean?
- Map local pieces to Euclidean spaces
- We don't want the dimension to vary much
- Sphere

HOMEOMORPHISMS

- $f : \mathbb{X} \to \mathbb{Y}$, 1-1, onto
- f is continuous (a map)
- f^{-1} is continuous
- f is a homeomorphism (bijective bicontinuous)
- $\mathbb{X}$ is homeomorphic to $\mathbb{Y}$
- $\mathbb{X} \approx \mathbb{Y}$
- $\mathbb{X}$ and $\mathbb{Y}$ have the same topological type

CHART



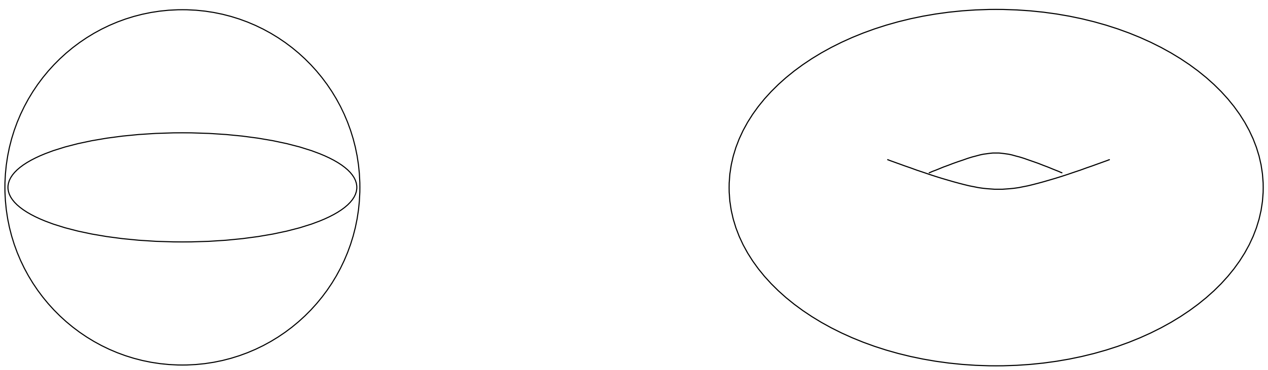
- $p \in U \subseteq \mathbb{X}$
- $\varphi : U \to U' \subseteq \mathbb{R}^d$ is a homeomorphism
- φ is a chart
- It has dimension d
- $u^i : \mathbb{R}^n \to \mathbb{R}$ standard coordinates on $\mathbb{R}^d$
- $x^i = u^i \circ \varphi : U \to \mathbb{R}$ are coordinate functions of φ

STRANGE SPACES

- Given two distinct points $x, y \in \mathbb{X}, x \neq y$
- U, a neighborhood of x
- V, a neighborhood of y
- $U \cap V = \emptyset$
- Then, $\mathbb{X}$ is Hausdorff.
- $\mathbb{X}$ is separable if it has a countable basis of neighborhoods.
- Metric spaces are always Hausdorff and separable (proof)

MANIFOLD

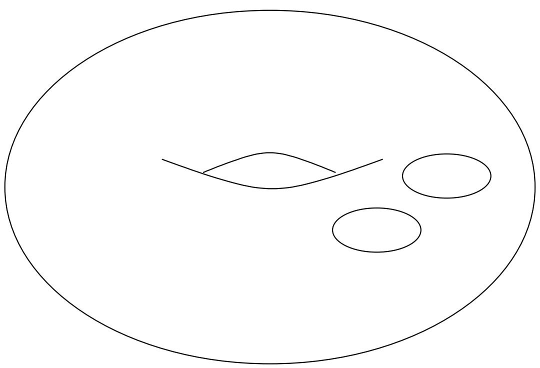
- $\mathbb{X}$ is separable, Hausdorff
- Has d-dimensional chart at every point $x \in \mathbb{X}$ (locally like $\mathbb{R}^d$)
- $\mathbb{X}$ is a (topological, abstract) d-manifold with dimension d



2-Manifolds

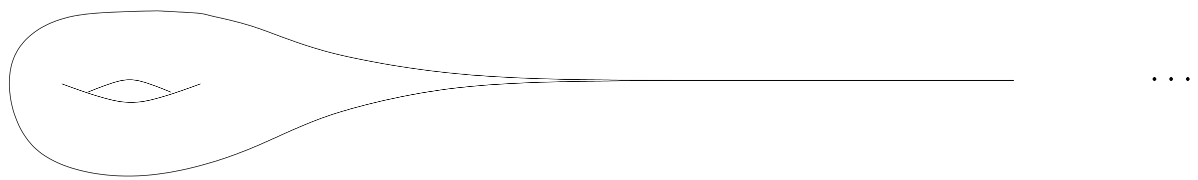
MANIFOLD WITH BOUNDARY

- $\mathbb{X}$ is separable, Hausdorff
- d-dimensional chart at most points
- Some points have neighborhoods homeomorphic to $\mathbb{H}^d = \{x \in \mathbb{R}^d \mid x_1 \geq 0\}$, $\mathbb{X}$ is a d-manifold with boundary
- The boundary $\partial\mathbb{X}$ of $\mathbb{X}$ is the set of points with neighborhood homeomorphic to $\mathbb{H}^d$.
- **(Theorem)** $\partial\mathbb{X}$ is a $(d-1)$-manifold.



