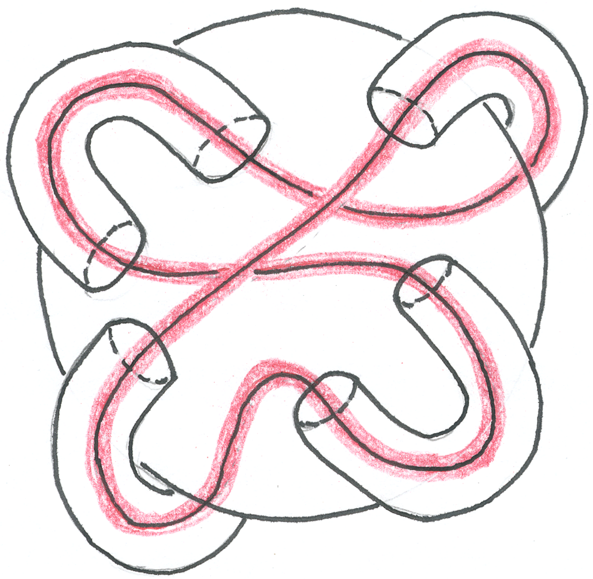
HOMOTOPY



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CS 468 – Lecture 5

2-11-4

Overview

- Review of Group Theory
- Homotopy
- Fundamental Group
- Markov's Proof

Group Theory Review

- Group
  - set S of objects
  - associative closed binary operation $*$ on S
  - identity, inverses wrto $*$

V_4	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

- Subgroup
  - subset that is a group
  - $\{e\}$, $\{e, a\}$, $\{e, b\}$, $\{e, c\}$, $\{e, a, b, c\}$ for V_4

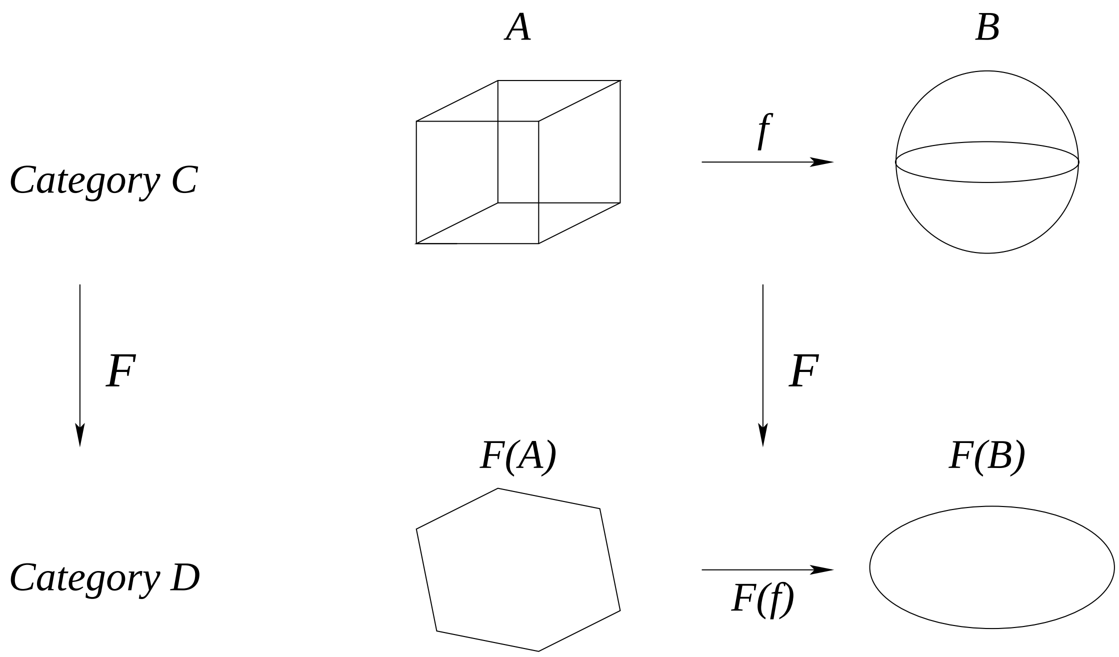
CATEGORIES

- A collection $\mathrm{Ob}(\mathcal{C})$ of objects
- Sets $\mathrm{Mor}(X, Y)$ of morphisms for each pair $X, Y \in \mathrm{Ob}(\mathcal{C})$
- An identity morphism $1 = 1_X \in \mathrm{Mor}(X, X)$ for each X.
- a composition of morphisms function $\circ : \mathrm{Mor}(X, Y) \times \mathrm{Mor}(Y, Z) \rightarrow \mathrm{Mor}(X, Z)$ for each triple $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, satisfying $f \circ 1 = 1 \circ f = f$, and $(f \circ g) \circ h = f \circ (g \circ h)$.
- A category $\mathcal{C}$

