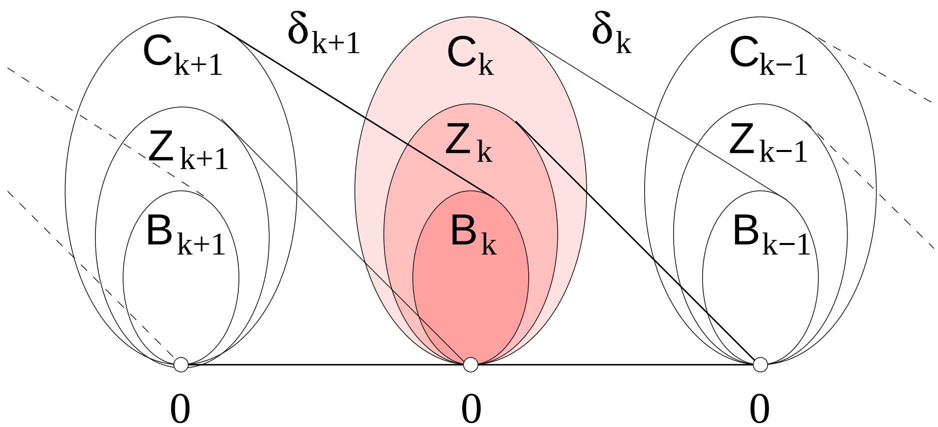
HOMOLOGY



Afra Zomorodian

CS 468 – Lecture 6

2-18-4

REINTERPRETING HOMOTOPY

- Let $l \in L$, where L is the set of all loops
- let $b \in B$, where B is the set of all boundaries
- $lb \simeq l$, so $lb \in [l]$, $\cdot$ is path product
- Define $lB = \{lb \mid b \in B\}$ and $Bl = \{bl \mid b \in B\}$
- But $lb \simeq l \simeq bl$, so $lB = [l] = Bl$ (up to homotopy)
- So, B is normal
- $\pi_1 = L/B$

GROUP THEORY REVIEW

V_4	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

- Cosets
  - equivalence sets induced by subgroup
  - $b + \{e, a\} = \{b, c\}$
  - $a + \{e, b\} = \{a, c\}$

Group Theory Review 2

- Normal
  - left and right cosets coincide
  - all subsets of abelian groups
- Factor Group G/H
  - group G with normal subgroup H
  - elements are cosets of H
  - operation is coset multiplication

Group Theory Review 3

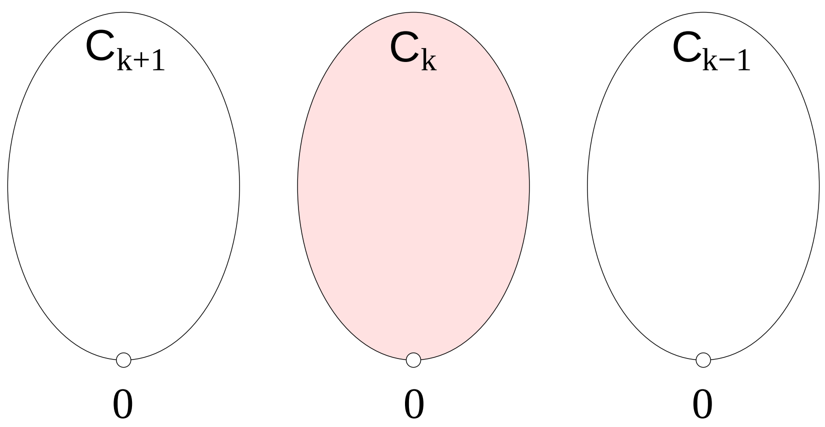
- Group V_4
- Normal subgroup $H = \{e, b\}$
- Cosets (classes) of $V_4/\{e, b\}$
  - $x + H$, where $x \in V_4$
  - $e + \{e, b\} = \{e, b\} = b + \{e, b\}$
  - $a + \{e, b\} = \{a, c\} = c + \{e, b\}$
- Addition: $(b + \{e, b\}) + (c + \{e, b\}) = (b + c) + \{e, b\} = a + \{e, b\}$
- Key idea: adding elements from subgroup does not change equivalence class. That is, $(x + H) + h = x + H$, where $h \in H$
- Example: $a + \{e, b\} + b = \{b, e\} = a + \{e, b\}$

WHY HOMOLOGY?