COMPACTNESS

- A covering of $A \subseteq X$ is a family $\{C_j \mid j \in J\}$ in 2^X, such that $A \subseteq \bigcup_{j \in J} C_j$.
- An open covering is a covering consisting of open sets.
- A subcovering of a covering $\{C_j \mid j \in J\}$ is a covering $\{C_k \mid k \in K\}$, where $K \subseteq J$.
- $\mathbb{A} \subseteq \mathbb{X}$ is compact if every open covering of A has a finite subcovering.

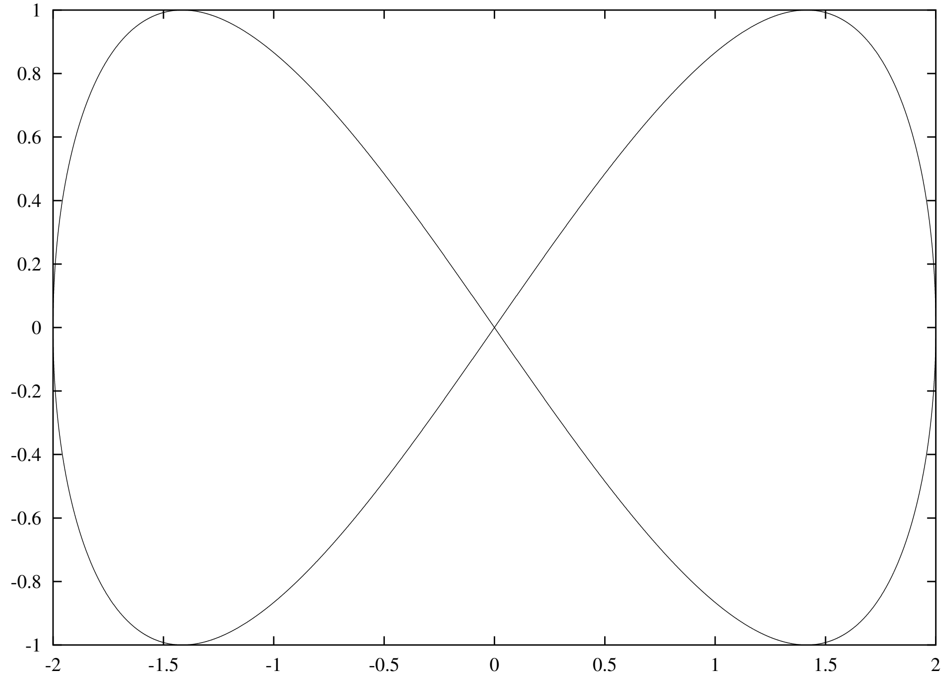


Finite area, not compact

EMBEDDING

- Homeomorphisms allow us to place one manifold within another
- $g : \mathbb{X} \to \mathbb{Y}$
- g is homeomorphism onto its image $g(\mathbb{X})$
- g is an embedding
- $g(\mathbb{X})$ is an embedded submanifold
- We give it the relative topology in $\mathbb{Y}$
- We are most familiar with embedded manifolds

