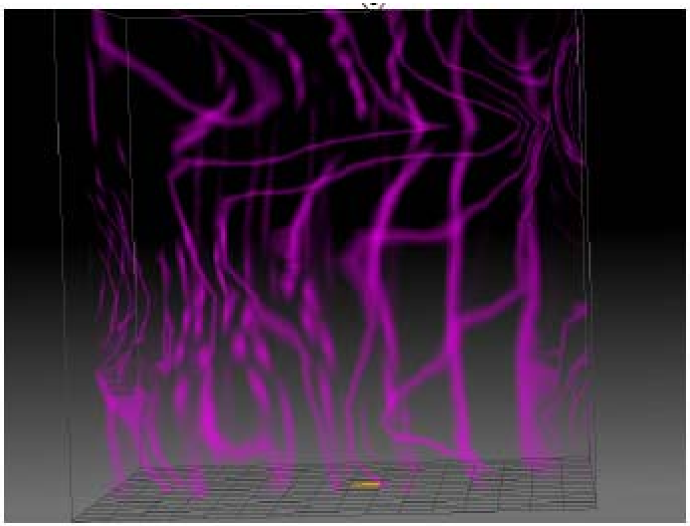
# Stable Fluid



Author: Jos Stam
Presenter: An Nguyen

# Goal

- Simulate behavior of gas (or liquid)
- Stable method

- Graphics applications
- Sacrifice accuracy for speed and control

# Navier-Stokes Equations

- Conservation of momentum

$$\begin{aligned}
\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} &= -\frac{\partial p}{\partial x} + g_x + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \\
\frac{\partial v}{\partial t} + \frac{\partial vu}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial vw}{\partial z} &= -\frac{\partial p}{\partial y} + g_y + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \\
\frac{\partial w}{\partial t} + \frac{\partial wu}{\partial x} + \frac{\partial wv}{\partial y} + \frac{\partial w^2}{\partial z} &= -\frac{\partial p}{\partial z} + g_z + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right),
\end{aligned}$$

- Conservation of mass

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

u, v, w: velocities in x, y, z directions
p: local pressure
g: gravity (local force in general)
$\nu$: viscosity

# Compact Form

Notation:

u: velocity
p: pressure
f: force
ν: viscosity
ρ: density

$$\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$$

$$\nabla^2 = \nabla \cdot \nabla$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f}$$

# Solving ODE/PDE

- Consider $du/dt = -\lambda u$, $u(0) = 1$, $\lambda > 0$
- Exact solution $u = e^{-\lambda t}$
- Explicit (Euler method):

$$(u_{i+1} - u_i)/\Delta t = -\lambda u_i, \quad u_i \approx u(i\Delta t)$$
$$u_{i+1} = (1 - \lambda\Delta t)\, u_i, \quad u_0 = 1$$
$$u_i = (1 - \lambda\Delta t)^i$$

- Implicit (Backward Euler method):

$$(u_{i+1} - u_i)/\Delta t = -\lambda u_{i+1}$$
$$u_{i+1} = (1 + \lambda\Delta t)^{-1}\, u_i, \quad u_0 = 1$$
$$u_i = (1 + \lambda\Delta t)^{-i}$$

In general: explicit method is cheap, implicit method is expensive

# Stability of a Numerical Method

Stability: Is the computed solution bounded?

(assuming that the true solution is bounded)

Explicit: $u_i = (1 - \lambda\Delta t)^i$

Stable when $|1 - \lambda\Delta t| \leq 1$

$\Delta t \leq 2/\lambda$

Implicit: $u_i = (1 + \lambda\Delta t)^{-i}$

Stable when $|1 + \lambda\Delta t| \geq 1$

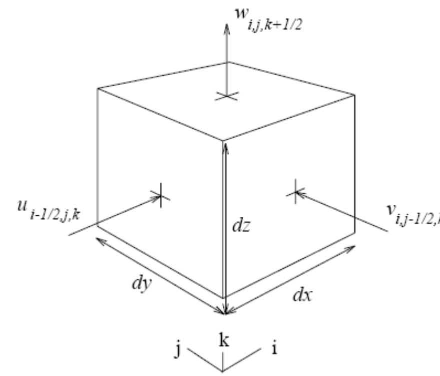
Unconditionally stable!

small time step when $\lambda$ is large!