Some Categories

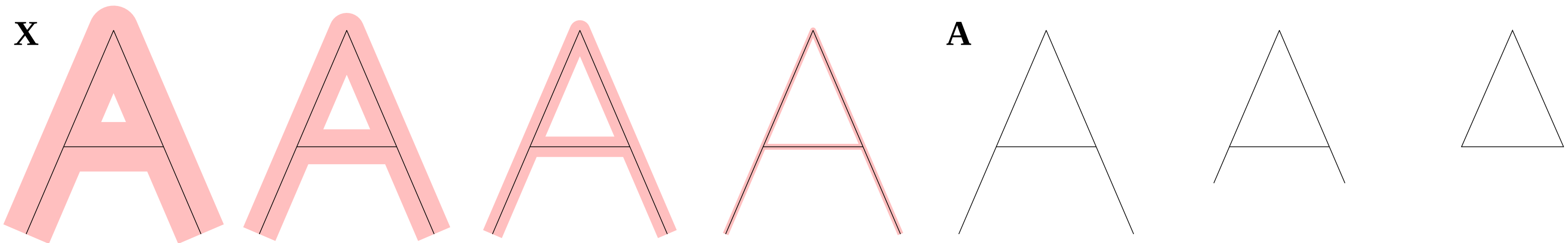
category	morphisms
sets	arbitrary functions
groups	homomorphisms
topological spaces	continuous maps
topological spaces	homotopy classes of maps

FUNCTORS



- $X \in \mathcal{C}$, $F(X) \in \mathcal{D}$,
- $f \in \text{Mor}(X, Y)$, $F(f) \in \text{Mor}(F(X), F(Y))$
- $F(1) = 1$ and $F(f \circ g) = F(f) \circ F(g)$
- F is a (covariant) functor

DEFORMATION RETRACTION

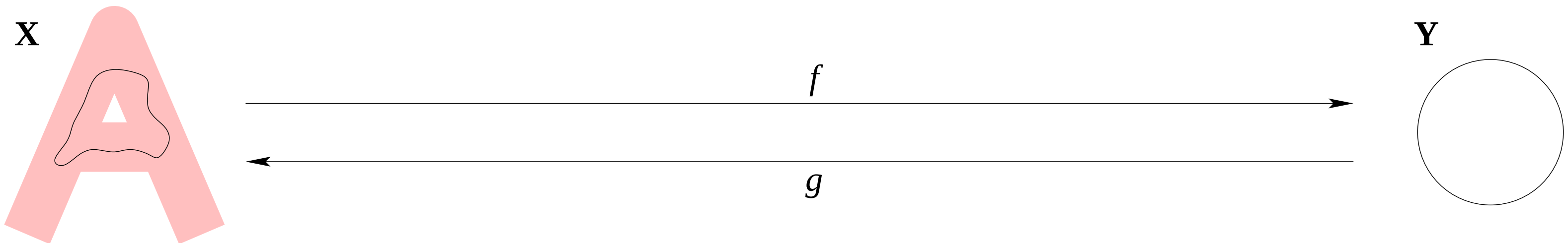


- Family of maps $f_t : \mathbb{X} \to \mathbb{X}, t \in [0, 1]$
- f_0 is the identity map
- $f_1(\mathbb{X}) = \mathbb{A}$
- $f_t|\mathbb{A}$ is the identity map, for all t
- Family is continuous, i.e. $\mathbb{X} \times [0, 1] \to \mathbb{X}$, $(x, t) \mapsto f_t(x)$ is continuous.
- f is a deformation retraction

HOMOTOPY

- Family of maps $f_t : \mathbb{X} \to \mathbb{Y}, t \in [0, 1]$
- Family is continuous, i.e. $\mathbb{X} \times [0, 1] \to \mathbb{Y}$, $(x, t) \mapsto f_t(x)$ is continuous.
- f is a homotopy
- f_0, f_1 are homotopic, $f_0 \simeq f_1$
- $\simeq$ between maps

