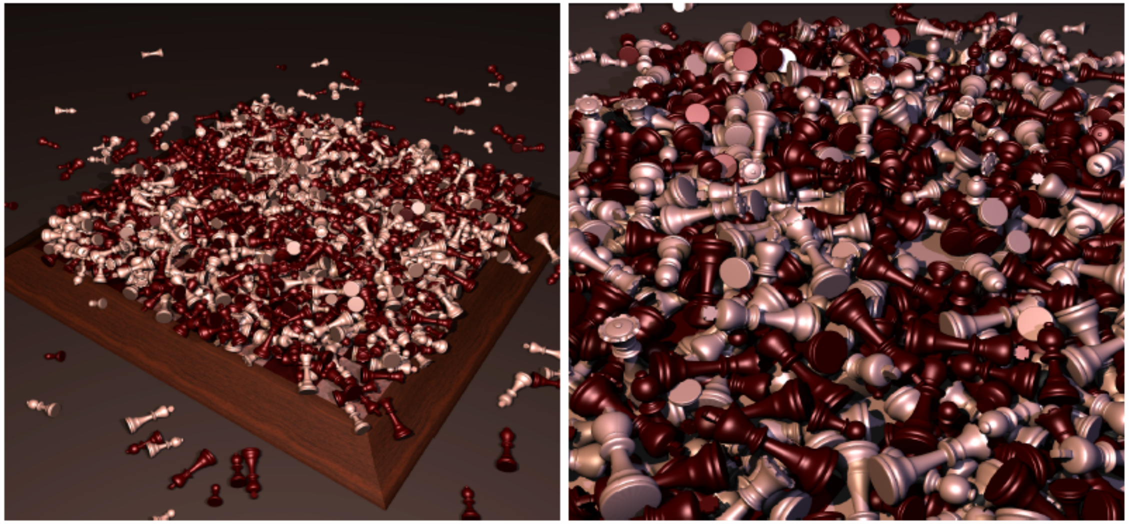
Fast Frictional Dynamics for Rigid Bodies



Danny Kaufman
Timothy Edmonds
Dinesh Pai

Rutgers University

Rigid Bodies

- Simplified Model
  - hard to know internal properties of all items
  - Rigid instead
- More complex simulation
  - Forces instantaneously change velocity of entire object
  - Either:
    - Allow interpenetration between object
    - Find earliest collision, hard if too many collisions
      - Ball bouncing

Task

- Quickly compute next time step configuration
  - Object not convex
  - Simulate
    - Collisions
    - Sliding friction
    - Rolling friction
    - External forces (i.e gravity)

Background

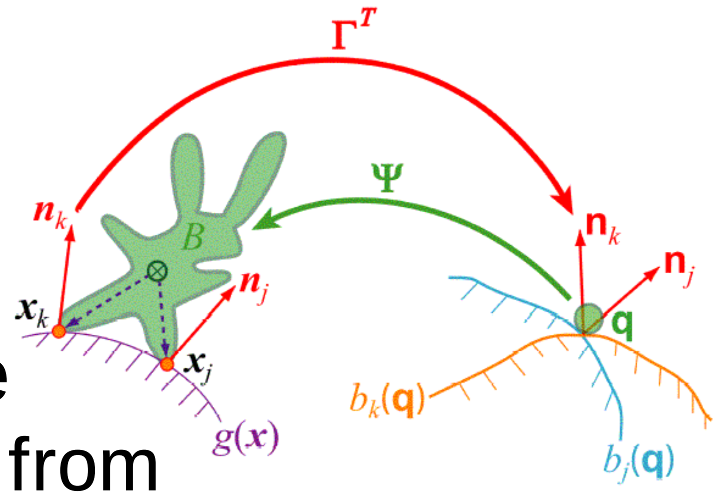
- Configuration of object $SE(3)$
- Mapping $\Psi_k : \mathrm{q} \in SE(3) \rightarrow \boldsymbol{x}_k \in \mathbb{R}^3$
- Homogeneous coordinates of x in frame i ${}^{i}x$
- To obtain relative to frame j left multiply by:

$$ {}^{j}_{i}\mathrm{E} = \begin{pmatrix} {}^{j}_{i}\boldsymbol{\Theta} & {}^{j}_{i}\boldsymbol{p} \\ 0 & 1 \end{pmatrix} $$

- Meaning: ${}^{j}\boldsymbol{x} = {}^{j}_{i}\mathrm{E}\, {}^{i}\boldsymbol{x}$

Background (cont.)