Stability is not about the accuracy of the approximation

Stability is necessary for a numerical method to be useful

# Previous Talk




$$\nabla \cdot \mathbf{u} = 0$$
$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f}$$

u: velocity
p: pressure
f: force
$\nu$: viscosity
$\rho$: density

- Advance u using explicit method

  $(u_{i+1} - u_i)/\Delta t = -(u_i \cdot \nabla)\, u_i - 1/\rho\, \nabla p_i + \nu\, \nabla^2 u_i + f_i$

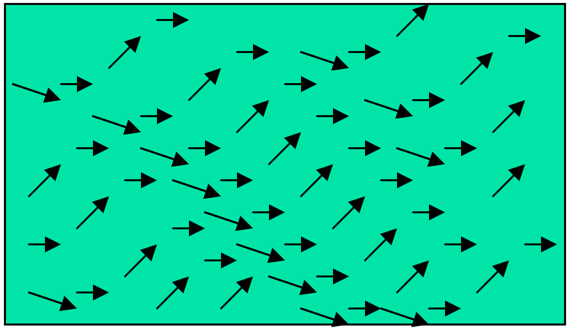
  $u_{i+1} = u_i + \Delta t\, [\, -(u_i \cdot \nabla)\, u_i - 1/\rho\, \nabla p_i + \nu\, \nabla^2 u_i + f_i\, ]$

- "Project" u to ensure conservation of mass and to compute new value for p

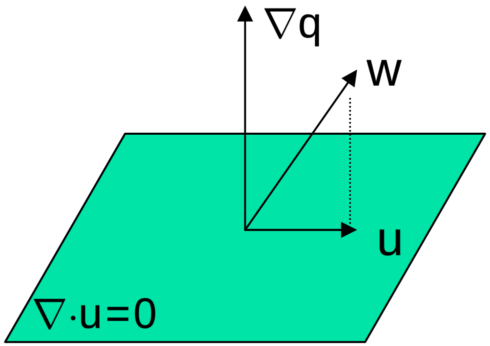
Stable when $\Delta t \approx \Delta x$

# Helmholtz-Hodge Decomposition

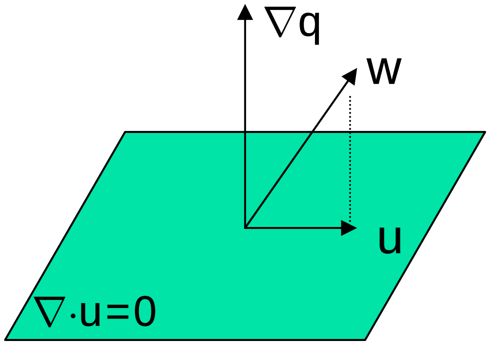
- Vector field w: $R^d \rightarrow R^d$, scalar field q: $R^d \rightarrow R$, where d = 2, 3



- Any vector field w can be written uniquely as $w = u + \nabla q$, where $\nabla \cdot u = 0$, and q is a scalar field



# Helmholtz-Hodge Decomposition




- Any vector field w can be written uniquely as $w = u + \nabla q$, where $\nabla \cdot u = 0$, and q is a scalar field

- Proof: Given w, the Poison equation $\nabla^2 q = \nabla \cdot w$ has a unique solution q

- Corollary: There is a projection operator P that projects any vector field into the space of "divergence free" vector fields
  if $w = u + \nabla q$ and $\nabla \cdot u = 0$, then $u = P(w)$

