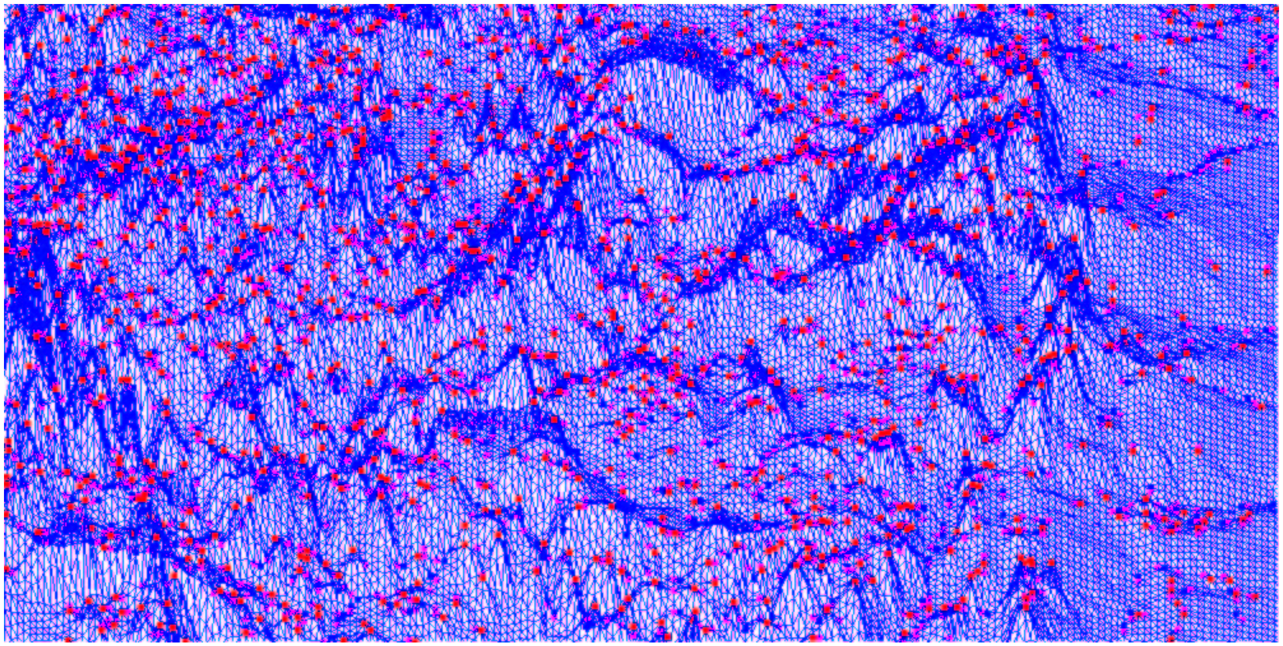
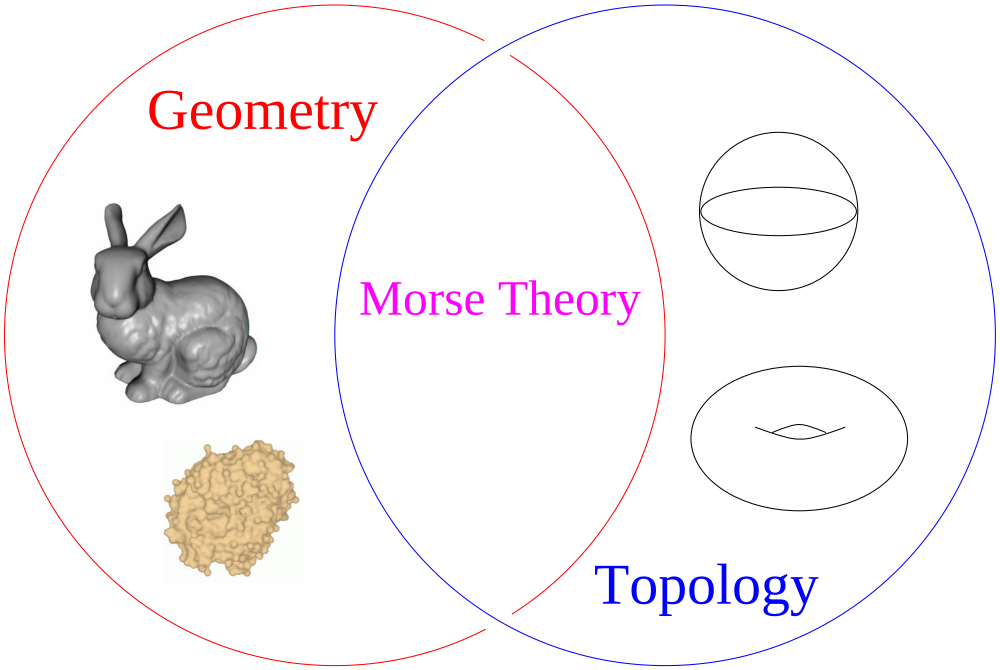
MORSE THEORY



