Moving Least Squares

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Outline

• The Approximation Power of Moving Least-Squares
  *D. Levin*

• Mesh-Independent Surface Interpolation
  *D. Levin*

• Defining point-set surfaces
  *N. Amenta and Y. Kil*
Problem

- Collection of point
  - Source of data: laser scanner
- Points are unorganized
- Usually no information about normal
  - But not always the case (next paper)
Applications

• Implicit surface definition
  • Projection operator
• Noise removal / Thinning
• Upsampling
• Ray tracing
Interpolation vs Smoothing
One Approach (Mesh based)

- Smooth interpolation by joining local patches each being an approximation in local reference domain.
- Piecewise polynomial patches.
- In most cases, result depends on the mesh defining the patches.
Example

350 pieces/patches
Alternative Approach (Meshless)

- Implicit definition of surface.
- $S = f(\{p_i\})$
Roadmap

Given

R = \{x_i\}

Goal

• Define a projection operator P

  \( x \in \mathbb{R}^d \quad P : x \rightarrow P(x) \in S \)

• Unique manifold \( S \equiv \{x \mid P(x) = x\} \)
MLS Approach

• Step 1
  • Define a local/reference domain (like a tangent plane)
  • Local parameterization
 MLS Approach

• Step 1
  • Define a local/reference domain

• Step 2
  • MLS approximation wrt reference domain (polynomial fitting)
Fitting Functions

Given *(functional setting)*

\{x_i, f_i\}

Goal

Find \( p \) in \( \Pi_m \) such that \( \{x_i, f_i\} \) satisfies

\[
\min_{p \in \Pi_m} \sum_{i} (p(x_i) - f_i)^2 \theta(\|x_i\|) \]

\[\text{error} \quad \text{weight}\]
θ : The Weight Function

- Non-negative decaying function
- Typical example
  - Gaussian kernel \( \theta(d) = \exp(-d^2/h^2) \)
Basic MLS

• For a given point $r$ near $R$, define a local approximating hyper-planer $H_r$
Equation of a line

$$H_{a,D} = \{ x \mid <a, x> - D = 0, x \in \mathbb{R}^d \}, a \in \mathbb{R}^d, \|a\| = 1$$
Basic MLS

• For a given point \( r \) near \( R \), define a local approximating hyper-planer \( H_r \)
Basic MLS

• For a given point $r$ near $R$, define $H_r$

$$\min_{a,D} \sum_i (a^T r_i - D)^2 \theta(||r_i - r||)$$

• In case of multiple local minima, the plane closest to $r$ is chosen.
Basic MLS

- For a given point \( r \) near \( R \), define \( H_r \)

Find a polynomial approx. of degree \( m \)

\[
\min_{p \in \Pi_m^{d-1}} \sum_i (p(x_i) - f_i)^2 \theta(|| r_i - r ||)
\]
MLS

Step 1

Step 2
Projection?

\[ \tilde{P}_m(\tilde{P}_m(r)) \neq \tilde{P}_m(r) \]

\[ \theta(||r_i - r||) \]

\[ \min_{a,D} \sum_i (<a, r_i > - D)^2 \theta(||r_i - r||) \]
Basic MLS

$$\tilde{P}_m(\tilde{P}_m(r)) \neq \tilde{P}_m(r)$$

- Doesn’t project points to a (d-1)-dim manifold.
- Doesn’t define a surface.
Simple fix

\[ \theta(\| r_i - r \|) \quad \theta(\| r_i - q \|) \]
Non-linear Optimization

\[ I(q, a) = \min_{a,D} \sum_i (\langle a, r_i \rangle - D)^2 \theta(\| r_i - q \|) \]

constraints

\[
\begin{align*}
(r - q) \parallel a(q) \\
J(q) &= I(q, a(q)) \\
\partial_{a(q)} J(q) &= 0
\end{align*}
\]
Basic MLS

- For a given point $r$ near $R$, define $H_r$

\[
\min_{a,D} \sum_i (\langle a, r_i \rangle - D)^2 \theta(\|r_i - q\|)
\]

- In case of multiple local minima, the plane closest to $r$ is chosen.
Given

\[ R = \{ x_i \} \]

**MLS**

- Define a projection operator \( P(P(x)) = P(x) \)
- Unique manifold \( S \equiv \{ x | P(x) = x \} \)
- Conjecture \( S \) is \( C^\infty \)
MLS surface
Computing $H_r$ and $p$

- Computing hyper-plane $H_r$
- Non-linear optimization problem
- Computed iteratively
- Computing $\theta()$: time consuming step
  - $O(N)$ for each iteration step
  - Approximate by doing a hierarchical clustering
- Fitting a polynomial $p(.)$, given $H_r$
  - Solve a linear system
    - Size depends on the order of approximation ($m$)
Applications: Denoising
Applications: Upsampling / Hole Filing
Applications: Ray Tracing
Sampling Condition?
Conclusions

- Surface is smooth and a manifold
- Adjustable feature size $h$ allows to smooth out noise

- The surface changes with addition of points.