# Removing Pressure Term

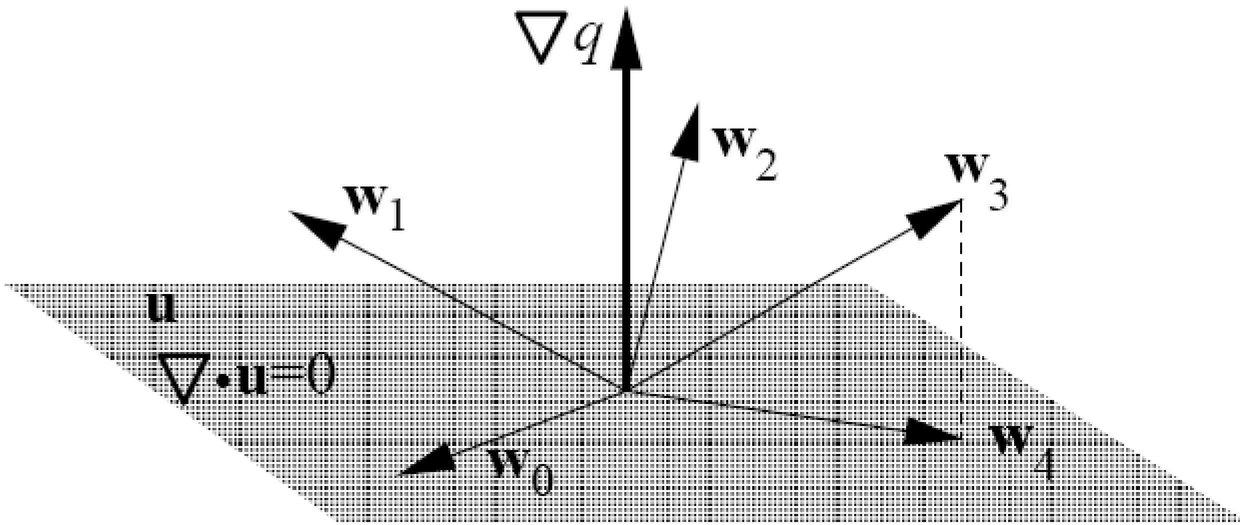
$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ \frac{\partial \mathbf{u}}{\partial t} &= -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f} \end{aligned}$$

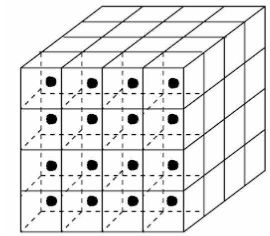
$$\Downarrow$$

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{P}\left(-(\mathbf{u} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \mathbf{u} + \mathbf{f}\right)$$

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{P}\left(-(\mathbf{u}\cdot\nabla)\mathbf{u} + \nu\nabla^2\mathbf{u} + \mathbf{f}\right)$$

# Strategy



$$\mathbf{w}_0(\mathbf{x}) \overset{\text{add force}}{\longrightarrow} \mathbf{w}_1(\mathbf{x}) \overset{\text{advect}}{\longrightarrow} \mathbf{w}_2(\mathbf{x}) \overset{\text{diffuse}}{\longrightarrow} \mathbf{w}_3(\mathbf{x}) \overset{\text{project}}{\longrightarrow} \mathbf{w}_4(\mathbf{x}).$$

$\frac{\partial u}{\partial t} = f$, $u(0) = w_0$, $w_1 = u(\Delta t)$

$\frac{\partial u}{\partial t} = -u \cdot \nabla u$, $u(0) = w_1$, $w_2 = u(\Delta t)$

$\frac{\partial u}{\partial t} = \nu \nabla^2 u$, $u(0) = w_2$, $w_3 = u(\Delta t)$

$w_4 = P(w_3)$

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{P}\left(-(\mathbf{u}\cdot\nabla)\mathbf{u} + \nu\nabla^2\mathbf{u} + \mathbf{f}\right)$$

# Step 1: Add Force

$$\frac{\partial u}{\partial t} = f$$

$u(x,0) = w_0(x)$

$w_1(x) = u(x, \Delta t)$

$$\mathbf{w}_1(\mathbf{x}) = \mathbf{w}_0(\mathbf{x}) + \Delta t\ \mathbf{f}(\mathbf{x}, t)$$