- Algebraization of first layer of geometry in structures
- How cells of dimension n attach to cells of dimension $n - 1$
- Our case: cells are simplices
- Less transparent, more machinery
- Combinatorial
- Finite description
- Computable

CHAIN GROUP

- Simplicial complex K
- k-chain: $c = \sum_i n_i[\sigma_i], n_i \in \mathbb{Z}, \sigma_i \in K$ (like a path)
- $[\sigma] = -[\tau]$ if $\sigma = \tau$ and σ and τ have different orientations.
- The kth chain group C_k of K is the free abelian group on its set of oriented k-simplices
- $\operatorname{rank} \mathsf{C}_k = ?$



BOUNDARY OPERATOR

- The boundary operator $\partial_k : \mathsf{C}_k \rightarrow \mathsf{C}_{k-1}$ is a homomorphism defined linearly on a chain c by its action on any simplex $\sigma = [v_0, v_1, \ldots, v_k] \in c$,

$$\partial_k \sigma = \sum_i (-1)^i [v_0, v_1, \ldots, \hat{v_i}, \ldots, v_k],$$

  where $\hat{v_i}$ indicates that v_i is deleted from the sequence.

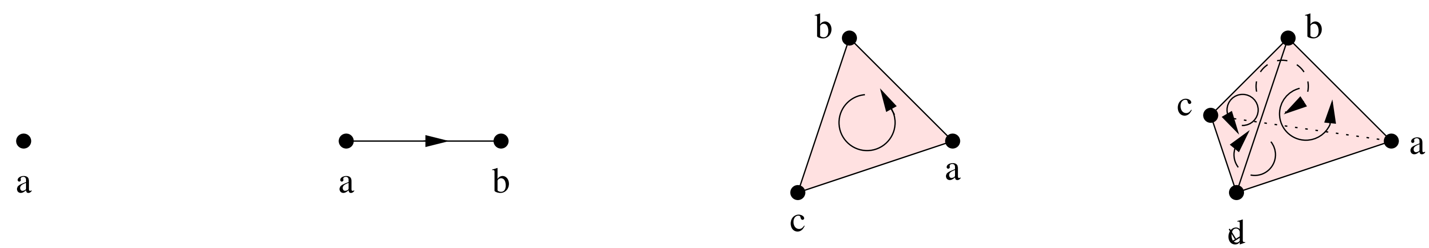
- $\partial_1 [a, b] = b - a$.

- $\partial_2 [a, b, c] = [b, c] - [a, c] + [a, b] = [b, c] + [c, a] + [a, b]$.

- $\partial_3 [a, b, c, d] = [b, c, d] - [a, c, d] + [a, b, d] - [a, b, c]$.

BOUNDARY OPERATOR

- $\partial_1[a, b] = b - a$.
- $\partial_2[a, b, c] = [b, c] - [a, c] + [a, b] = [b, c] + [c, a] + [a, b]$.
- $\partial_3[a, b, c, d] = [b, c, d] - [a, c, d] + [a, b, d] - [a, b, c]$.
- $\partial_1 \partial_2[a, b, c] = [c] - [b] - [c] + [a] + [b] - [a] = 0$.



vertex a | edge [a, b] | triangle [a, b, c] | tetrahedron [a, b, c, d]

Boundary Theorem

- (Theorem) $\partial_{k-1}\partial_k = 0$, for all k.
- Proof:

  $\partial_{k-1}\partial_k[v_0, v_1, \ldots, v_k] =$

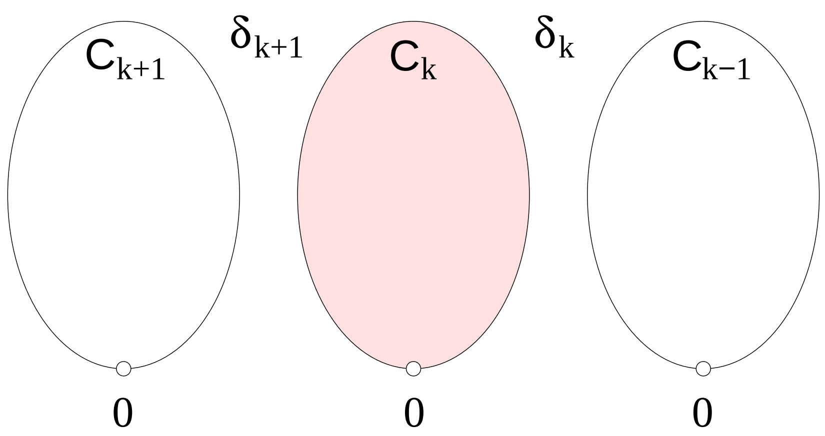
$$\begin{aligned}
&= \partial_{k-1}\sum_i (-1)^i [v_0, v_1, \ldots, \hat{v_i}, \ldots, v_k] \\
&= \sum_{j<i} (-1)^i (-1)^j [v_0, \ldots, \hat{v_j}, \ldots, \hat{v_i}, \ldots, v_k] \\
&\quad + \sum_{j>i} (-1)^i (-1)^{j-1} [v_0, \ldots, \hat{v_i}, \ldots, \hat{v_j}, \ldots, v_k] \\
&= 0,
\end{aligned}$$

  as switching i and j in the second sum negates the first sum.

CHAIN COMPLEX

- The boundary operator connects the chain groups into a chain complex C_*:

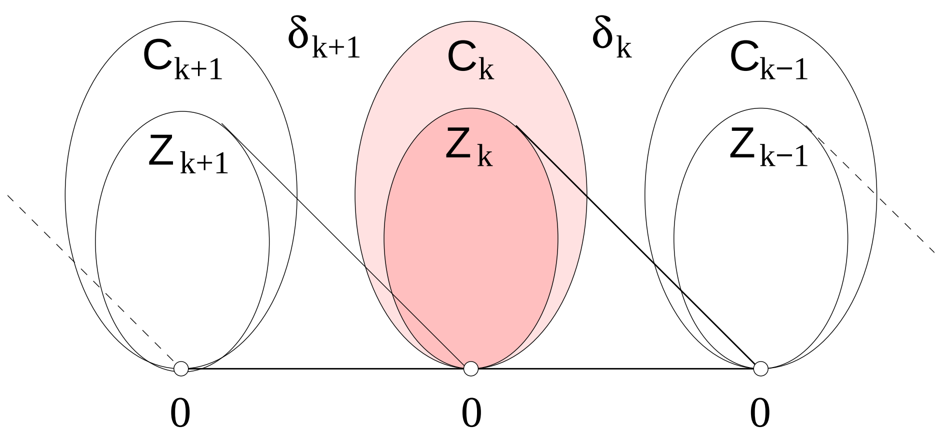
$$\ldots \to \mathsf{C}_{k+1} \xrightarrow{\partial_{k+1}} \mathsf{C}_k \xrightarrow{\partial_k} \mathsf{C}_{k-1} \to \ldots .$$



CYCLE GROUP

- Let c be a k-chain
- If it has no boundary, it is a k-cycle
- $\partial_k c = \emptyset$, so $c \in \ker \partial_k$
- The kth cycle group is

