“Mean Value Coordinates for Closed Triangular Meshes”

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Outline

- Abstract
- Preliminaries
- Previous work
- Mean value Interpolation
- 3D Mean value coordinates for closed triangular meshes
- Applications
- Questions
Abstract

- Search for a function that can interpolate a set of values at the vertices of a mesh smoothly into its interior
- Mean value coordinates have been used as an interpolant for closed 2D polygons.
Abstract

- This paper generalizes the mean value coordinates to closed triangular meshes
- Interesting applications to surface deformation and volumetric textures
For any function that is continuous on \([a, b]\) and differentiable on \((a, b)\) there exists some \(c\) in the interval \((a, b)\) such that the secant joining the endpoints of the interval \([a, b]\) is parallel to the tangent at \(c\).
Harmonic functions and Mean Value property

A harmonic function is twice continuously differentiable function $f: U \rightarrow \mathbb{R}$ which satisfies the Laplace’s equation

$$
\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0
$$

everywhere on $U$. This is also often written as

$$
\nabla^2 f = 0 \quad \text{or} \quad \Delta f = 0.
$$
Harmonic functions and Mean Value property

- They attain there maxima/minima only at the boundaries.
- Let $B(x,r)$ be a ball with center $x$ and radius $r$, contained totally in $U$. 

Harmonic functions and Mean Value property

Then,

\[
u(x) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x,r)} u \, dS = \frac{n}{\omega_n r^n} \int_{B(x,r)} u \, dV
\]

where \( \omega_n \) is the surface area of the unit sphere in \( n \) dimensions.
Barycentric Coordinates
(Mobius, 1827)

Given \( \mathbf{v} \) find weights \( w_i \) such that

\[
\mathbf{v} = \frac{\sum_i w_i \mathbf{p}_i}{\sum_i w_i}
\]

\[
\frac{w_i}{\sum_j w_j}
\]

are barycentric coordinates.
Boundary Value Interpolation

Given $p_i$, compute $w_i$ such that

$$v = \frac{\sum_i w_ip_i}{\sum_i w_i}$$
Boundary Value Interpolation

Given $p_i$, compute $w_i$ such that

$$v = \frac{\sum_i w_i p_i}{\sum_i w_i}$$

Given values $f_i$ at $p_i$, construct a function

$$\hat{f}[v] = \frac{\sum_i w_i f_i}{\sum_i w_i}$$

Interpolates values at vertices

Linear on boundary
Boundary Value Interpolation

Given $p_i$, compute $w_i$ such that

$$ v = \frac{\sum_i w_i p_i}{\sum_i w_i} $$

Given values $f_i$ at $p_i$, construct a function

$$ \hat{f}[v] = \frac{\sum_i w_i f_i}{\sum_i w_i} $$

Interpolates values at vertices
Linear on boundary
Smooth on interior
Previous work: Wachpress’s solution (1975)

Star-shaped polygon.

weights $\lambda_1, \ldots, \lambda_k \geq 0$

$$
\sum_{i=1}^{k} \lambda_i v_i = v_0, \quad \sum_{i=1}^{k} \lambda_i = 1
$$

$$
\lambda_i = \frac{w_i}{\sum_{j=1}^{k} w_j}, \quad w_i = \frac{A(v_{i-1}, v_i, v_{i+1})}{A(v_{i-1}, v_i, v_0)A(v_i, v_{i+1}, v_0)} = \frac{\cot \gamma_{i-1} + \cot \beta_i}{\|v_i - v_0\|^2}
$$
Barycentric coordinates for arbitrary polygons in the plane

\[ A_i(v), \quad -B_i(v), \quad A_{i-1}(v) \]

\[ A_i(v)(v_{i-1} - v) - B_i(v)(v_i - v) + A_{i-1}(v)(v_{i+1} - v) = 0 \]  \hspace{1cm} \text{(Coxeter, 1969)}

define

\[ w_i(v) = b_{i-1}(v)A_{i-2}(v) - b_i(v)B_i(v) + b_{i+1}(v)A_{i+1}(v) \]