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{P}\left(-(\mathbf{u}\cdot\nabla)\mathbf{u} + \nu\nabla^2\mathbf{u} + \mathbf{f}\right)$$

# Step 2: Advection

$$\frac{\partial u}{\partial t} = -u \cdot \nabla u$$

$u(0) = w_1$
$w_2 = u(\Delta t)$

- "Propagate" disturbance
- Linearized, solving $\partial u/\partial t = - w_1 \cdot \nabla u$ instead

# Method of Characteristics

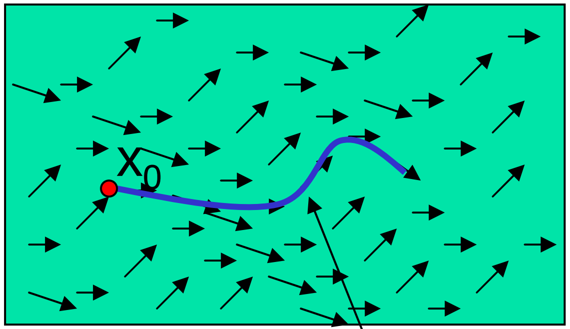
- Problem: solving $\partial a(x,t)/\partial t = -v(x) \cdot \nabla a(x,t)$, where $a(x,0) = a_0(x)$

function of t, "characteristic curve at $x_0$"

- Approach: Fix $x_0$, and let $p(x_0, t)$ be such that $p(x_0, 0) = x_0$ and that $dp(x_0, t)/dt = v(x_0)$.

- Let $b(t) = a(p(x_0, t), t)$, then by chain rule $db/dt = \nabla a \cdot dp/dt + \partial a/\partial t = 0$

  i.e. $b(t)$ = constant, $a(.,t)$ is a constant along the characteristic curves



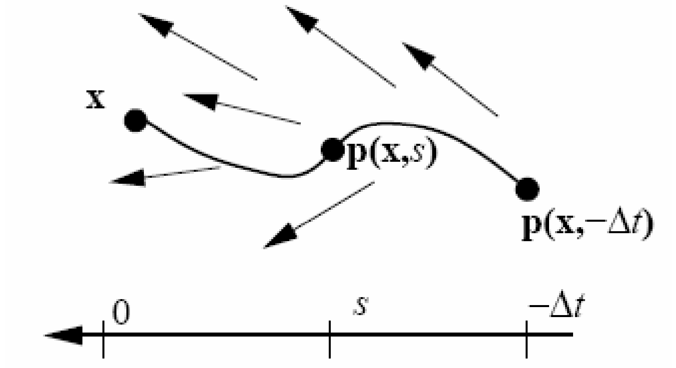

$p(x_0, t)$

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{P}\left( -(\mathbf{u}\cdot\nabla)\mathbf{u} + \nu\nabla^2\mathbf{u} + \mathbf{f} \right)$$

# Step 2: Advection

$$\frac{\partial u}{\partial t} = -w_1 \cdot \nabla u$$

$u(0) = w_1$
$w_2 = u(\Delta t)$




$$\mathbf{w}_2(\mathbf{x}) = \mathbf{w}_1(\mathbf{p}(\mathbf{x}, -\Delta t))$$

$p(x, -\Delta t) = x - \Delta t\, v(x)$ if t is small
integrate $v(x)$ if $\Delta t$ is large

max $w_2(x) \leq$ max $w_1(x)$

explicit method requires small time step $\Delta t \approx \Delta x$ here

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{P}\left(-(\mathbf{u}\cdot\nabla)\mathbf{u} + \nu\nabla^2\mathbf{u} + \mathbf{f}\right)$$

# Step 3: Diffusion

$$\frac{\partial u}{\partial t} = \nu\nabla^2 u$$

$u(0) = w_2$

$w_3 = u(\Delta t)$

previous, explicit method: $w_3(x) = (I + \nu\Delta t\nabla^2)w_2(x)$

$$\left(\mathbf{I} - \nu\Delta t\nabla^2\right)\mathbf{w}_3(\mathbf{x}) = \mathbf{w}_2(\mathbf{x})$$