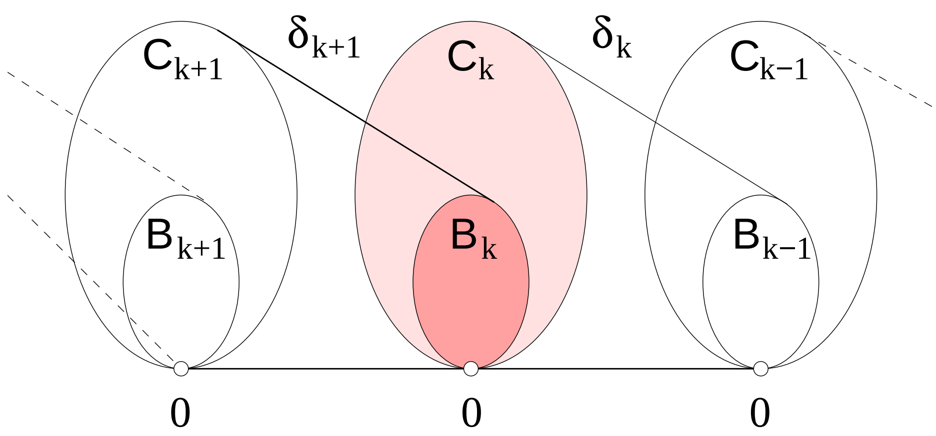
$$\mathsf{Z}_k = \ker \partial_k = \{c \in \mathsf{C}_k \mid \partial_k c = \emptyset\}.$$



Boundary Group

- Let b be a k-chain
- If b is a boundary of something, it is a k-boundary.
- The kth boundary group is

$$\mathsf{B}_k = \operatorname{im} \partial_{k+1} = \{c \in \mathsf{C}_k \mid \exists d \in \mathsf{C}_{k+1} : c = \partial_{k+1} d\}.$$



Relationship

- Let b be a k-boundary.
- Then, $\exists c \in \mathsf{C}_{k+1}$, such that $b = \partial_{k+1} c$.
- What is the boundary of b?

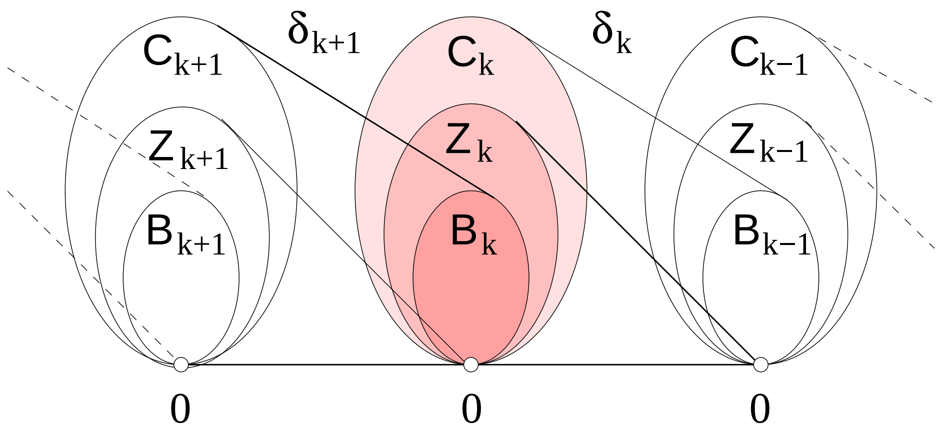
$$\partial_k b = \partial_k \partial_{k+1} c = \emptyset,$$

  by the boundary theorem.

- That is, every boundary is a cycle!
- What is the point-set theoretic version?