- q is the orientation
- W is world frame
- Then ${}^{W}\Psi_k(\mathsf{q}) = {}^{W}_{B}\mathsf{E}\,{}^{B}\boldsymbol{x}_k$
- Using that, obtain time derivative for changes from frame i to j ${}^{j}\dot{\boldsymbol{x}} = {}^{j}_{i}\dot{\mathsf{E}}\,{}^{i}\boldsymbol{x}$



Background

- Define the skew-symmetric matrix

$$[\boldsymbol{\omega}] \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -\omega_2 & \omega_1 \\ \omega_2 & 0 & -\omega_0 \\ -\omega_1 & \omega_0 & 0 \end{pmatrix}$$

- A multiply by $[\omega]$ gives a cross product, therefore:

$${}^{i}_{j}\mathsf{E}\,{}^{j}_{i}\dot{\mathsf{E}} = \begin{pmatrix} \Theta^T \dot{\Theta} & \Theta^T \dot{\boldsymbol{p}} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} [\boldsymbol{\omega}] & \boldsymbol{v} \\ 0 & 0 \end{pmatrix}$$

- Frame i with respect to frame j in i's coordinates

Background

- Element in tangent space to $SE(3)$ denoted $se(3)$
- Determines spatial velocity vector, called a *twist*
- Linear operator $\mathfrak{U}$ extracts it.

$$ {}^{i}\phi(j,i) \stackrel{\text{def}}{=} (\boldsymbol{\omega}, \boldsymbol{v})^T = \mathfrak{U}({}^{i}_{j}\mathsf{E}\,{}^{j}_{i}\dot{\mathsf{E}}) $$

- Its inverse is the bracket operator

$$ [\,{}^{i}\phi(j,i)] = \begin{pmatrix} [\boldsymbol{\omega}] & \boldsymbol{v} \\ 0 & 0 \end{pmatrix} $$

- Top equation describes relative motion of frame i with respect to frame j in i's coordinates

Background

- We have $^{i}\dot{\boldsymbol{x}}_k = \begin{pmatrix} [\boldsymbol{\omega}] & \boldsymbol{v} \\ 0 & 0 \end{pmatrix} {}^{i}\boldsymbol{x}_k$
- Those are in the tangent space of x_k

$$T_{\boldsymbol{x}_k}\mathbb{R}^3$$

- Providing the mapping from twist to elements of $se(3)$

$$^{i}\Gamma_k : {}^{i}\phi \rightarrow {}^{i}\dot{\boldsymbol{x}}_k$$

$$^{i}\Gamma_k = \begin{pmatrix} -[{}^{i}\boldsymbol{x}_k] & I \end{pmatrix}$$

Background

- Change in frames induces change in spatial velocity coordinates

$$ {}^{j}_{i}\mathsf{Ad} = \begin{pmatrix} \Theta & 0 \\ [p]\Theta & \Theta \end{pmatrix} \qquad {}^{j}\phi = {}^{j}_{i}\mathsf{Ad}\,{}^{i}\phi $$

- $SE(3)$ equipped with kinetic metric

$$ \mathsf{a}\cdot\mathsf{b} = \mathsf{a}^{T}\mathsf{M}(\mathsf{q})\,\mathsf{b} = \mathsf{a}^{T}\begin{pmatrix} \mathscr{I} & 0 \\ 0 & \mathscr{M} \end{pmatrix}\mathsf{b},\ \forall\, \mathsf{a},\mathsf{b} \in se(3) $$

  - M(q) is frame appropriate inertial matrix at q.