Sparse matrix

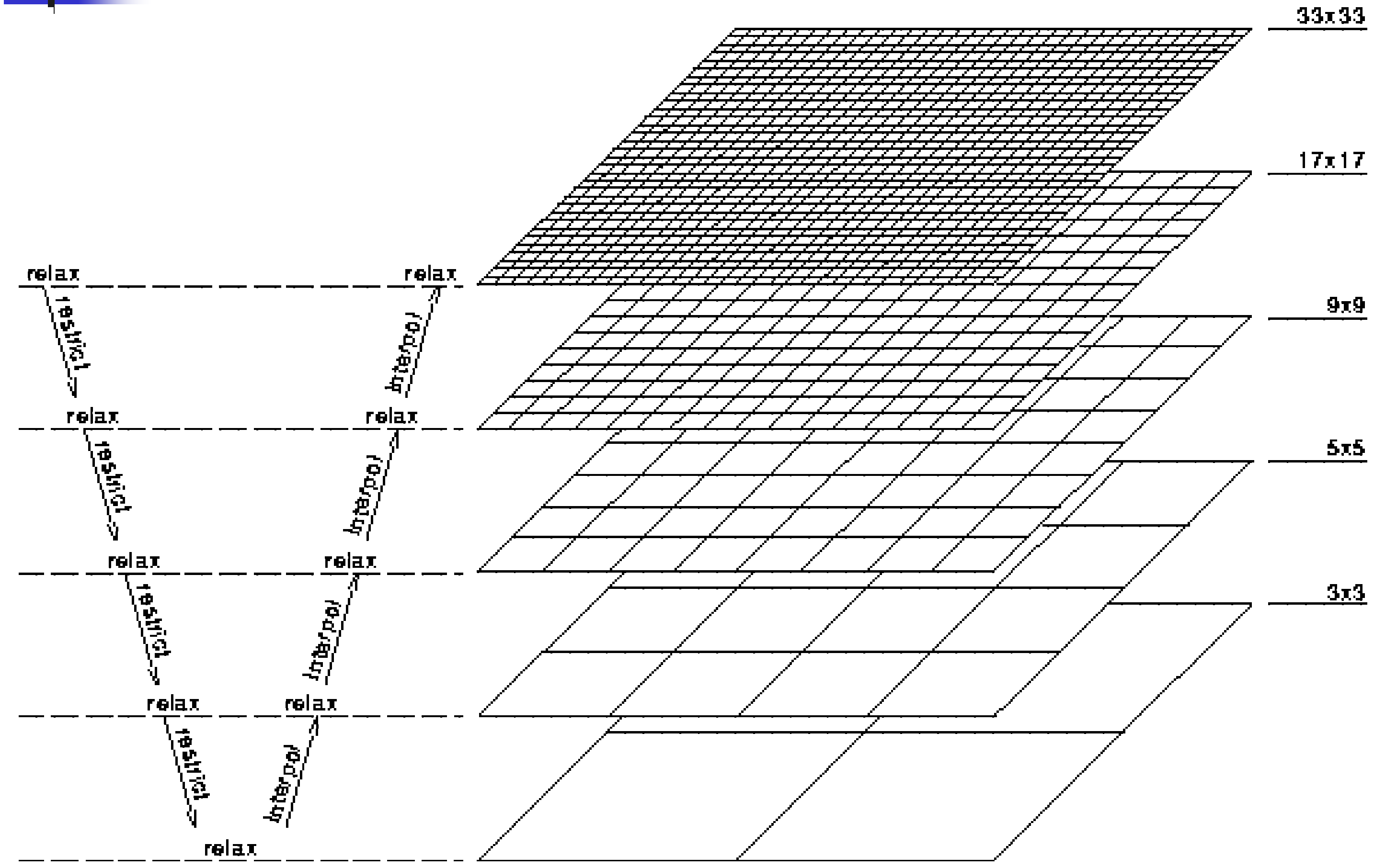
Solved using multi-grid

$\nabla^2 u = \partial^2 u/\partial x^2 + \partial^2 v/\partial x^2 + \partial^2 w/\partial z^2$

