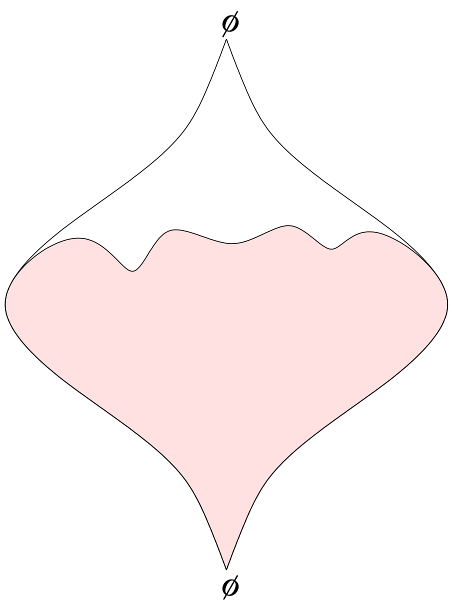
SIMPLICIAL COMPLEXES



Afra Zomorodian

CS 468 – Lecture 3

1-28-4

OVERVIEW

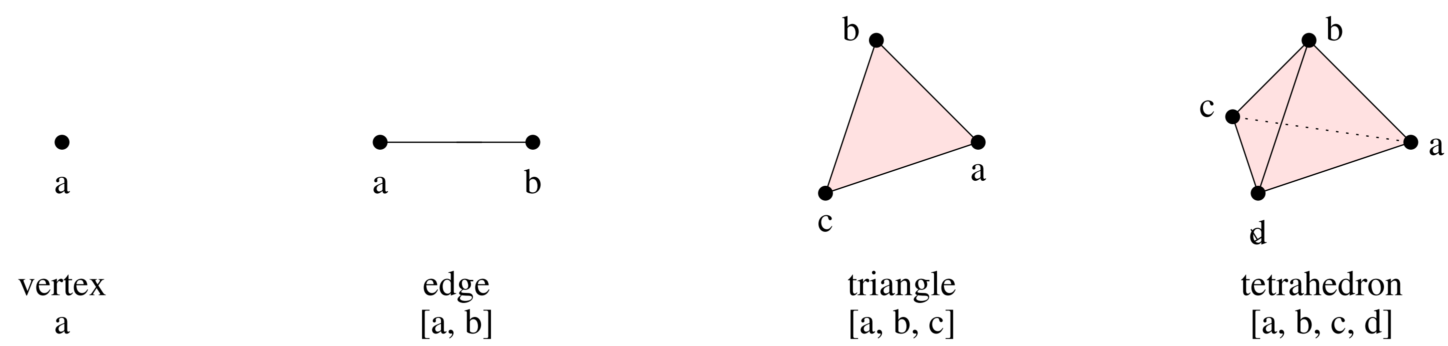
- Point set topology
  - Manifolds: locally Euclidean
  - Homeomorphisms: bijective bi-continuous maps
  - Classification Theorem for 2-Manifolds
  - How to compute?
- Combinatorial topology
  - Simplicial Complexes
  - Triangulations
  - 2-Manifold Homeomorphism problem (revisited)

GEOMETRIC DEFINITION:
COMBINATIONS

- $S = \{p_0, p_1, \ldots, p_k\} \subseteq \mathbb{R}^d$.
- linear combination: $x = \sum_{i=0}^{k} \lambda_i p_i$, for some $\lambda_i \in \mathbb{R}$.
- affine combination: linear combination with $\sum_{i=0}^{k} \lambda_i = 1$.
- convex combination: an affine combination with $\lambda_i \geq 0$, for all i.
- The set of all convex combinations is the convex hull.
- S is linearly (affinely) independent if no point in S is a linear (affine) combination of the other points in S.