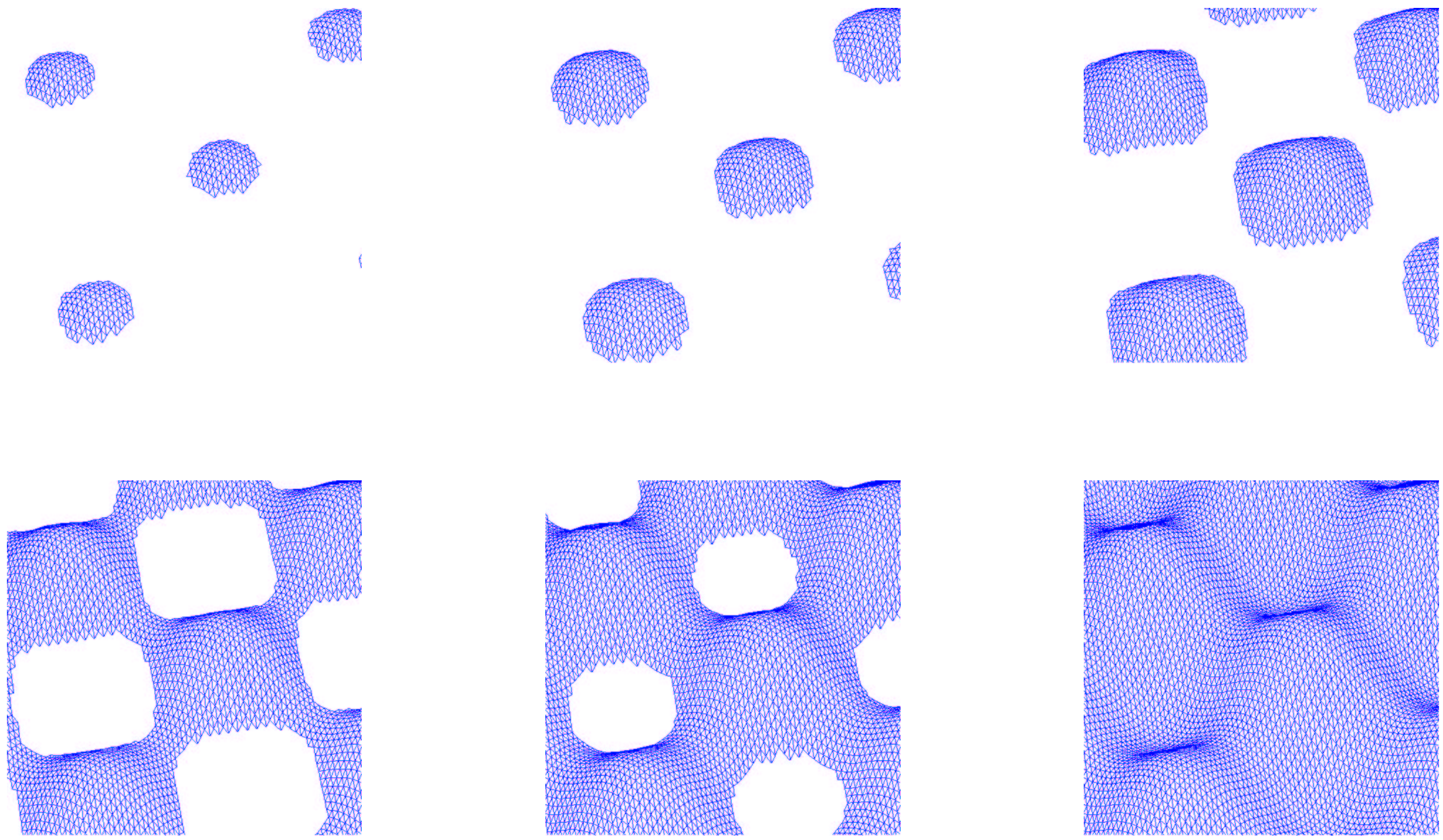
CS 468 – Lecture 9

3/10/4

SHAPE



EXCURSIONS



OVERVIEW

- Relationship between Geometry and Topology
- Tangent Spaces
- Derivatives
- Critical points
- Persistence