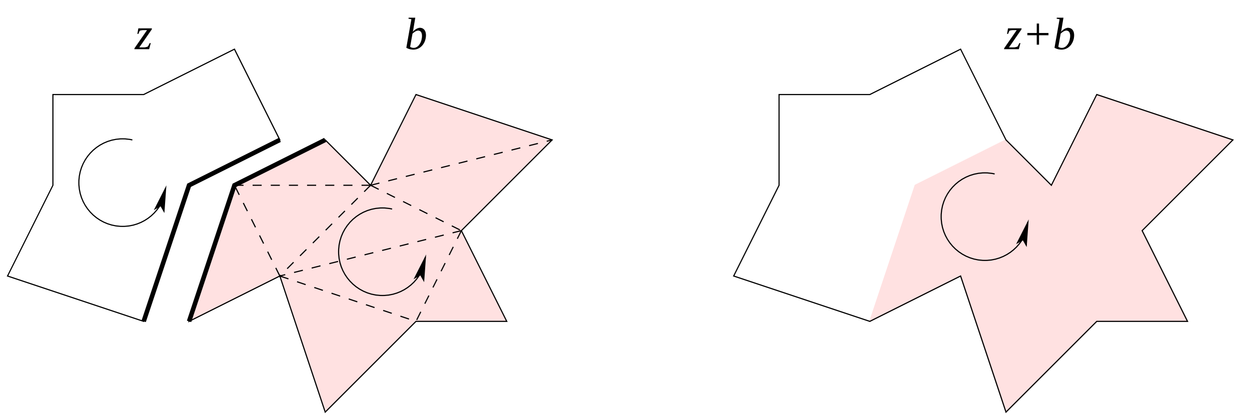
NESTING

- $\mathsf{B}_k \subseteq \mathsf{Z}_k \subseteq \mathsf{C}_k$
- Chains are analogs of paths
- Cycles are analogs of loops
- Boundaries are analogs of bounding loops
- We need a simplicial analog of homotopy!



ADDING CYCLES

- z is a k-cycle
- b is a k-boundary
- We would like to have $z + b$ be equivalent to z
- That is, if $z_1 - z_2 = b$ where b is a boundary, then $z_1 \sim z_2$
- Any boundary would do!

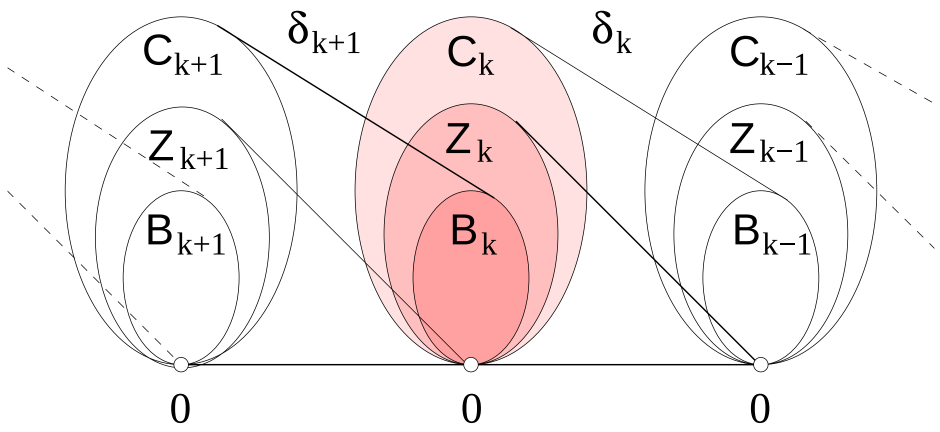


SIMPLICIAL HOMOLOGY

- The kth homology group is

$$\mathsf{H}_k = \mathsf{Z}_k / \mathsf{B}_k = \ker \partial_k / \operatorname{im} \partial_{k+1}.$$

- If $z_1 = z_2 + \mathsf{B}_k$, $z_1, z_2 \in \mathsf{Z}_k$, we say z_1 and z_2 are homologous
- $z_1 \sim z_2$.