HOMOTOPY EQUIVALENCE

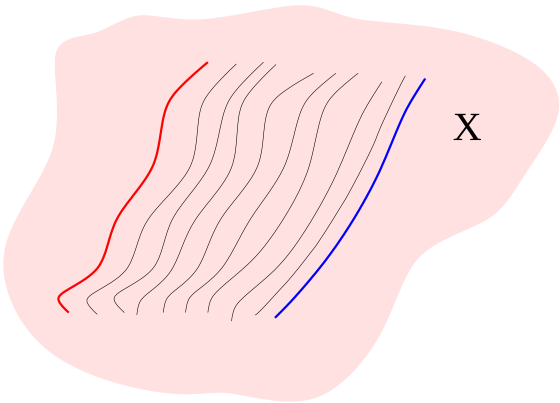


- $f : \mathbb{X} \to \mathbb{Y}$, $g : \mathbb{Y} \to \mathbb{X}$

Homeomorphism:	$g \circ f = 1_{\mathbb{X}}$	$f \circ g = 1_{\mathbb{Y}}$
Homotopy:	$g \circ f \simeq 1_{\mathbb{X}}$	$f \circ g \simeq 1_{\mathbb{Y}}$

- $\mathbb{X} \simeq \mathbb{Y}$ are homotopy equivalent, have the same homotopy type
- $\simeq$ is an equivalence relation
- Contracible spaces $\simeq$ a point
- (Theorem) $\mathbb{X} \approx \mathbb{Y} \Rightarrow \mathbb{X} \simeq \mathbb{Y}$

LOOPS



- A path in $\mathbb{X}$ is a continuous map $f : [0, 1] \to \mathbb{X}$.
- A loop is a path f with $f(0) = f(1)$ which is the base-point.
- The equivalence class of a path f under the equivalence relation of homotopy is $[f]$.

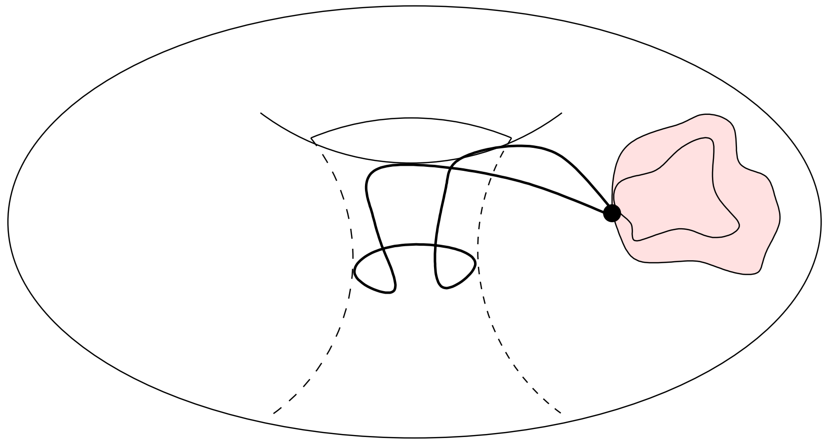
PRODUCT PATH

- Given two paths $f, g : [0, 1] \to \mathbb{X}$, the product path $f \cdot g$ is a path which traverses f and then g
- Over loops, the product is closed and associative
- Trivial loop
  - homotopy equivalent to a point
  - contractible
  - must be the boundary of a disk
  - bounding (otherwise, non-bounding)
- Inverse $-f$: go backwards

FUNDAMENTAL GROUP

- elements: homotopy classes of loops $[f]$ in $\mathbb{X}$ based at x_0
- binary operation: $[f][g] = [f \cdot g]$
- identity: trivial loop
- inverses: $-[f] = [-f]$
- The fundamental group $\pi_1(\mathbb{X}, x_0)$ of $\mathbb{X}$ and basepoint x_0
- The fundamental group is a functor from the category of topological spaces to the category of groups