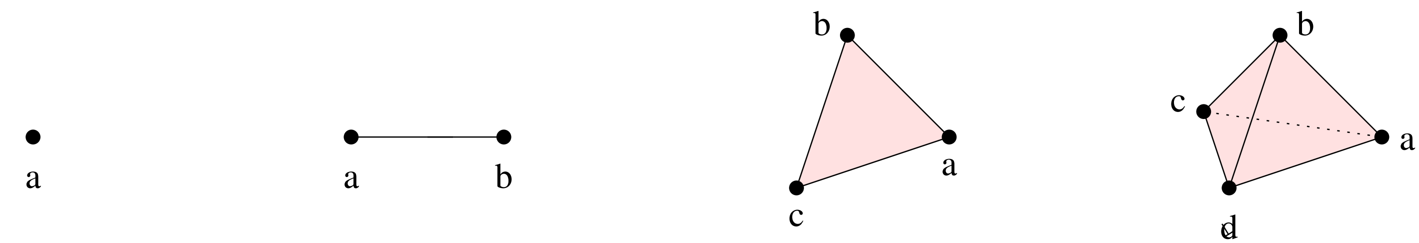
Geometric Definition:
Simplex

- A k-simplex is the convex hull of $k+1$ affinely independent points $S = \{v_0, v_1, \ldots, v_k\}$.
- The points in S are the vertices of the simplex.
- A k-simplex is a k-dimensional subspace of $\mathbb{R}^d$, $\dim \sigma = k$.



Geometric Definition:
Faces

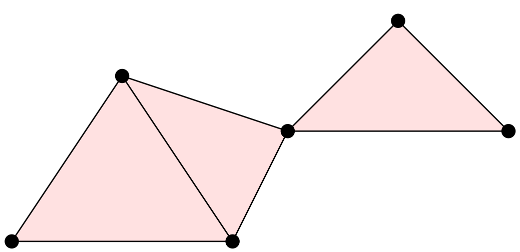
- σ: a k-simplex defined by $S = \{v_0, v_1, \ldots, v_k\}$.
- τ defined by $T \subseteq S$ is a face of σ
- σ is its coface.
- $\sigma \geq \tau$ and $\tau \leq \sigma$.
- $\sigma \leq \sigma$ and $\sigma \geq \sigma$.

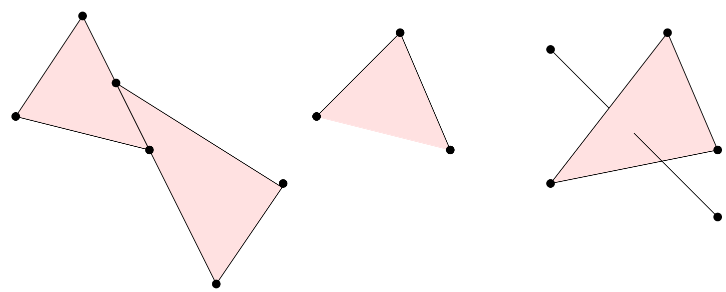


vertex a | edge [a, b] | triangle [a, b, c] | tetrahedron [a, b, c, d]

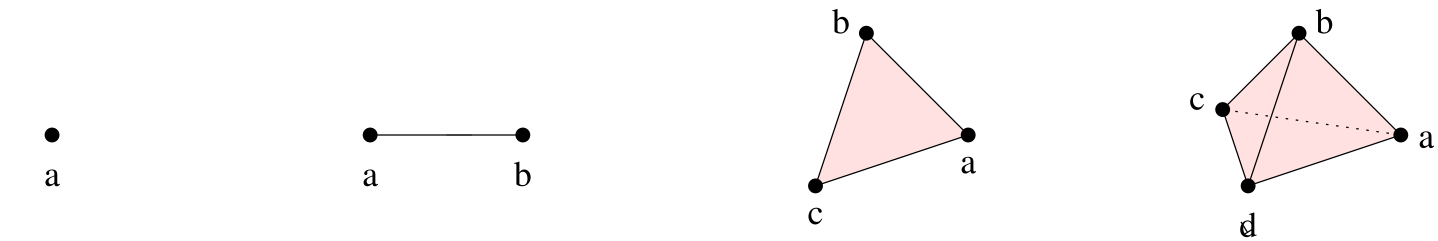
Geometric Definition: Simplicial Complex

- A simplicial complex K is a finite set of simplices such that
  1. $\sigma \in K, \tau \leq \sigma \Rightarrow \tau \in K$,
  2. $\sigma, \sigma' \in K \Rightarrow \sigma \cap \sigma' \leq \sigma, \sigma'$ or $\sigma \cap \sigma' = \emptyset$.
- The dimension of K is $\dim K = \max\{\dim \sigma \mid \sigma \in K\}$.
- The vertices of K are the zero-simplices in K.
- A simplex is principal if it has no proper coface in K.