Description

- Homology groups are finitely generated abelian.
- (Theorem) Every finitely generated abelian group is isomorphic to product of cyclic groups of the form

$$\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \ldots \times \mathbb{Z}_{m_r} \times \mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z},$$

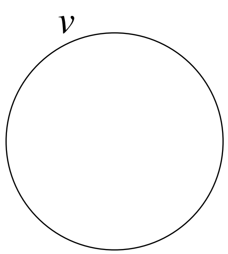
- The kth Betti number β_k of a simplicial complex K is $\beta_k = \beta(\mathsf{H}_k)$, the rank of the free part of H_k.
- Torsion coefficients $m_1, m_2, \ldots, m_r$

INTERPRETATION

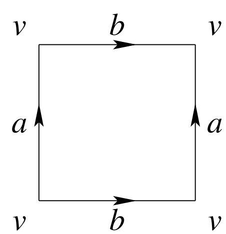
- Compactify $\mathbb{R}^3$ via a one point compactification to get $\mathbb{S}^3$
- Subcomplexes are torsion-free
- Alexander Duality:
  - β_0 measures the number of components of the complex.
  - β_1 is the rank of a basis for the tunnels.
  - β_2 counts the number of voids in the complex.

HOMOLOGY OF 2-MANIFOLDS

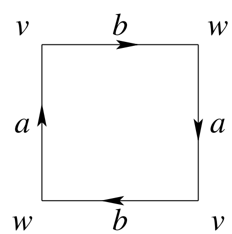
2-manifold	H_0	H_1	H_2
(a) sphere	$\mathbb{Z}$	$\{0\}$	$\mathbb{Z}$
(b) torus	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z}$
(c) projective plane	$\mathbb{Z}$	$\mathbb{Z}_2$	$\{0\}$
(d) Klein bottle	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\{0\}$



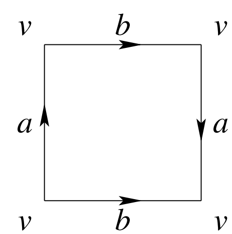
(a) Sphere



(b) Torus



(c) Projective plane



(d) Klein bottle

INVARIANCE

- (Hauptvermutung) Any two triangulations of a topological space have a common refinement (Poincaré 1904)
  - True for polyhedra of dimension ≤ 2 (Papakyriakopoulos 1943)
  - True for 3-manifolds (Moïse 1953)
  - False in dimensions ≥ 6 (Milnor 1961)
  - False for manifolds of dimension ≥ 5 (Kirby and Siebenmann 1969)
- Singular homology
- Axiomatization
- (Theorem) $\mathbb{X} \simeq \mathbb{Y} \Rightarrow \mathsf{H}_*(\mathbb{X}) = \mathsf{H}_*(\mathbb{Y})$

Euler Revisited

- Let K be a simplicial complex and $s_i = |\{\sigma \in K \mid \dim \sigma = i\}|$. The Euler characteristic $\chi(K)$ is

$$\chi(K) = \sum_{i=0}^{\dim K} (-1)^i s_i = \sum_{\sigma \in K - \{\emptyset\}} (-1)^{\dim \sigma}.$$

- We have new language!
- Let C_* be the chain complex on K
- $\text{rank}(\mathsf{C}_i) = |\{\sigma \in K \mid \dim \sigma = i\}|$
- $\chi(K) = \chi(\mathsf{C}_*) = \sum_i (-1)^i \, \text{rank}(\mathsf{C}_i)$.

EULER-POINCARÉ

- Chain complex C_*:

$$\ldots \rightarrow \mathsf{C}_{k+1} \xrightarrow{\partial_{k+1}} \mathsf{C}_k \xrightarrow{\partial_k} \mathsf{C}_{k-1} \rightarrow \ldots.$$

- $\mathsf{H}_*(\mathsf{C}_*)$ is a chain complex:

$$\ldots \rightarrow \mathsf{H}_{k+1} \xrightarrow{\partial_{k+1}} \mathsf{H}_k \xrightarrow{\partial_k} \mathsf{H}_{k-1} \rightarrow \ldots.$$

- What is its Euler characteristic?
- (Theorem) $\chi(K) = \chi(\mathsf{C}_*) = \chi(\mathsf{H}_*(\mathsf{C}_*))$.
- $\sum_i (-1)^i s_i = \sum_i (-1)^i \operatorname{rank}(\mathsf{H}_i) = \sum_i (-1)^i \beta_i$
- Sphere: $2 = 1 - 0 + 1$
- Torus: $0 = 1 - 2 + 1$