$\pi_1(\mathbb{T}^2)$



- $\pi_1(\mathbb{T}^2) \cong \mathbb{Z} \times \mathbb{Z}$
- donut vs. ball

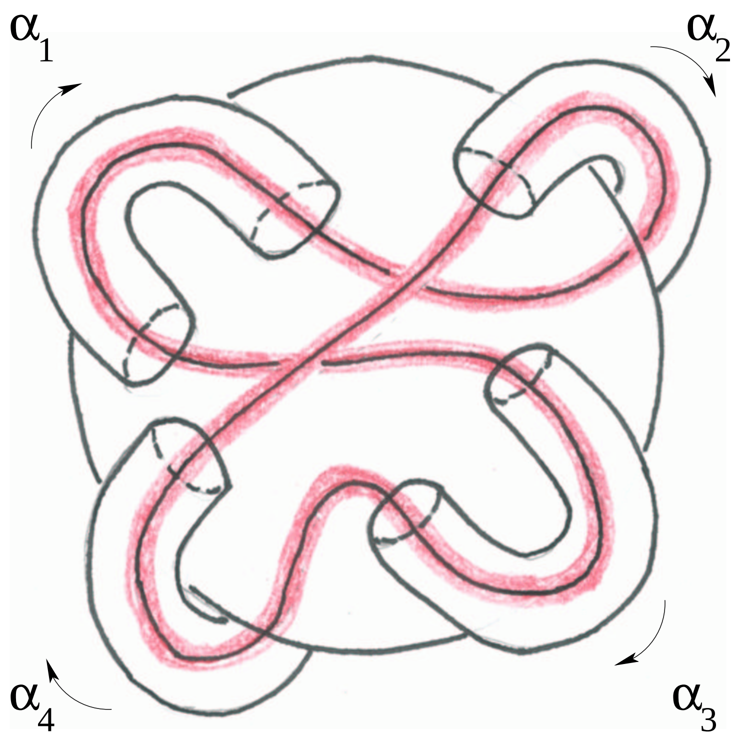
HOMOTOPY GROUPS

- $\pi_1(\mathbb{X})$ is the first of many homotopy groups $\pi_n(\mathbb{X})$ for a space $\mathbb{X}$.
- n-dimensional cycles
- $\mathbb{X} \approx \mathbb{Y}$ implies $\pi_n(\mathbb{X}) = \pi_n(\mathbb{Y})$, for all n
- *Not* the other way around (as usual)
- Problems:
  1. not combinatorial
  2. very complicated, not directly computable from simplicial complexes
  3. may be an infinite description of space

MARKOV'S PROOF

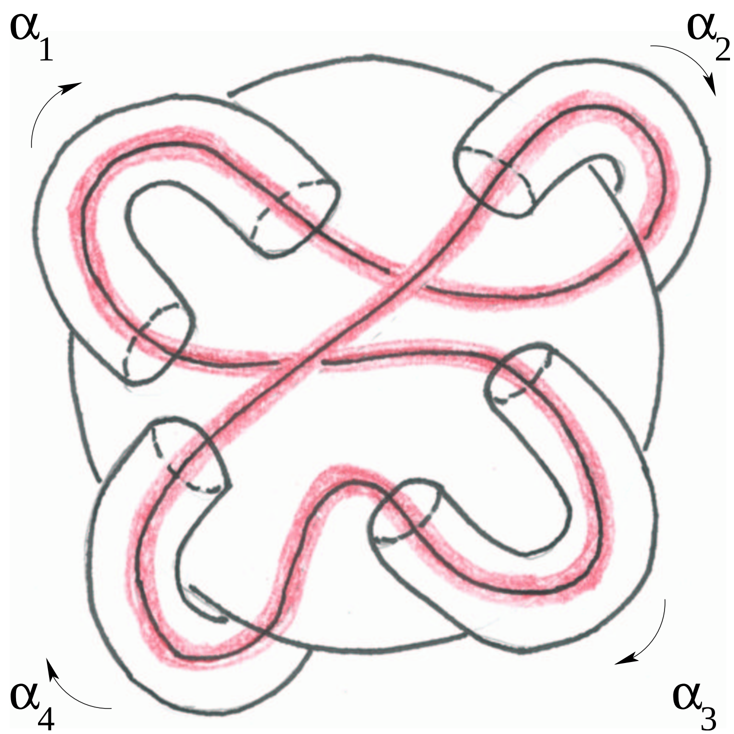
- Fundamental group is a functor from the category of topological spaces to the category of groups
- Reduce the homeomorphism problem to the isomorphism problem of groups
- (Dehn 1912) Given two finitely presented groups, decide whether or not they are isomorphic.
- (Adyan 1955) For any fixed group, Dehn's problem is undecidable.
- Given a finitely presented group $G : (a_1, \ldots, a_n : r_1, \ldots, r_m)$
- Build a manifold whose fundamental group is G