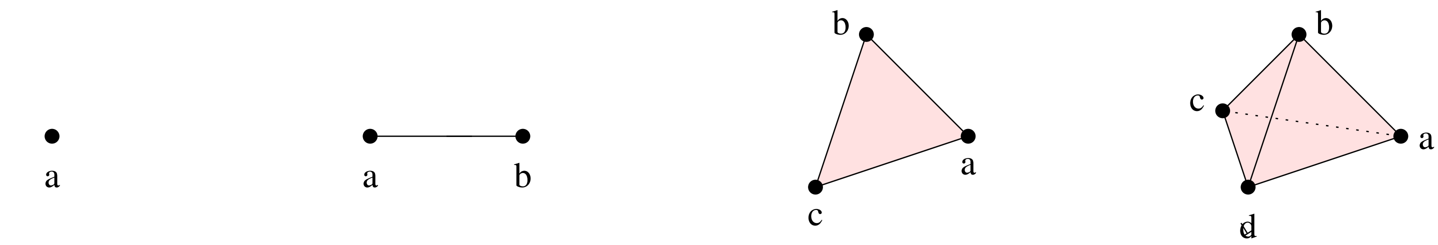
Size of a Simplex: Low Dimensions



vertex a | edge [a, b] | triangle [a, b, c] | tetrahedron [a, b, c, d]

k/l	0	1	2	3
0	1	0	0	0
1	2	1	0	0
2	3	3	1	0
3	4	6	4	1
4	?	?	?	?

SIZE OF A SIMPLEX:
ANOTHER VIEW



vertex | edge | triangle | tetrahedron
a | [a, b] | [a, b, c] | [a, b, c, d]

1

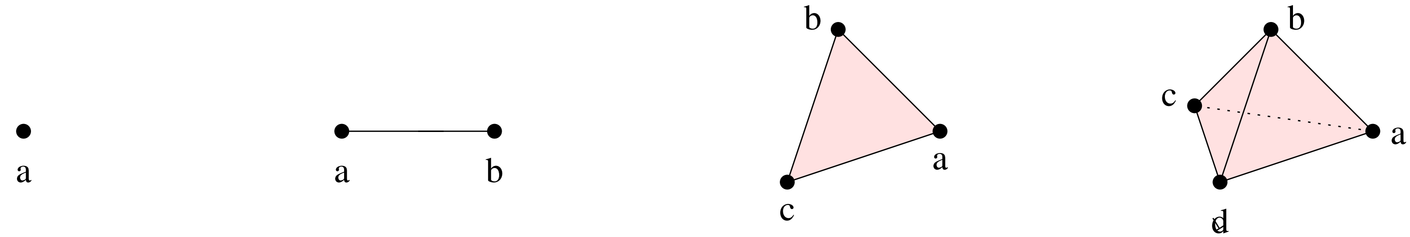
2 1

3 3 1

4 6 4 1

5 10 10 5 1

Size of a Simplex:
Pascal's Triangle



vertex a | edge [a, b] | triangle [a, b, c] | tetrahedron [a, b, c, d]

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Size of a Simplex: Binomial Coefficients

- $\emptyset$ is the (-1)-simplex.
- A k-simplex has $\binom{k+1}{l+1}$ faces of dimension l
- Total size is:

$$\sum_{l=-1}^{k} \binom{k+1}{l+1} = 2^{k+1}$$

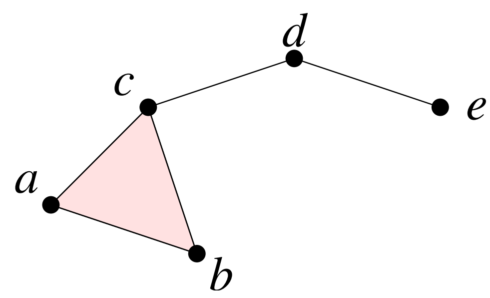
- Large, simple, uniform

ABSTRACT DEFINITION:
SIMPLICIAL COMPLEX

- Given a set K
- Collection of subsets $\mathcal{S}$ of K ((abstract) simplices)
- $(K, \mathcal{S})$ is an abstract simplicial complex provided that:
  1. For all $v \in K, \{v\} \in \mathcal{S}$. We call the sets $\{v\}$ the vertices of K.
  2. If $\tau \subseteq \sigma \in \mathcal{S}$, then $\tau \in \mathcal{S}$.
- σ is a k-simplex of dimension k if $|\sigma| = k + 1$.
- face, coface as before.
- We call $\mathcal{S}$ the complex.