Background

- Explicit Euler step

$$ {}^{j}_{i}\mathsf{E}^{h} = {}^{j}_{i}\mathsf{E}^{0} \exp([\,{}^{i}\phi]h) $$

Constraints and forces

- Interpenetration allowed between steps
- Take configuration half-step using the last know velocity
- Each contact (including penetrations) are used to calculate set of constraints

Constraints

- For rigid body $\boldsymbol{B}$ $\quad g(\boldsymbol{x}) \geq 0, \ \forall \boldsymbol{x} \in B \subset \mathbb{R}^3$
- Constraints on $\boldsymbol{B}$ in $SE(3)$

$$b(\mathsf{q}) \geq 0, \ \mathsf{q} \in SE(3)$$

- From the $\mathbf{R}^3$ constraint gradient obtain the $se^*(3)$ constraint gradient using the differential transpose matrix

$$\nabla b_k(\mathsf{q}) = {\Gamma_k}^T \nabla g(\boldsymbol{x}_k) = \begin{pmatrix} [\boldsymbol{x}_k] \\ I \end{pmatrix} \nabla g(\boldsymbol{x}_k)$$

Constraints and contact forces

- A contract will impart normal or tangential force
- Find span of forces and convert into wrenches

$$\boldsymbol{n}_k \stackrel{\text{def}}{=} \frac{\nabla g(\boldsymbol{x}_k)}{||\nabla g(\boldsymbol{x}_k)||}$$

- Tangential forces span constraint's tangent plane

$$\boldsymbol{S}_k = \{\boldsymbol{s}_k \in \mathbb{R}^3 : {\boldsymbol{s}_k}^T \boldsymbol{n}_k = 0\}$$

Contact Forces

- Wrenches generated give:

$$_i\mathsf{n}_k = {}_i\Gamma_k{}^T \boldsymbol{n}_k = \begin{pmatrix} [\,{}^i\boldsymbol{x}_k] \\ \boldsymbol{I} \end{pmatrix} \boldsymbol{n}_k$$

$$_i\mathsf{S}_k = {}_i\Gamma_k{}^T \boldsymbol{S}_k = \begin{pmatrix} [\,{}^i\boldsymbol{x}_k] \\ \boldsymbol{I} \end{pmatrix} \boldsymbol{S}_k$$

- Embed wrenches into $se(3)$

$$^i\mathsf{f} = \mathsf{M}(\mathsf{q})^{-1}\ {}_i\mathsf{f}$$

- New twists ${}^i\mathrm{n}_k$ and ${}^i\mathrm{s}_k$ in ${}^i\mathrm{S}_k$

Contact Forces

- These are not generally orthogonal
- Observe inner product

$$
{}^{i}\mathsf{n}_k^T \mathsf{M}\, {}^{i}\mathsf{s}_k = (\mathsf{M}^{-1} \begin{pmatrix} [\,{}^{i}\boldsymbol{x}_k]\boldsymbol{n}_k \\ \boldsymbol{n}_k \end{pmatrix})^T \mathsf{M}\ (\mathsf{M}^{-1} \begin{pmatrix} [\,{}^{i}\boldsymbol{x}_k]\boldsymbol{s}_k \\ \boldsymbol{s}_k \end{pmatrix})
$$

$$
= \boldsymbol{n}_k^T \left([\,{}^{i}\boldsymbol{x}_k]^T \mathscr{I}^{-1} [\,{}^{i}\boldsymbol{x}_k] + \mathscr{M}^{-1} \right) \boldsymbol{s}_k .
$$