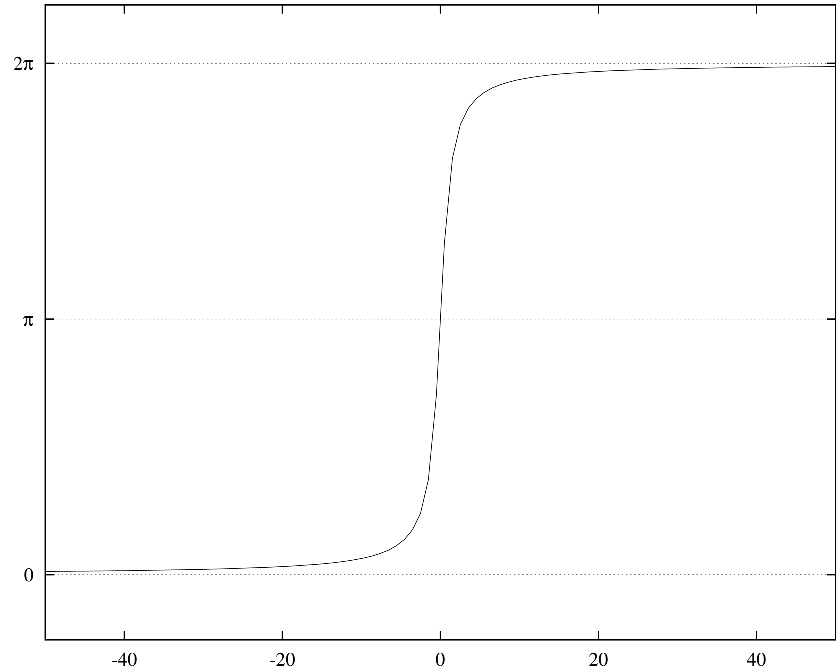
Non-Embedding



$$F\colon \mathbb{R} \longrightarrow \mathbb{R}^2$$

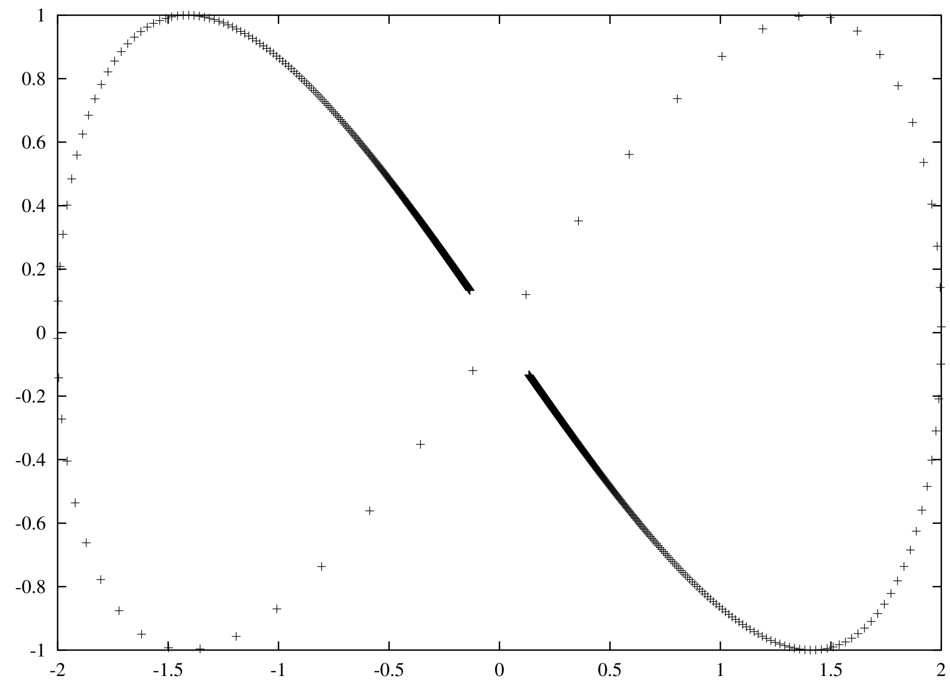
$$F(t) = 2\cos(t - \pi/2), \sin(2(t - \pi/2))$$

A Fix



$$g\colon \mathbb{R} \longrightarrow (0, 2\pi)$$
$$g(t) = \pi + 2\tan^{-1}(t)$$

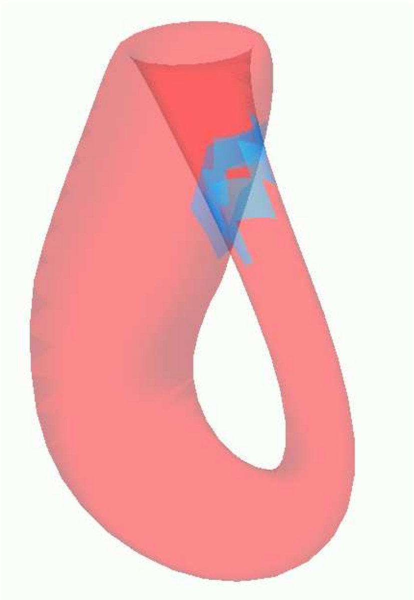
EMBEDDING?



$$\hat{F}(t) = F(g(t))$$

IMMERSIONS

- Nasty things can happen for non-compact manifolds
- Definition requires smooth notions
- For compact manifolds, an immersion allows self-intersection



Standard immersion of the Klein bottle

WHAT TO REMEMBER

- Topology worries about connectivity
- A topology is a set of open sets that define neighborhoods
- A manifold is locally Euclidean
- Homeomorphisms map manifolds to each other
- Natural question: which spaces are homeomorphic to each other?