ABSTRACT DEFINITION: VERTEX SCHEME

- Geometric $\longrightarrow$ Abstract
- Let K be a simplicial complex with vertices V
- Define $\mathcal{S}$ to be all subsets $\{v_0, v_1, \ldots, v_k\}$ of V such that the vertices $v_0, v_1, \ldots, v_k$ span a simplex of K.
- $\mathcal{S}$ is the vertex scheme of K.
- $(V, \mathcal{S})$ is an abstract simplicial complex.



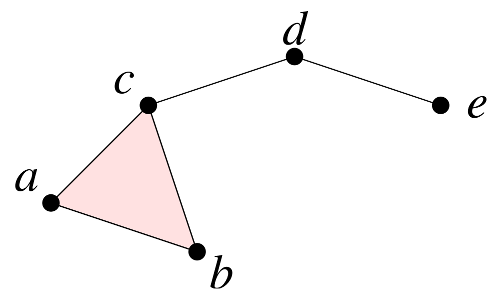
ABSTRACT DEFINITION:
GEOMETRIC REALIZATION

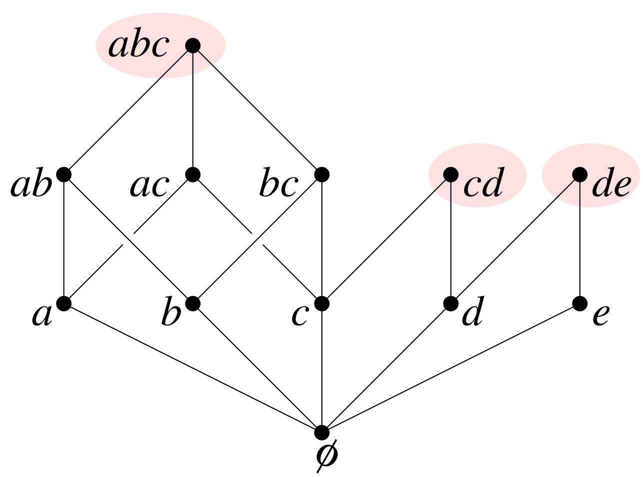
- Abstract $\longrightarrow$ Geometric
- Two abstract simplicial complexes are isomorphic if we can go from one to the other by renaming vertices.
- (Theorem) Every abstract complex $\mathcal{S}$ is isomorphic to the vertex scheme of some simplicial complex K.
- We call K a geometric realization of $\mathcal{S}$.
- In practice, a map of vertices

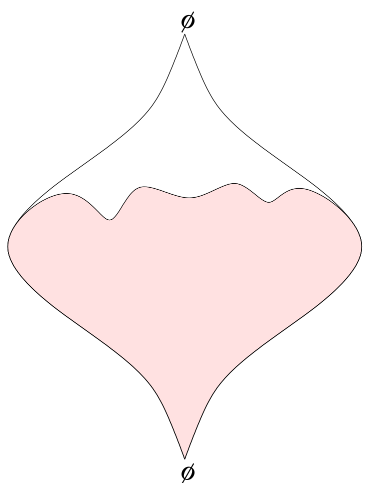
Wavefront's OBJ Format

```
v -0.269616 0.228466 0.077226
v -0.358878 0.240631 0.044214
v -0.657287 0.527813 0.497524
v 0.186944 0.256855 0.318011
v -0.074047 0.212217 0.111664
...
f 19670 20463 20464
f 8936 8846 14300
f 4985 12950 15447
f 4985 15447 15448
...
```