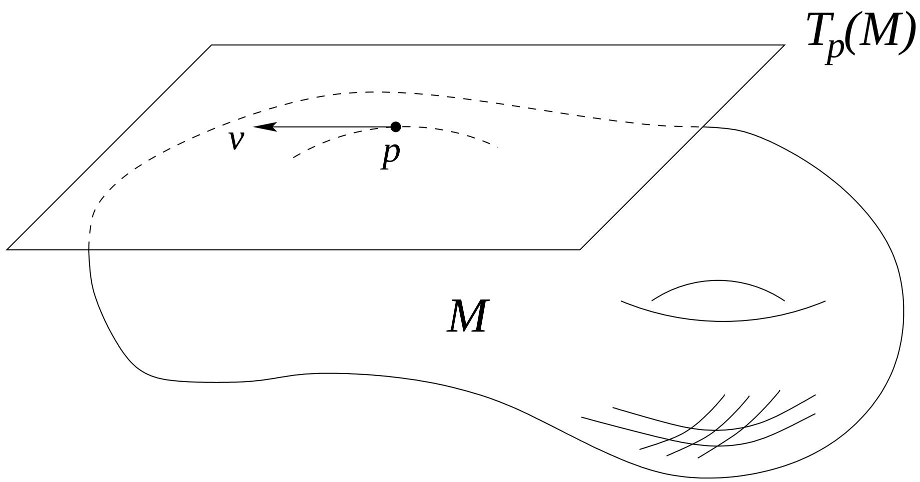
TANGENT SPACE $T_p(\mathbb{R}^3)$

- A tangent vector v_p to $\mathbb{R}^3$ consists of two points of $\mathbb{R}^3$:
  - vector part v
  - point of application p.
- The set $T_p(\mathbb{R}^3)$ consists of all tangent vectors to $\mathbb{R}^3$ at p, and is called the tangent space of $\mathbb{R}^3$ at p.

TANGENTS SPACE $T_p(\mathbb{M})$

- $\mathbb{M}$ is a smooth, compact, 2-manifold without boundary
- $\mathbb{M} \subset \mathbb{R}^3$ is embedded (not necessary, extends)
- Let p be a point on $\mathbb{M}$ in $\mathbb{R}^3$.
- A tangent vector v_p to $\mathbb{R}^3$ at p is tangent to $\mathbb{M}$ at p if v is the velocity of some curve in $\mathbb{M}$.
- The set of all tangent vectors to M at p is called the tangent plane of M at p and is denoted by $T_p(\mathbb{M})$.

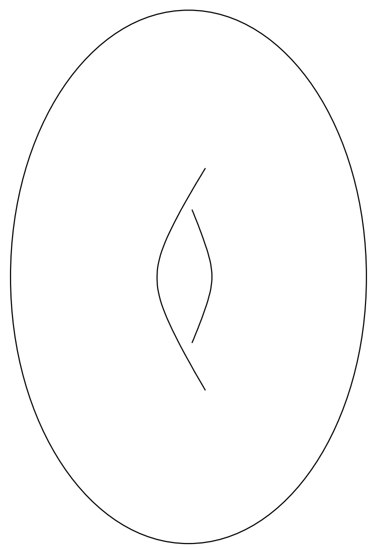
TANGENT PLANE



- A patch is the inverse of a chart.
- Let $p \in \mathbb{M} \subset \mathbb{R}^3$, and let φ be a patch in $\mathbb{M}$ such that $\varphi(u_0, v_0) = p$.
- A tangent vector v to $\mathbb{R}^3$ at p is tangent to $\mathbb{M}$ iff v can be written as a linear combination of $\varphi_u(u_0, v_0)$ and $\varphi_v(u_0, v_0)$.
- $T_p(\mathbb{M})$ is the best linear approximation of the surface M near p.

Functions on Manifolds

- A vector: direction for moving
- Real valued smooth function $h : \mathbb{M} \to \mathbb{R}$.
- How does h vary in direction v_p?