weight functions \( b_i : \mathbb{R}^2 \to \mathbb{R} \) can be chosen arbitrarily

\[ b_i(v) = \frac{||v_i - v||}{A_{i-1}(v)A_i(v)} \quad \text{guarantee } \sum_{i=1}^{n} w_i(v) \neq 0 \text{ for any } v \in \mathbb{R}^2 \]

\[ w_i(v) = \frac{\tan(\alpha_{i-1}(v)/2) + \tan(\alpha_i(v)/2)}{r_i(v)} \]  \hspace{1cm} \text{(Hormann 2004)}
Floater: Mean Value Coordinates

\[ \lambda_i = \frac{w_i}{\sum_{j=1}^{k} w_j}, \quad w_i = \frac{\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2)}{\|v_i - v_0\|} \]

- These weights were derived by application of mean value theorem for harmonic functions.
- They depend smoothly on the vertices.
Previous Work

convex polygons
[Wachspress 1975]

closed polygons
[Floater 2003, Hormann 2004]
Previous Work

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Previous Work

convex polygons
[Wachspress 1975]

closed polygons
[Floater 2003, Hormann 2004]
convex polygons  
[Wachspress 1975]

3D convex polyhedra  

closed polygons  
[Floater 2003, Hormann 2004]

3D closed triangle meshes  
[Floater et al (to appear in CAGD)]
Continuous Barycentric Coordinates

Discrete

\[ \hat{f}[v] = \frac{\sum_i w_i f_i}{\sum_i w_i} \]

Continuous

\[ \hat{f}[v] = \frac{\int_x w[x,v]f[x]dx}{\int_x w[x,v]dx} \]
Continuous Barycentric Coordinates

Discrete

\[ \hat{f}[v] = \frac{\sum_i w_i f_i}{\sum_i w_i} \]

Continuous

\[ \hat{f}[v] = \frac{\int_x w[x,v] f[x] dx}{\int_x w[x,v] dx} \]
Mean Value Interpolation

\[
\hat{f}[v] = \frac{\int_X \frac{f[x]}{|p[x] - v|} dS_v}{\int_X 1 dS_v}
\]

- Continuous form of mean value coordinates
- Consider evaluation of the numerator
Mean Value Interpolation

$p[x]$  

$f[x]$
Mean Value Interpolation

\[ v \]

\[ S_v (v) \]
Mean Value Interpolation

- Project the function $f[x]$ onto the boundary of this circle
Mean Value Interpolation

\[ \hat{f}[v] = \frac{\int_{S_v} \frac{f[x]}{|p[x]-v|} dS_v}{\int_{S_v} \frac{1}{|p[x]-v|} dS_v} \]

- Integrate the projected function divided by \((p[x]-v)\) over the circle \(S_v\) and then normalize.
Mean Value Interpolation

\[ \hat{f}(v) = \frac{\int_x \frac{f(x)}{|p(x) - v|} dS_v}{\int_x \frac{1}{|p(x) - v|} dS_v} \]

Generates smooth function
Interpolates boundary
Reproduces linear functions
Mean Value Interpolation

\[ \hat{f}(v) = \frac{\int_{x}^{p[x]} \frac{f(x)}{|p(x) - v|} \, dS_v}{\int_{x}^{p[x]} \frac{1}{|p(x) - v|} \, dS_v} \]

Generates smooth function
Interpolates boundary
Reproduces linear functions
Mean Value Interpolation

\[ \hat{f}[v] = \frac{\int_{\mathcal{V}} \frac{f(x)}{|p(x) - v|} dS_v}{\int_{\mathcal{V}} \frac{1}{|p(x) - v|} dS_v} \]

Generates smooth function
Interpolates boundary
Reproduces linear functions
Relation to Discrete Coordinates

MV coordinates $\rightarrow$ closed-form solution of continuous interpolant for piecewise linear shapes

Discrete

Continuous
Relation to Discrete Coordinates

MV coordinates $\rightarrow$ closed-form solution of continuous interpolant for piecewise linear shapes