SUBCOMPLEXES: EXAMPLE





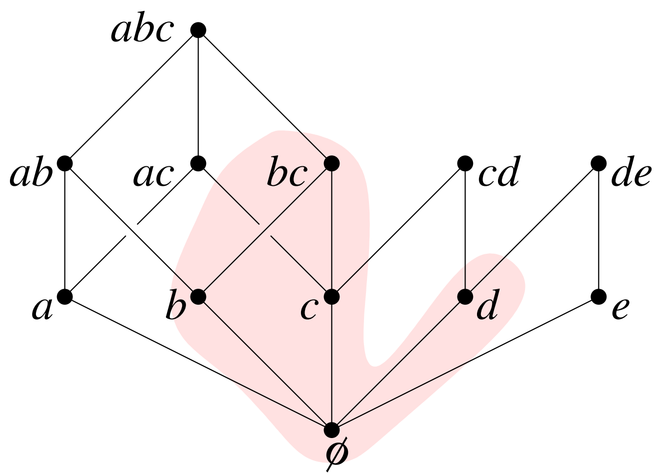


SUBCOMPLEXES:
POSETS

- Let S be a finite set. A partial order is a binary relation on S such that all $x, y, z \in S$,
  1. (Reflexive) $x \leq x$,
  2. (Antisymmetric) $x \leq y$ and $y \leq x$ implies $x = y$,
  3. (Transitive) $x \leq y$ and $y \leq z$ implies $x \leq z$.
- A set with a partial order is a partially ordered set or poset for short.
- Partial vs. full orders
- Acha!

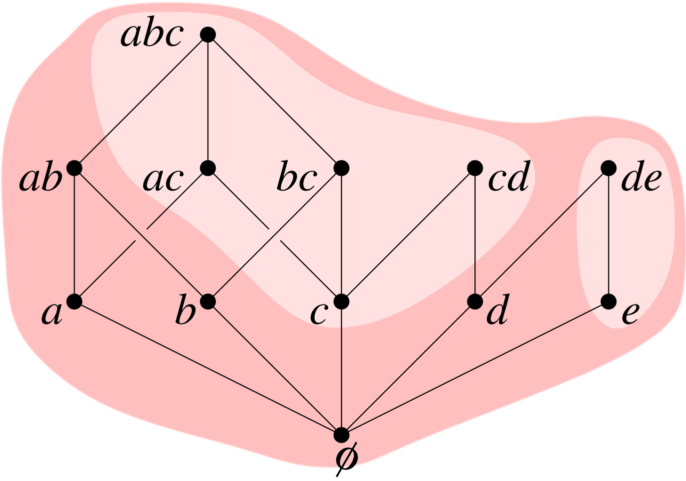
SUBCOMPLEXES: CLOSURE

- A subcomplex is a simplicial complex $L \subseteq K$.
- The smallest subcomplex containing a subset $L \subseteq K$ is its closure, $\mathrm{Cl}\, L = \{\tau \in K \mid \tau \leq \sigma \in L\}$.
- Everything "below" is included.



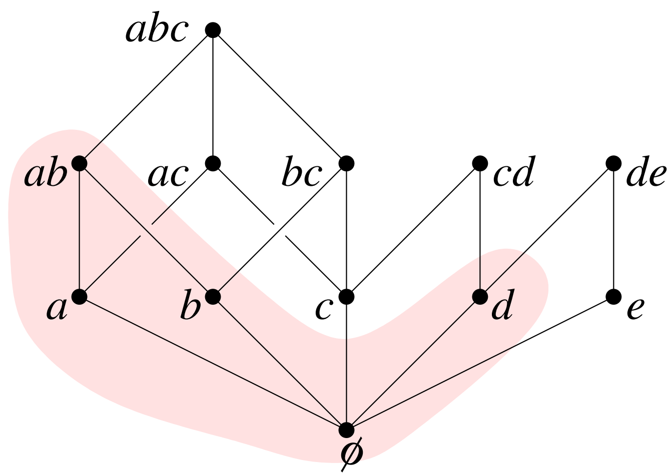
SUBCOMPLEXES: STAR

- The star of L contains all of the cofaces of L,
  $\mathrm{St}\, L = \{\sigma \in K \mid \sigma \geq \tau \in L\}$.
- Everything “above” is included.
- Stars are analogs of neighborhoods (open).



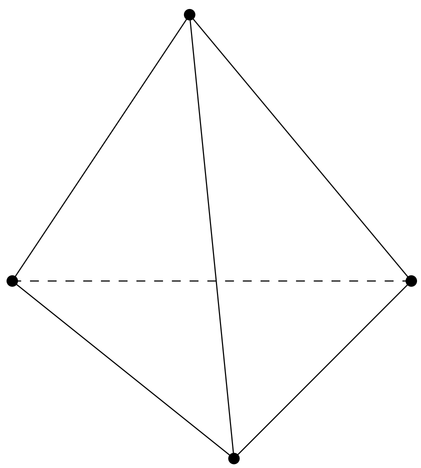
SUBCOMPLEXES:
LINK

- The link of L is the boundary of its star, $\text{Lk}\, L = \text{Cl}\,\text{St}\, L - \text{St}\,(\text{Cl}\, L - \{\emptyset\})$.