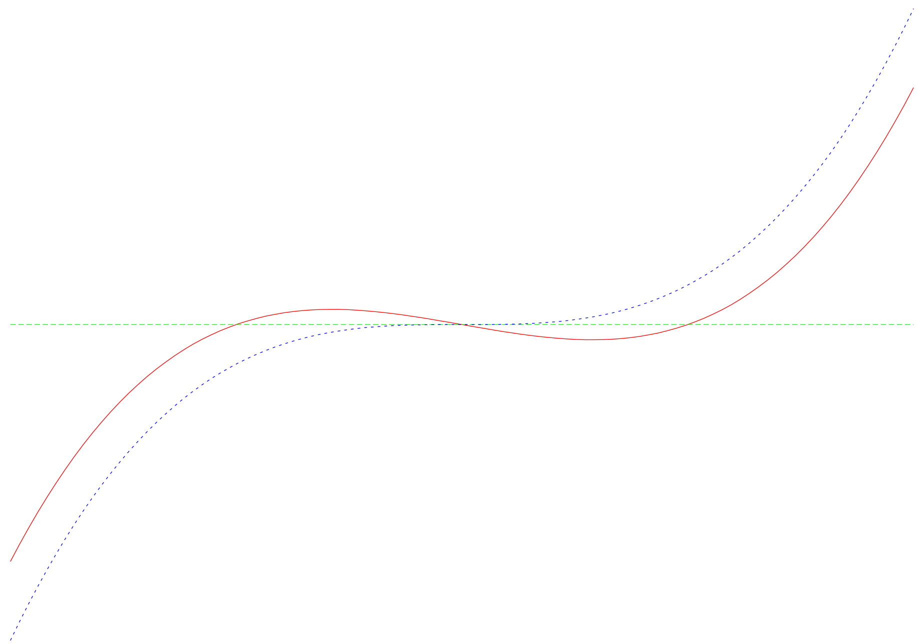


DERIVATIVES

- A vector field or flow on V is a function that assigns a vector $v_p \in T_p(\mathbb{M})$ to each point $p \in \mathbb{M}$.
- The derivative $v_p[h]$ of h with respect to v_p is the common value of $(d/dt)(h \circ \gamma)(0)$, for all curves $\gamma \in \mathbb{M}$ with initial velocity v_p.
- The differential dh_p of $h : \mathbb{M} \to \mathbb{R}$ at $p \in \mathbb{M}$ is a linear function on $T_p(\mathbb{M})$ such that $dh_p(v_p) = v_p[h]$, for all tangent vectors $v_p \in T_p(\mathbb{M})$.
- A differential converts vector fields to real-valued functions

CRITICAL POINTS

- Travel in all directions in $T_p(\mathbb{M})$
- A point $p \in \mathbb{M}$ is critical for map $h : \mathbb{M} \to \mathbb{R}$ if dh_p is the zero map.
- Otherwise, p is regular.



DEGENERACIES

- Let x, y be a patch on $\mathbb{M}$ at p.
- The Hessian of $h : \mathbb{M} \to \mathbb{R}$ is:

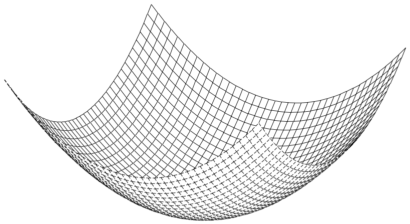
$$H(p) = \begin{bmatrix} \frac{\partial^2 h}{\partial x^2}(p) & \frac{\partial^2 h}{\partial y \partial x}(p) \\ \frac{\partial^2 h}{\partial x \partial y}(p) & \frac{\partial^2 h}{\partial y^2}(p) \end{bmatrix}.$$

- Basis $(\frac{\partial}{\partial x}(p), \frac{\partial}{\partial y}(p))$ for $T_p(\mathbb{M})$.
- A critical point $p \in \mathbb{M}$ is non-degenerate if the Hessian is non-singular at p, i.e. $\det H(p) \neq 0$.
- Otherwise, it is degenerate.

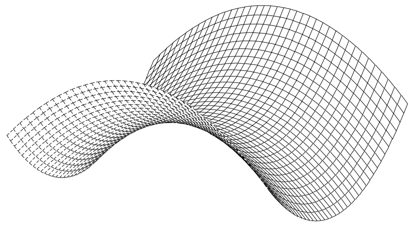
Morse Functions

- A smooth map $h : \mathbb{M} \rightarrow \mathbb{R}$ is a Morse function if all its critical points are non-degenerate.
- Any twice differentiable function h may be unfolded to a Morse function.
- As close to h as we specify!
- Morse functions are dense