$\partial^2 u_{i,j,k}/\partial x^2 = 1/(\Delta x)^2\,(u_{i,j,k+1} - 2\,u_{i,j,k} + u_{i,j,k-1})$

# Multi-grid method

$$\left(\mathbf{I} - \nu \Delta t \nabla^2\right) \mathbf{w}_3(\mathbf{x}) = \mathbf{w}_2(\mathbf{x})$$




approximate linear time algorithm for sparse matrix system

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{P}\left(-(\mathbf{u}\cdot\nabla)\mathbf{u} + \nu\nabla^2\mathbf{u} + \mathbf{f}\right)$$

# Step 4: Projection

Find $w_4$ such that $\nabla \cdot w_4 = 0$ and $w_3 = w_4 + \nabla q$

$$\nabla^2 q = \nabla \cdot \mathbf{w}_3 \qquad \mathbf{w}_4 = \mathbf{w}_3 - \nabla q.$$

Sparse matrix
Solved using multi-grid
Can be solved in linear time

# Periodic Boundary Condition

$\text{FourierStep}(\mathbf{w}_0, \mathbf{w}_4, \Delta t)$:

- add force: $\mathbf{w}_1 = \mathbf{w}_0 + \Delta t\ \mathbf{f}$
- advect: $\mathbf{w}_2(\mathbf{x}) = \mathbf{w}_1(\mathbf{p}(\mathbf{x}, -\Delta t))$
- transform: $\hat{\mathbf{w}}_2 = \text{FFT}\{\mathbf{w}_2\}$
- diffuse: $\hat{\mathbf{w}}_3(\mathbf{k}) = \hat{\mathbf{w}}_2(\mathbf{k})/(1 + \nu \Delta t k^2)$
- project: $\hat{\mathbf{w}}_4 = \hat{\mathbf{P}} \hat{\mathbf{w}}_3$
- transform: $\mathbf{w}_4 = \text{FFT}^{-1}\{\hat{\mathbf{w}}_4\}$

k: wave number

# Substances in the Fluid

- Simulate dust density, smoke/water droplets, fluid temperature, texture coordinate

- Propagate scalar quantity a using

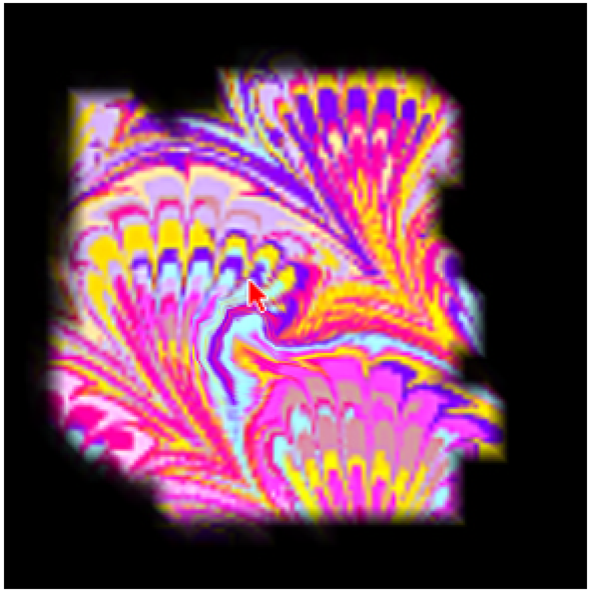
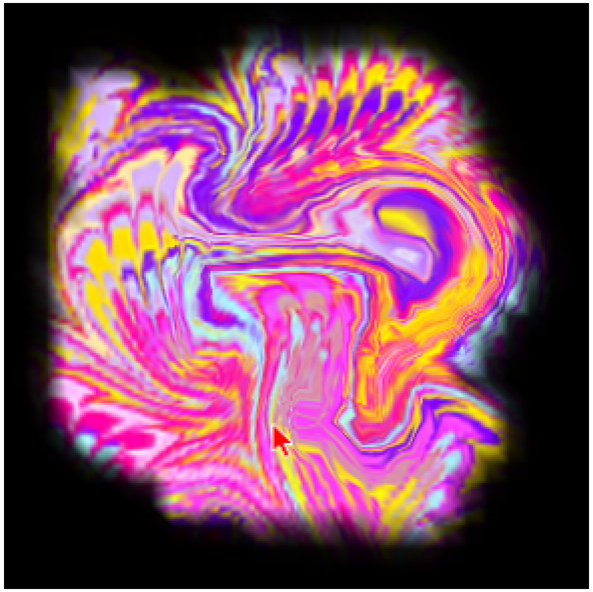
$$\frac{\partial a}{\partial t} = -\mathbf{u} \cdot \nabla a + \kappa_a \nabla^2 - \alpha_a a + S_a,$$