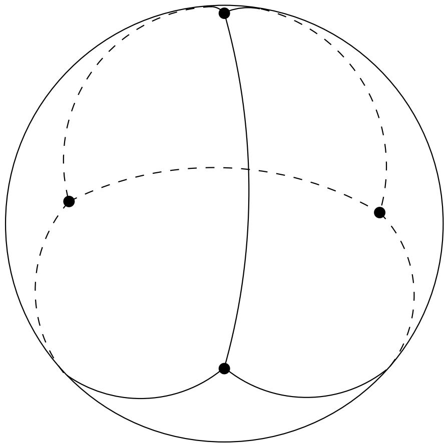


TRIANGULATIONS

- The underlying space $|K|$ of a simplicial complex K is $|K| = \cup_{\sigma \in K} \sigma$.
- $|K|$ is a topological space.
- A triangulation of a topological space $\mathbb{X}$ is a simplicial complex K such that $|K| \approx \mathbb{X}$.



$\approx$



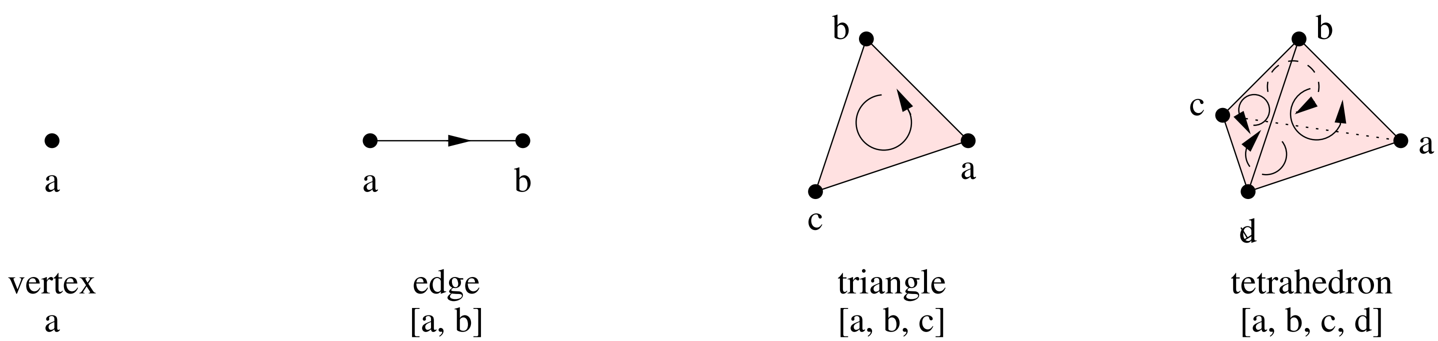
ORIENTABILITY

- An orientation of a k-simplex $\sigma \in K$, $\sigma = \{v_0, v_1, \ldots, v_k\}, v_i \in K$ is an equivalence class of orderings of the vertices of σ, where

$$(v_0, v_1, \ldots, v_k) \sim (v_{\tau(0)}, v_{\tau(1)}, \ldots, v_{\tau(k)})$$

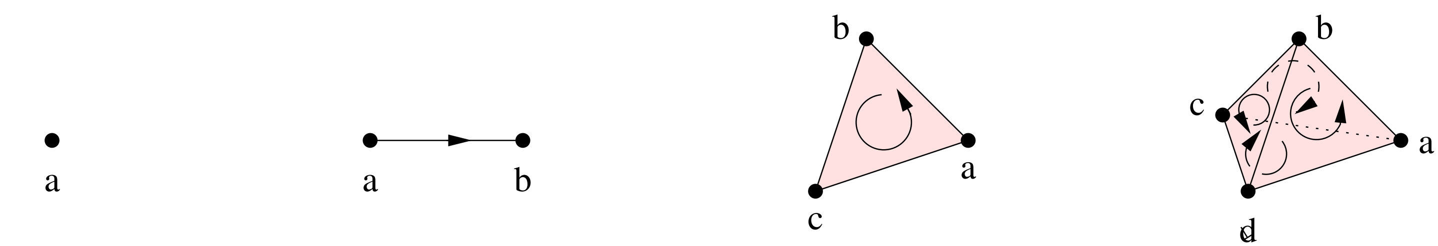
are equivalent orderings if the parity of the permutation τ is even.

- We denote an oriented simplex, a simplex with an equivalence class of orderings, by $[\sigma]$.