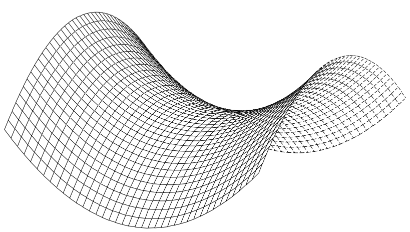
Morse Lemma



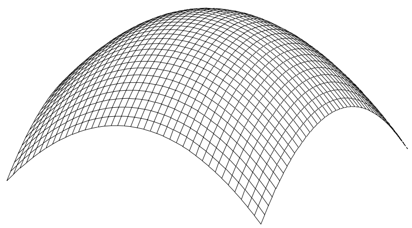
(a) $x^2 + y^2$



(b) $-x^2 + y^2$



(c) $x^2 - y^2$



(d) $-x^2 - y^2$

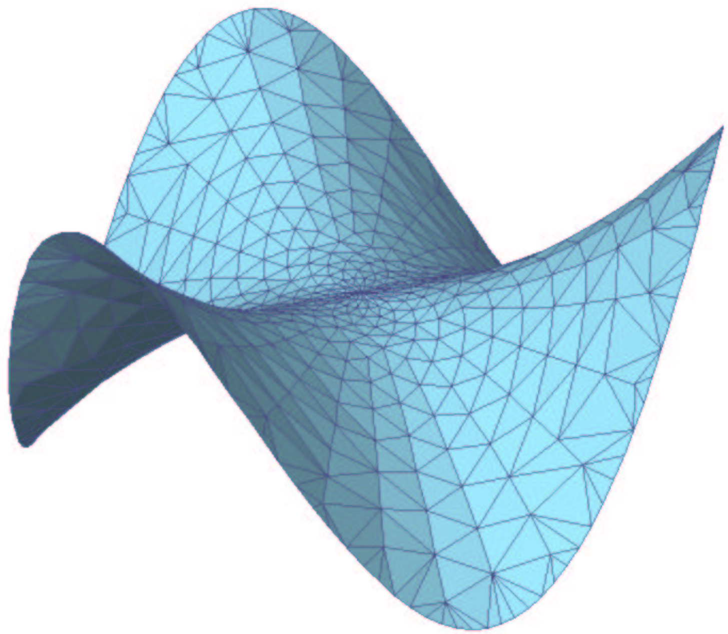
INDICES

- (Lemma) It is possible to choose local coordinates x, y at a critical point $p \in \mathbb{M}$, so that a Morse function h takes the form:

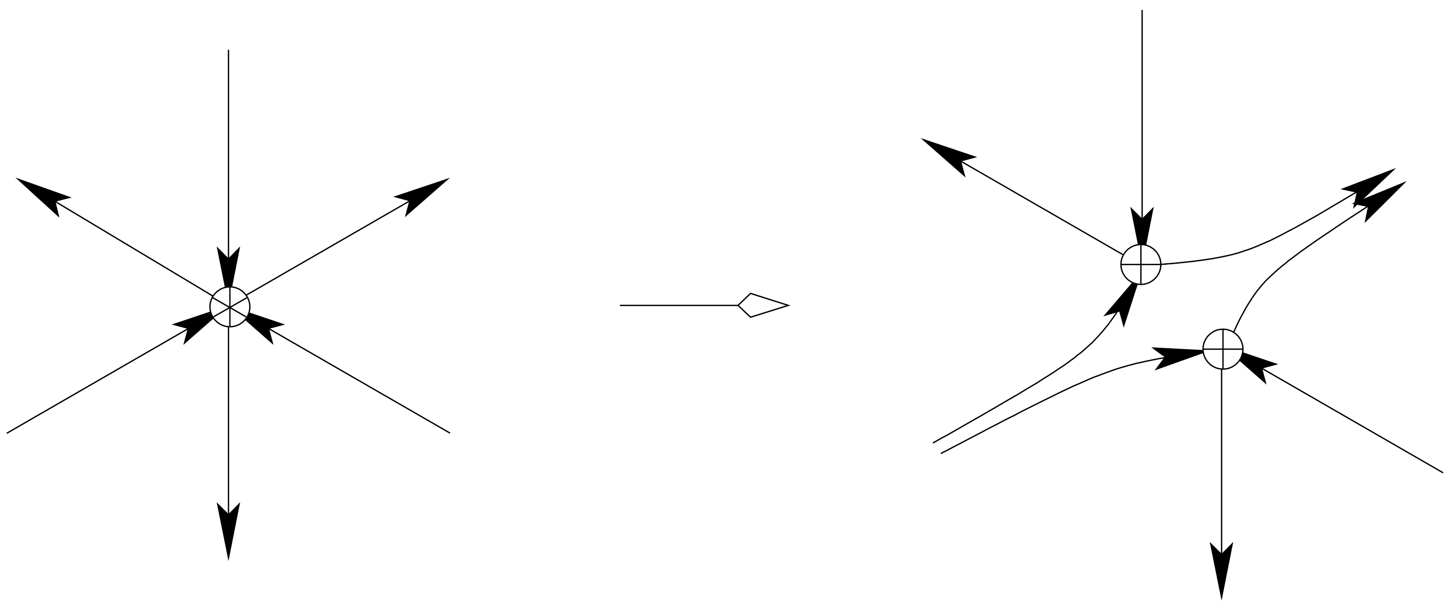
$$h(x, y) = \pm x^2 \pm y^2$$

- The index i(p) of h at critical point $p \in \mathbb{M}$ is the number of minuses.
- Equivalently, the index at p is the number of the negative eigenvalues of $H(p)$.
- A critical point of index 0, 1, or 2, is called a minimum, saddle, or maximum, respectively.