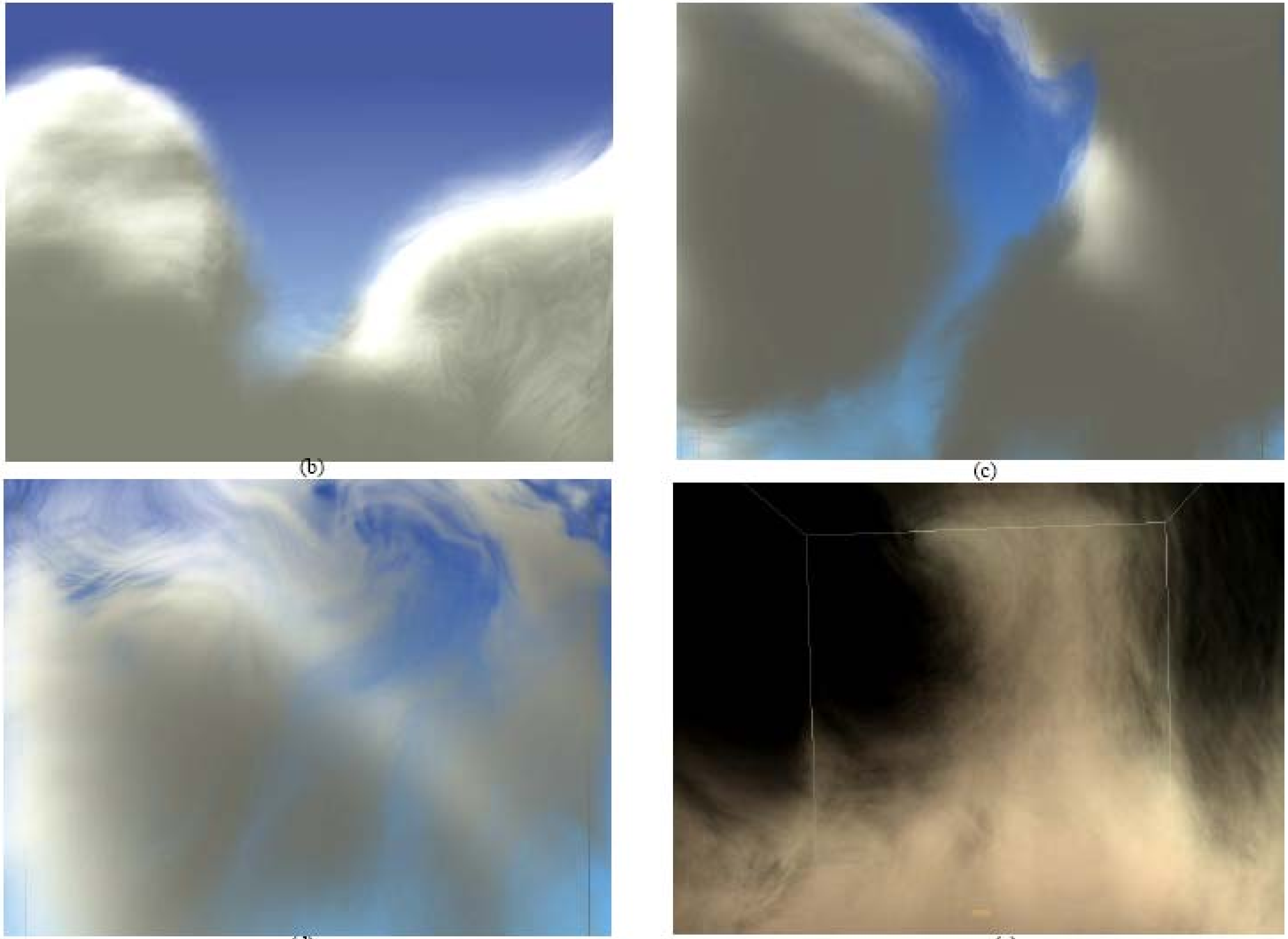
fluid velocity

$\kappa_a$: diffusion constant
$\alpha_a$: dissipation term
$S_a$: source

# Results

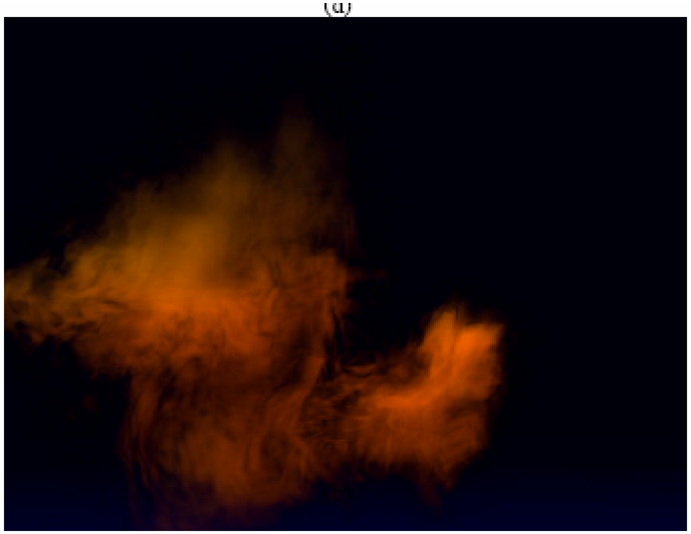
- $16^3$ – $30^3$ grids
- Texture map is used for rendering