- Therefore cannot make assumption that a friction cone in $se(3)$ is orthogonal

Multi-point Contact

- Contact includes points that penetrate constraints
- Set of collision points

$$C(\mathbf{q}) = \{k \in \mathbb{Z} : b_k(\mathbf{q}) \leq 0\}$$

- Define normal cone by span of all twists from contact points

$$\mathsf{N}(\mathbf{q}) = \left\{ \sum_{k \in C(\mathbf{q})} \lambda_k \ {}^{i}\mathsf{n}_k , \ \lambda_k \geq 0 \right\}$$

Multi-point Contact

- Sliding cone of embedded wrenches spans the entire range of possible contributions

$$\mathsf{S}(\mathbf{q}) = \bigoplus_{k \in C(\mathbf{q})} {}^{i}\mathsf{S}_k$$

Multi-point Contact

- For a twist to be a feasible velocity at x_k its projection along ${}^i n_k$ >= 0
- Inner product in $se(3)$ ${}^i\phi^T M(q)\, {}^i n_k$
- Subspace of feasible twists

$$T_k = \{t \in se(3) : t^T M(q)\, {}^i n_k \geq 0\}$$

- For all C(q) $$T(q) = \bigcap_{k \in C(q)} T_k$$

Non-Smooth Dynamics

- Without contacts

$$\mathsf{M}\,{}^{i}\dot{\phi} = {}_{i}\mathsf{f} + {}_{i}\mathsf{c}$$

- Where (${}_{i}\mathsf{c} = ({}^{i}\phi_{\times})^{T}\,\mathsf{M}\,{}^{i}\phi$)
- If there are contacts then

$$\mathsf{M}\,{}^{i}\dot{\phi} - {}_{i}\mathsf{f} - {}_{i}\mathsf{c} \in \left\{ \sum_{k \in C(\mathsf{q})} \lambda_k \; {}_{i}\mathsf{n}_k,\; \lambda_k \geq 0 \right\}$$

- Embedding into $se(3)$

$${}^{i}\dot{\phi} - {}^{i}\mathsf{f} - {}^{i}\mathsf{c} \in \mathsf{N}(\mathsf{q})$$

Dynamics

- Using first order discretization with step h

$$({}^{i}\phi^{t+1} - {}^{i}\phi^{t}) = h({}^{i}\mathsf{f} + {}^{i}\mathsf{c}) + {}^{i}\mathsf{r}, \quad {}^{i}\mathsf{r} \in \mathsf{N}(\mathsf{q})$$

- Using pre and post-resolution velocity defined

$${}^{i}\phi^{-} \stackrel{\text{def}}{=} {}^{i}\phi^{t} + h({}^{i}\mathsf{f} + {}^{i}\mathsf{c})$$

$${}^{i}\phi^{+} \stackrel{\text{def}}{=} {}^{i}\phi^{t+1} = {}^{i}\phi^{-} + {}^{i}\mathsf{r}$$

Contact Resolution

- We know what subspace ${}^{i}\mathrm{r}$ must be in, but not hwo to select it.
- Moreau - ${}^{i}\mathrm{r}$ is the twist formed by the minimum spanning vector is se(3) between pre-resolution velocity, and the subspace of feasible velocities

$$ {}^{i}\mathrm{r} = \mathrm{proj}_{\mathrm{N(q)}}(-{}^{i}\phi^{-}) $$

Contact Resolution

- Equivalent to directly projecting ${}^{i}\phi^{-}$ onto the subpace of feasible velocities, T(q⁻)

$$ {}^{i}\phi^{+} = \mathrm{proj}_{\mathsf{T}(\mathsf{q})}\left({}^{i}\phi^{-}\right) \in \partial \mathsf{T}(\mathsf{q}) $$

- The boundary to T(q)
- We can now restrict the point to:

$$ A(\mathsf{q}) = \{ k \in C(\mathsf{q}) : \mathsf{n}_k^T \, \mathsf{M} \; {}^{i}\phi^{-} \leq 0 \} $$