Discrete

Continuous
3D Mean Value Coordinates

Find weights $w_i$ which allow us to represent any $v$ as a weighted combination of the vertices of a closed triangular mesh and satisfy mean value interpolation

$$v = \frac{\sum_i w_i p_i}{\sum_i w_i} \quad \rightarrow \quad \sum_i w_i (p_i - v) = 0$$
3D Mean Value Coordinates

- Given a triangular mesh and a vertex $v$ in its interior
- Consider a unit sphere centered at vertex $v$
3D Mean Value Coordinates

- Project the mesh onto the surface of the sphere
- Planar triangles -> spherical triangles
Define $m$ as the mean vector = integral of unit normal over spherical triangle.
3D Mean Value Coordinates

Given \( m \), represent it as a weighted combination of the vertex \( v \) to the vertices \( p_k \) of the triangle

\[
m = \sum_{k=1}^{3} w_k (p_k - v)
\]
3D Mean Value Coordinates

\[ m = \sum_{k=1}^{3} w_k (p_k - v) \]

Stokes' Theorem \( \sum_j m_j = 0 \)

\[ \sum_i w_i (p_i - v) = 0 \]
Computing The Mean Vector

Given spherical triangle, compute mean vector $m$ (integral of unit normal)
Computing The Mean Vector

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Build wedge with face normals $n_k$
Computing The Mean Vector

Given spherical triangle, compute mean vector $m$ (integral of unit normal)

Build wedge with face normals $n_k$

Apply Stokes’ Theorem,

$$\sum_{k=1}^{3} \frac{1}{2} \theta_k n_k + m = 0$$
Interpolant Computation

Compute mean vector

\[ \sum_{k=1}^{3} \frac{1}{2} \theta_k n_k + m = 0 \]
Interpolant Computation

Compute mean vector
\[ \sum_{k=1}^{3} \frac{1}{2} \theta_k n_k + m = 0 \]

Calculate weights
\[ w_k = \frac{n_k \cdot m}{n_k \cdot (p_k - v)} \]
Interpolant Computation

Compute mean vector
$$\sum_{k=1}^{3} \frac{1}{2} \theta_k n_k + m = 0$$

Calculate weights
$$w_k = \frac{n_k \cdot m}{n_k \cdot (p_k - v)}$$

Sum over all triangles
$$\hat{f}[v] = \frac{\sum_j \sum_{k=1}^{3} w_k^j f_k^j}{\sum_j \sum_{k=1}^{3} w_k^j}$$
Implementation Considerations

**Special cases**
- \( v \) on boundary

**Numerical stability**
- Small spherical triangles
- Large meshes

Pseudo-code provided in paper
Application: Surface Deformation

\[ v = \frac{\sum_i w_i p_i}{\sum_i w_i} \]
Application: Surface Deformation

\[ v = \frac{\sum_i w_i p_i}{\sum_i w_i} \]
Application: Surface Deformation

\[ v = \frac{\sum_i w_i p_i}{\sum_i w_i} \]

\[ \hat{v} = \frac{\sum_i w_i \hat{p}_i}{\sum_i w_i} \]
Application: Surface Deformation

\[ \nu = \frac{\sum_i w_i p_i}{\sum_i w_i} \]

\[ \hat{\nu} = \frac{\sum_i w_i \hat{p}_i}{\sum_i w_i} \]
Applications Boundary Value Problems
Applications Solid Textures
## Applications Surface Deformation

<table>
<thead>
<tr>
<th>Control Mesh</th>
<th>Surface</th>
<th>Computing Weights</th>
<th>Deformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>216 triangles</td>
<td>30,000 triangles</td>
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<td>0.03 seconds</td>
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Real-time!
# Applications Surface Deformation

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<tr>
<td>98 triangles</td>
<td>96,966 triangles</td>
<td>3.3 seconds</td>
<td>0.09 seconds</td>
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Summary

Integral formulation for closed surfaces
Closed-form solution for triangle meshes
  - Numerically stable evaluation
Applications
  - Boundary Value Interpolation
  - Surface Deformation
Thank You

- Questions?