MONKEY SADDLE



UNFOLDING



PL FUNCTIONS

- Let K be a triangulation of a compact 2-manifold without boundary $\mathbb{M}$.
- Let $h : \mathbb{M} \rightarrow \mathbb{R}$ be a function that is linear on every triangle.
- The function is defined by its values at the vertices of K.
- Assume $h(u) \neq h(v)$ for all vertices $u \neq v \in K$.
- Sometimes called a height function over a 2-manifold.

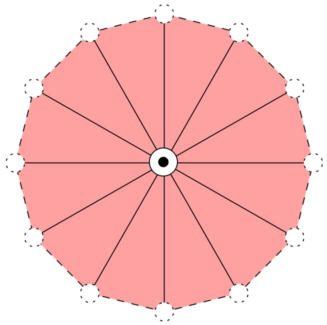
STARS

- Recall: the star of a vertex u in a triangulation K is $\text{St}\, u = \{\sigma \in K \mid u \leq \sigma\}$.
- The lower and upper stars of u for a height function h are

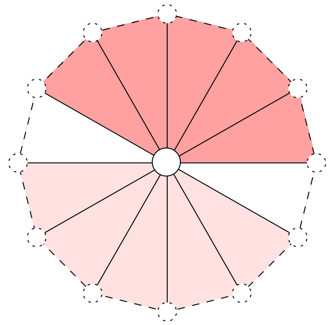
$$\begin{aligned} \underline{\text{St}}\, u &= \{\sigma \in \text{St}\, u \mid h(v) \leq h(u), \forall \text{ vertices } v \leq \sigma\} \\ \overline{\text{St}}\, u &= \{\sigma \in \text{St}\, u \mid h(v) \geq h(u), \forall \text{ vertices } v \leq \sigma\} \end{aligned}$$

- Suppose u is a maximum. What's $\underline{\text{St}}\, u$? What's $\overline{\text{St}}\, u$?
- $K = \dot{\bigcup}_u \underline{\text{St}}\, u = \dot{\bigcup}_u \overline{\text{St}}\, u$.