Friction

- Modify the dynamics rule to have both normal and tangential impulses

$$ {}^{i}\phi^{t+1} - {}^{i}\phi^{t} = h\,({}^{i}\mathsf{f} + {}^{i}\mathsf{c}) + {}^{i}\mathsf{r} + {}^{i}\delta,\ \ {}^{i}\mathsf{r} \in \mathsf{N}(\mathsf{q}),\ \ {}^{i}\delta \in \mathsf{S}(\mathsf{q}) $$

- Normalize tangent vector at each contact, so that

$$ {}^{i}\mathsf{s}_k = \mathsf{M}^{-1}\ {}_{i}\Gamma_k{}^{T}\,\boldsymbol{s}_k, \text{ where } \boldsymbol{s}_k^T \boldsymbol{n}_k = 0,\ and\ ||\boldsymbol{s}_k|| = 1 $$

Frictional Impulse Constraitns

- Generalize frictional coefficient μ_k
- We let ${}^i\phi^\tau$ be the twist from first projection
- Find the frictional impulse which can be set to a convex QP.

$$ {}^i\delta = \underset{y}{\operatorname{argmin}} \| y + {}^i\phi^\tau \|^2 $$

subject to: $y \in S$.

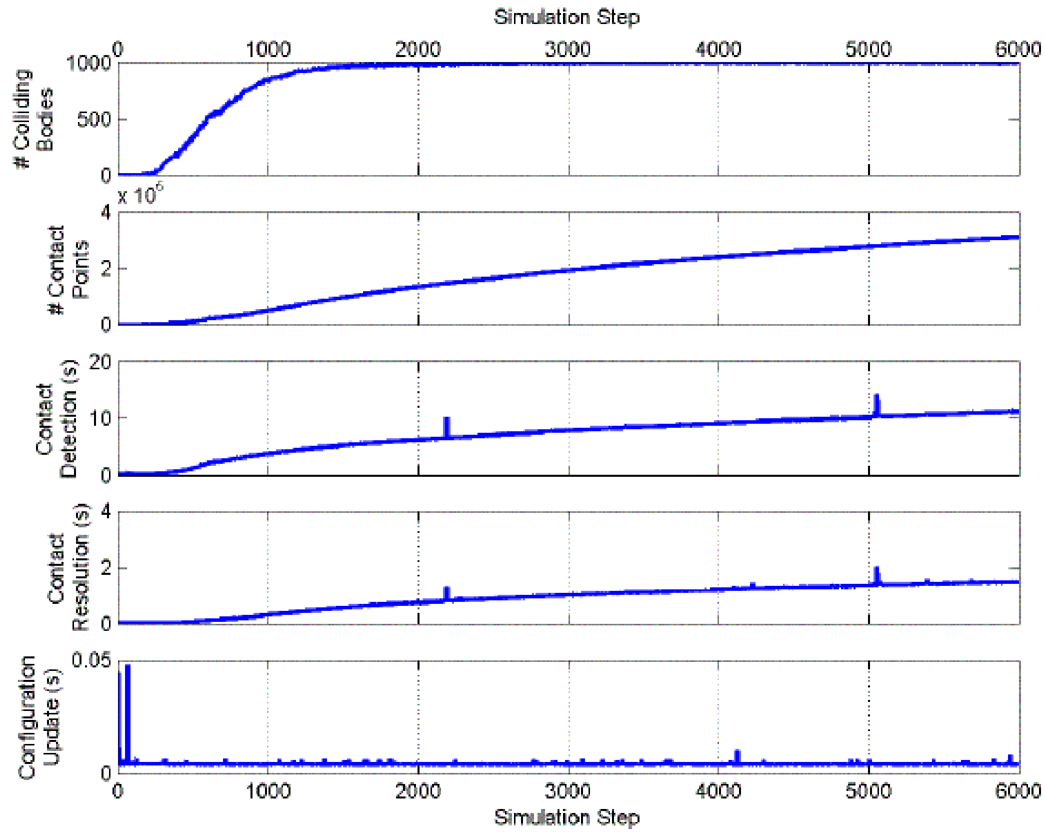
$$ {}^i s_k^T \, M \, y \le \mu_k \, {}^i n_k^T M \, {}^i r, \; \forall \, {}^i s_k \in {}^i S_k, \;\; \forall k \in A(q). $$