- Fast enough for interactive control of fluid



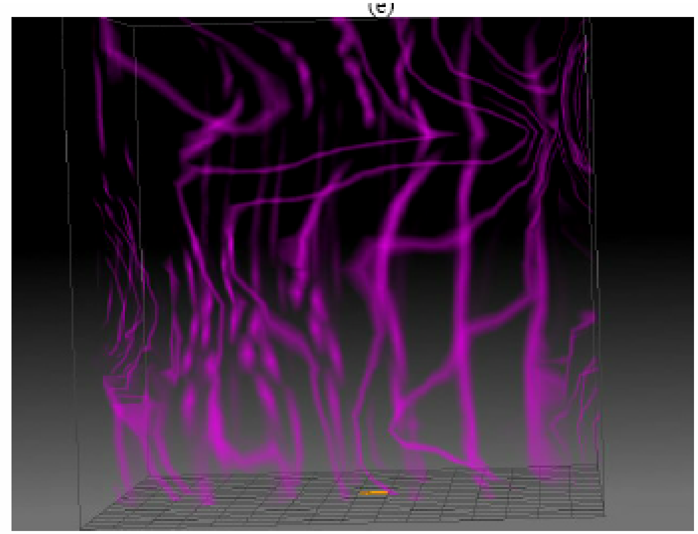
(b) (c) (d) (e)

(d)



(f)

(e)



(g)

# Summary

- Unconditional stable algorithm to solve Navier Stokes Equations
- Allowing fast simulation of fluid