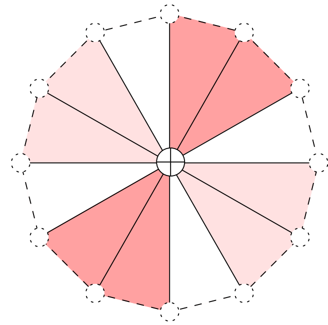
PL STARS



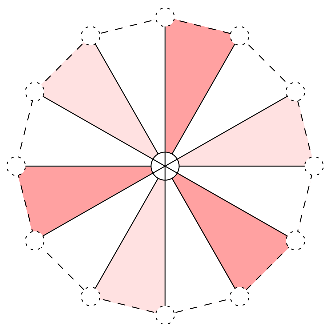
(a) minimum



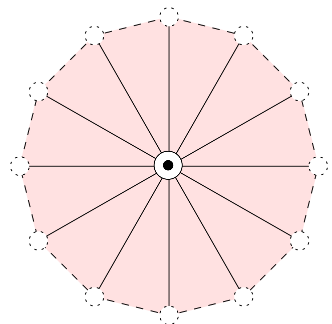
(b) regular



(c) saddle



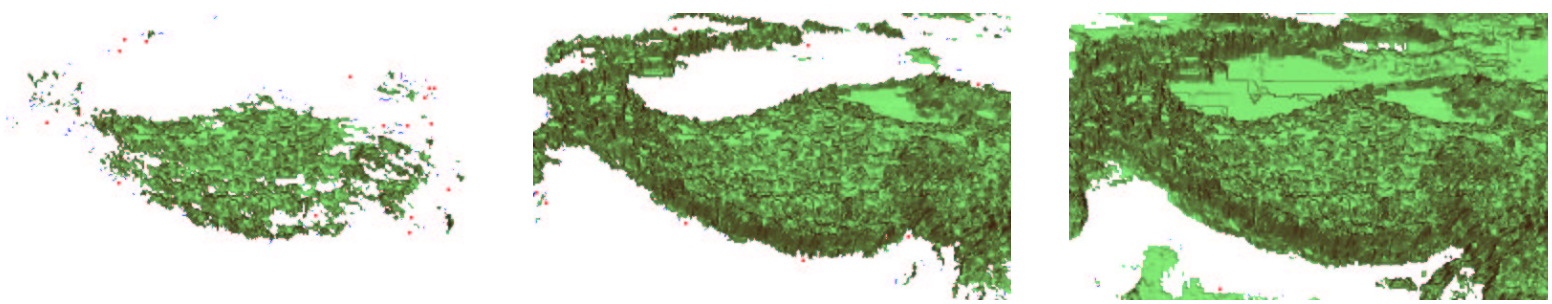
(d) monkey



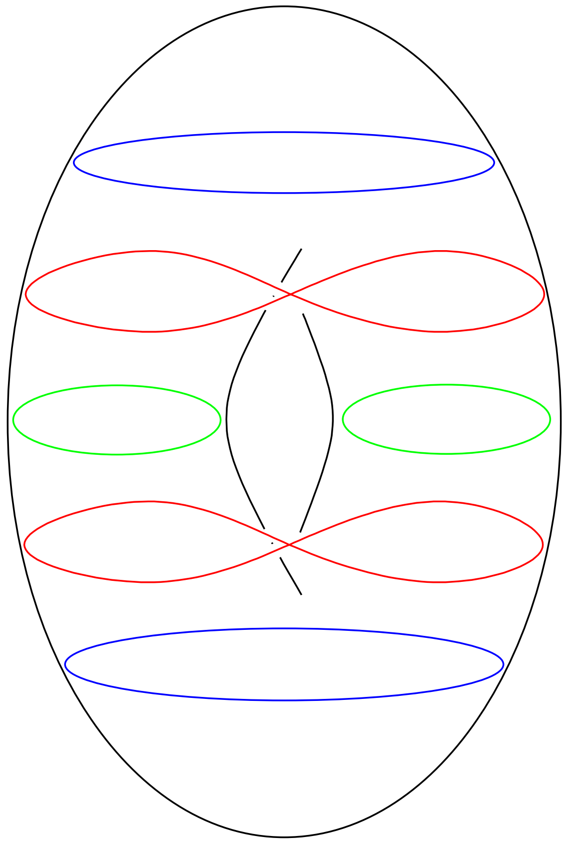
(e) maximum

FILTRATIONS

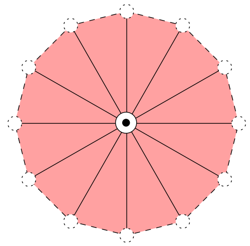
- Sort the n vertices of K in order of increasing height to get the sequence $u^1, u^2, \ldots, u^n, h(u^i) < h(u^j)$, for all $1 \leq i < j \leq n$.
- Let K^i be the union of the first i lower stars, $K^i = \bigcup_{1 \leq j \leq i} \underline{\mathrm{St}}\, u^j$.
- Same idea with upper stars
- Recall $\chi = v - e + t = \beta_0 - \beta_1 + \beta_2$



LEVELS OF TORUS

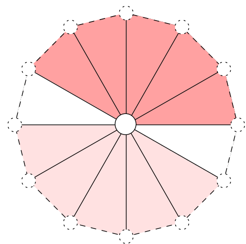


MINIMUM



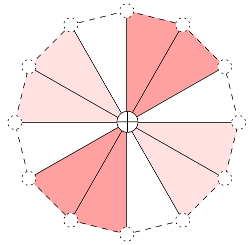
- $\underline{\text{St}}\, u^i = u^i$, so a minimum vertex is a new component and $\chi^i = \chi^{i-1} + 1$.
- $\beta_0^i = \beta_0^{i-1} + 1$, $\beta_1^i = \beta_1^{i-1}$, $\beta_2^i = \beta_2^{i-1}$
- Therefore, $\chi^i = \beta_0^{i-1} + 1 - \beta_1^{i-1} + \beta_2^{i-1} = \chi^{i-1} + 1$
- So, a minimum creates a new 0-cycle and acts like a positive vertex in the filtration of a complex.
- The vertex is unpaired at time i.

REGULAR



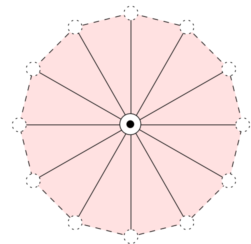
- $\underline{\text{St}}\, u^i$ is a single wedge, so $\chi^i = \chi^{i-1} + 1 - 1 = \chi^{i-1}$.
- $\underline{\text{St}}\, u^i \neq \emptyset$ so $\beta_0^i = \beta_0^{i-1}$
- $\overline{\text{St}}\, u^i \neq \emptyset$ so $\beta_2^i = \beta_2^{i-1}$ (also duality!)
- Using Euler-Poincaré, we get $\beta_1^i = \beta_1^{i-1}$.
- No topological changes!
- All the cycles created at time i are also destroyed at time i.
- The positive and negative simplices in $\underline{\text{St}}\, u^i$ all paired and cancel

SADDLE



- $\underline{\text{St}}\, u^i$ has two wedges, bringing in two more edges than triangles.
- $\chi^i = \chi^{i-1} + 1 - 2 = \chi^{i-1} - 1$.
- If this saddle connects two components, it destroys a 0-cycle and $\beta_0^i = \beta_0^{i-1} - 1$.
- Otherwise, it creates a new 1-cycle and $\beta_1^i = \beta_1^{i-1} + 1$.
- All the simplices in a saddle are paired, except for a single edge whose sign corresponds to the action of the saddle.
- We have $\chi^i = \chi^{i-1} - 1$ in either case.