$$ {}^i n_k^T \, M \, y \ge 0, \; \forall \, k \in A(q). $$

Friction

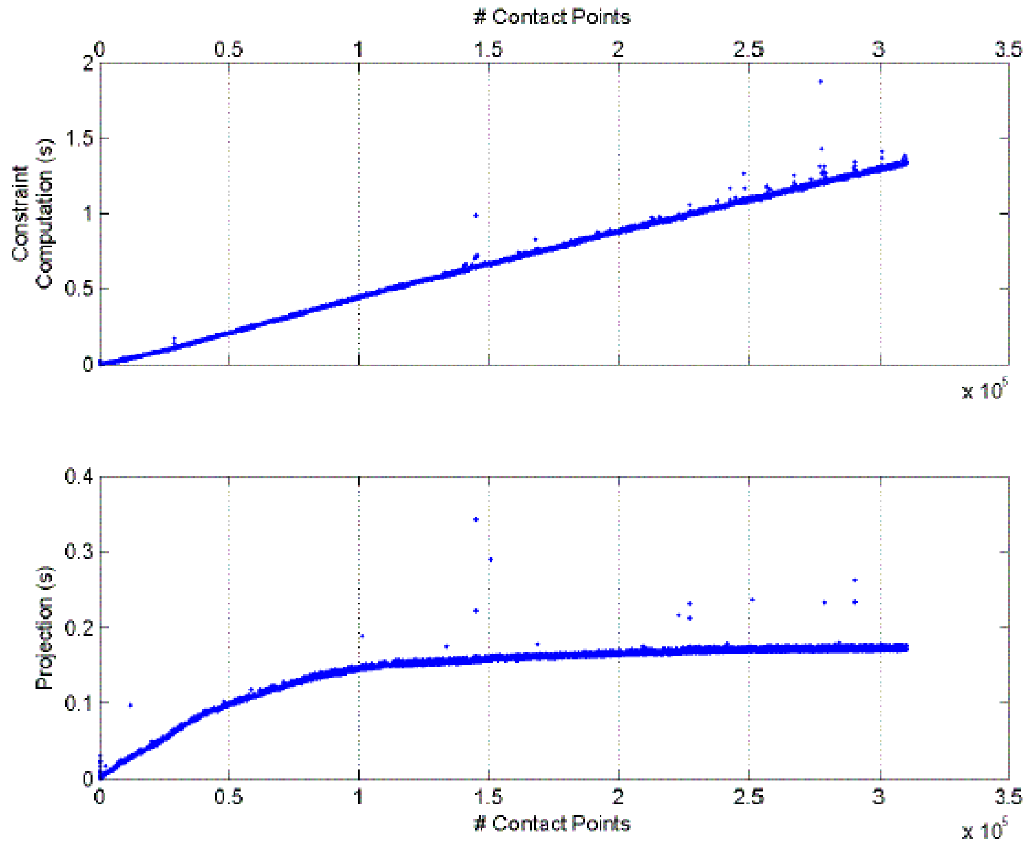
- Adding the frictional reaction to the tangential velocity gives the post-revolution vecocity

Results



Simulation Step
0
1000
2000
3000
4000
5000
6000
Colliding Bodies
1000
500
0
x 10^5
Contact Points
4
2
0
Contact Detection (s)
20
10
0
Contact Resolution (s)
4
2
0
Configuration Update (s)
0.05
0
0
1000
2000
3000
4000
5000
6000
Simulation Step

Results



Contact Points
0
0.5
1
1.5
2
2.5
3
3.5
2
1.5
1
0.5
0
Constraint Computation (s)
x 10^5
0.4
0.3
0.2
0.1
0
Projection (s)
0
0.5
1
1.5
2
2.5
3
3.5
Contact Points
x 10^5

Movies