Orientability

- Two k-simplices sharing a $(k-1)$-face σ are consistently oriented if they induce different orientations on σ.
- A triangulable d-manifold is orientable if all d-simplices can be oriented consistently.
- Otherwise, the d-manifold is non-orientable



vertex
a

edge
[a, b]

triangle
[a, b, c]

tetrahedron
[a, b, c, d]

Invariants

- A (topological) invariant is a map f that assigns the same object to spaces of the same topological type.
- $\mathbb{X} \approx \mathbb{Y} \implies f(\mathbb{X}) = f(\mathbb{Y})$
- $f(\mathbb{X}) \neq f(\mathbb{Y}) \implies \mathbb{X} \not\approx \mathbb{Y}$ (contrapositive)
- $f(\mathbb{X}) = f(\mathbb{Y}) \implies$ nothing
- "coarser" differentiation

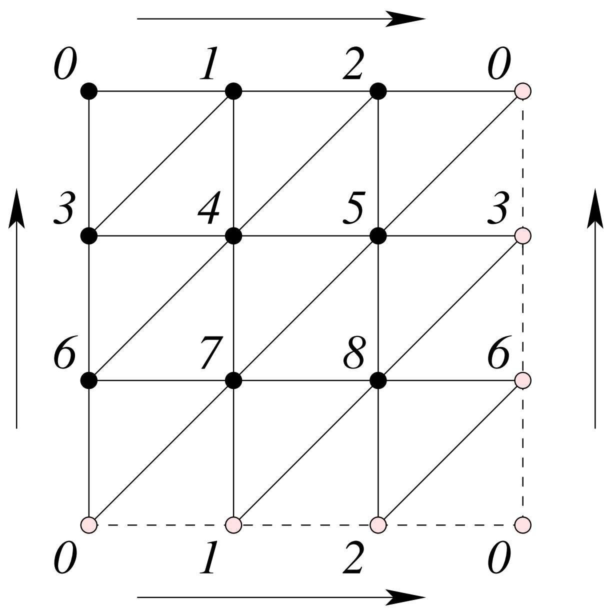
Euler Characteristic: Definition

- K a simplicial complex with s_k k-simplices.
- The Euler characteristic $\chi(K)$ is

$$\chi(K) = \sum_{i=0}^{\dim K} (-1)^i s_i = \sum_{\sigma \in K - \{\emptyset\}} (-1)^{\dim \sigma}.$$

- $v - e + f = 1$ (Graph Theory)
- Invariant for $|K|$
- Any triangulation gives the same answer!
- Intrinsic property

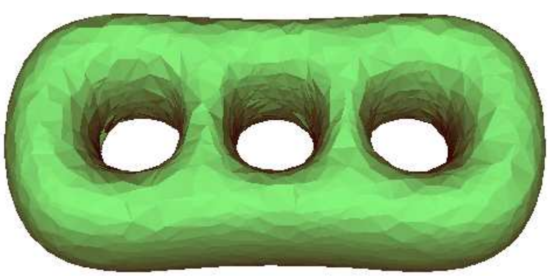
EULER CHARACTERISTIC:
BASIC 2-MANIFOLDS



2-Manifold	χ
Sphere $\mathbb{S}^2$	2
Torus $\mathbb{T}^2$	0
Klein bottle $\mathbb{K}^2$	0
Projective plane $\mathbb{R}\mathrm{P}^2$	1

Euler Characteristic:
Connected Sums

- (Theorem) For compact surfaces $\mathbb{M}_1, \mathbb{M}_2$, $\chi(\mathbb{M}_1 \# \mathbb{M}_2) = \chi(\mathbb{M}_1) + \chi(\mathbb{M}_2) - 2$.
- $\chi(g\mathbb{T}^2) = 2 - 2g$
- $\chi(g\mathbb{R}\mathrm{P}^2) = 2 - g$
- The connected sum of g tori is called a surface with genus g.



Euler Characteristic: Homeomorphism Problem

- (Theorem) Closed compact surfaces $\mathbb{M}_1$ and $\mathbb{M}_2$ are homeomorphic, $\mathbb{M}_1 \approx \mathbb{M}_2$ iff
  1. $\chi(\mathbb{M}_1) = \chi(\mathbb{M}_2)$ and
  2. either both surfaces are orientable or both are non-orientable.
- "iff" so full answer. We're done!
- Higher dimensions?