Maximum



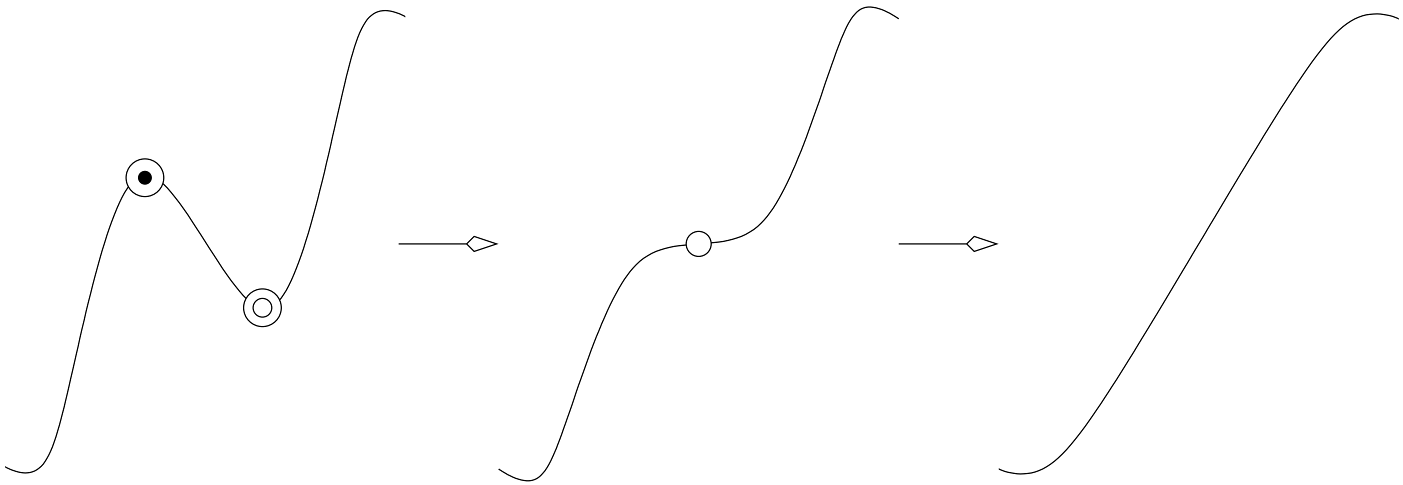
- $\underline{\mathrm{St}}\, u^i = \mathrm{St}\, u^i$ and has the same number of edges and triangles.
- So, $\chi^i = \chi^{i-1} + 1$ for the single vertex.
- In our case, one global minimum
- If global minimum, $\beta_2^i = \beta_2^{i-1} + 1 = 1$.
- Otherwise, the lower star covers a 1-cycle and $\beta_1^i = \beta_1^{i-1} - 1$.
- Single unpaired triangle (positive or negative)
- We have $\chi^i = \chi^{i-1} + 1$ in both cases.

CORRESPONDENCE

critical	unpaired	action
minimum	vertex	$\boxed{\beta_{0^{++}}}$
saddle	edge	$\boxed{\beta_{0^{--}}}$ or $\beta_{1^{++}}$
maximum	triangle	$\beta_{1^{--}}$ or $\beta_{2^{++}}$

- Correspondence allows us to talk about persistent critical points
- Let m_i be the number of index i critical points in K
- $\chi(K) = \sum_i (-1)^i s_i = \sum_i (-1)^i \beta_i = \sum_i (-1)^i m_i$

CANCELLATION



- Pairs of critical points annihilate each other
- Inverse unfolding plus smoothing
- Need additional structure (Morse-Smale Complex) to do this geometrically

CONCLUSION

- Geometry and topology are deeply related.
- If you want to modify one, you need to pay attention to the other.
- Morse theory relates critical points of a function on a manifold (geometry) to the topology of the manifold
- $\chi(K) = \sum_i (-1)^i s_i = \sum_i (-1)^i \beta_i = \sum_i (-1)^i m_i$

Class Recap

1. Point set topology
2. Surface topology and ZIP
3. Simplicial complexes
4. Group theory
5. Homotopy and Markov's undecidability
6. Homology
7. Computing homology
8. Topology of PCD
9. Morse theory