GENERATORS



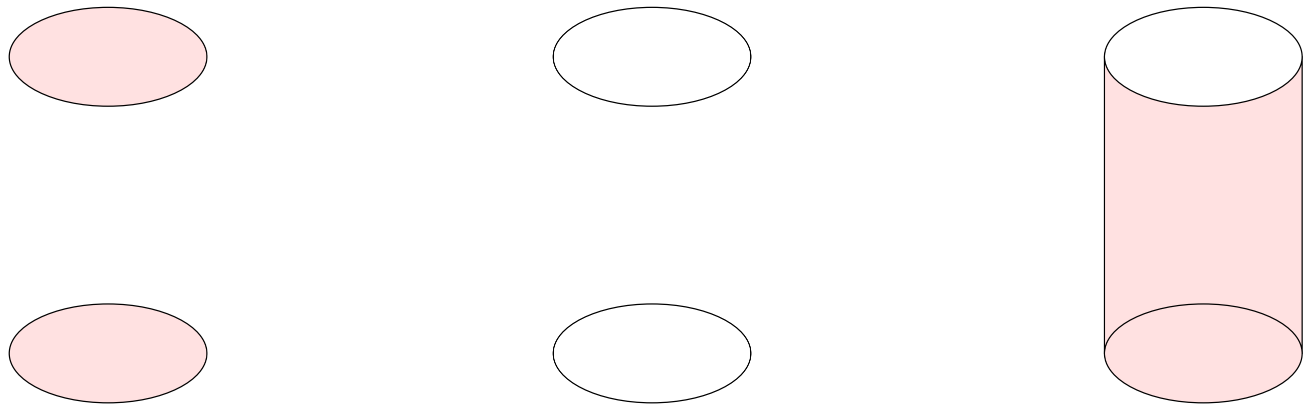
- Take a four-dimensional closed ball $\mathbb{B}^4$
- Attach n handles, one per generator, to get $\mathbb{M}$
- A word is a loop in the manifold: $\alpha_1^{-1}\alpha_3\alpha_4\alpha_2$, $\alpha_2\alpha_1^{-1}\alpha_3\alpha_4$

RELATIONS



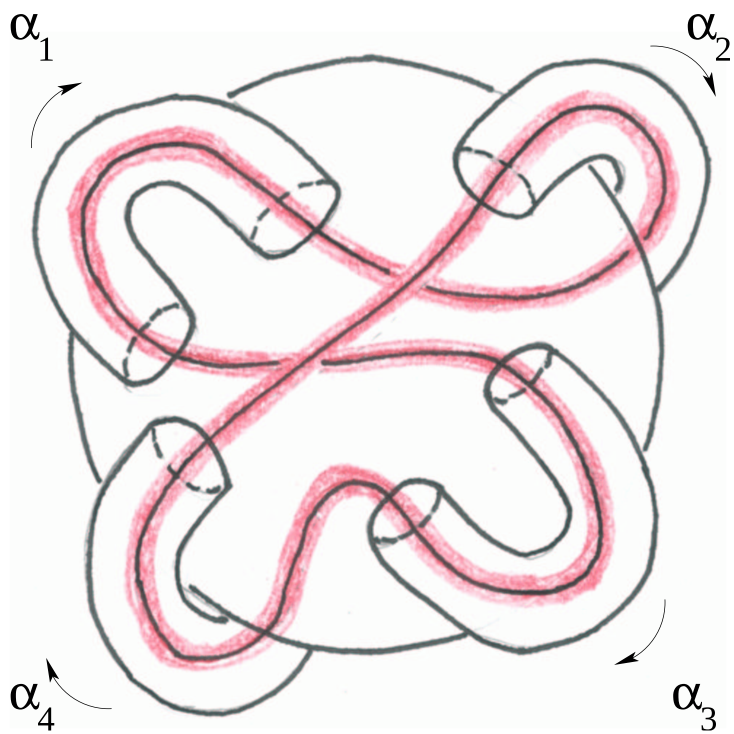
- $r_i = 1$ means loop C_i in $\mathbb{M}$ should be trivial
- Need a disk to make C_i bounding

PRODUCT SPACES



(a) $\mathbb{S}^0 \times \mathbb{B}^2$ (b) $\mathbb{S}^0 \times \mathbb{S}^1$ (c) $\mathbb{B}^1 \times \mathbb{S}^1$

DEHN SURGERY



- Carve out $N_i \approx \mathbb{S}^1 \times \mathbb{B}^3$
- $\partial N_i \approx \mathbb{S}^1 \times \mathbb{S}^2$
- $\partial(\mathbb{B}^2 \times \mathbb{S}^2) \approx \mathbb{S}^1 \times \mathbb{S}^2$, so sew it in

The Reduction

- Use Dehn surgery to kill all r_i and get $\mathbb{M}_m$
- If the group was trivial, the additional relations don't change that
- We reduced $\approx$ to $\cong$ for the fixed space $\mathbb{S}^4$ and the trivial group
- Works for $n \geq 4$
- Works for homotopy $n \geq 4$
- Works for any "interesting property" (Rice Theorem)

Recap

- Homotopy: coarser definition of topological equivalence
- Homotopic loops form a group
- We use functors to move between categories
- Fundamental group is a functor
- Homotopy groups
- Markov's construction
  - Homeomorphism problem undecidable for $n \geq 4$
  - Homotopy problem undecidable for $n \geq 4$
- Poincaré Conjecture: Let V be a 3-manifold without boundary whose fundamental group is trivial. Then $V \approx \mathbb{S}^3$.