Realistic Animation of Fluids

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Overview

- Problem Statement
- Previous Work
- Navier-Stokes Equations
- Explicit solving
- Results

Problem Statement

- Simulate fluids for graphics applications
- Scalability
- Usability
  - Easy to setup
  - Controllable

Related Work

- Non-physics based
  - Paramtric functions
  - Sinusoidal phase functions
- Physics based
  - CFD
    - Too expensive
    - Not controllable
  - Kass and Miller
    - Shallow water
  - Chen and Lobo
    - Navier-Stokes in 2-D
    - Fluid zero depth
    - Use instantaneous pressure

Navier-Stokes Equation

- Conservation of Momentum

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

- Conservation of Mass

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Conservation of Momentum

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{V}$$

- Vector form

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- Vector form
- Normalize for ρ
- LHS
  - Intertial force

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  - Intertial force
- RHS
  - Surface forces due to pressure differences

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  - Sheer force due to viscosity

LHS

- Begin with Newton's Second Law

$$\mathbf{F} = m\frac{d\mathbf{V}}{dt}$$

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$$\frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{V}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{V}}{\partial z}\frac{dz}{dt} + \frac{\partial \mathbf{V}}{\partial t}$$

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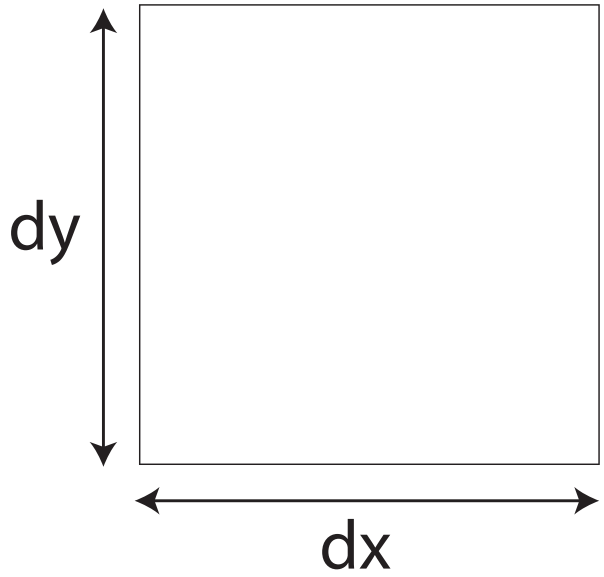
$$= u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} + \frac{\partial \mathbf{V}}{\partial t}$$

RHS

- Pressure
- Gravity
- Sheer forces

$$F_{yx} = \mu \frac{\partial^2 u}{\partial y^2}$$

$$F_{xx} = \mu \frac{\partial^2 u}{\partial x^2}$$



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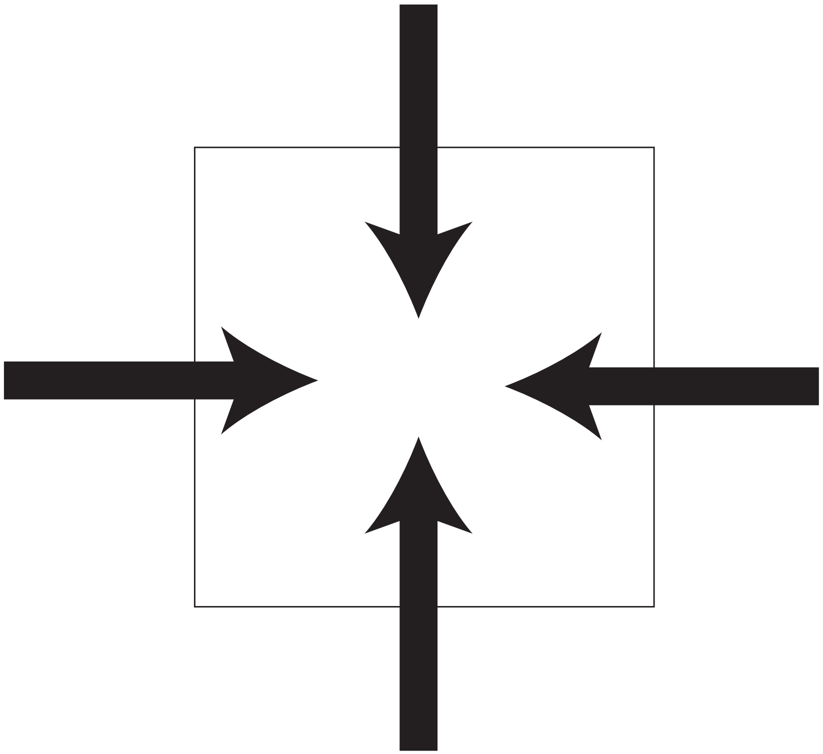

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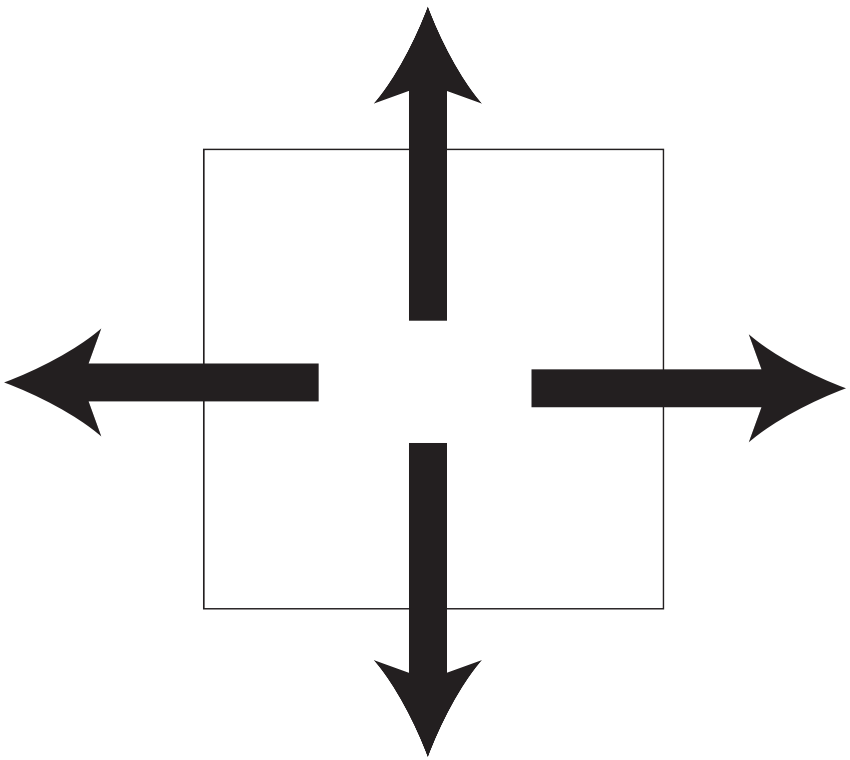

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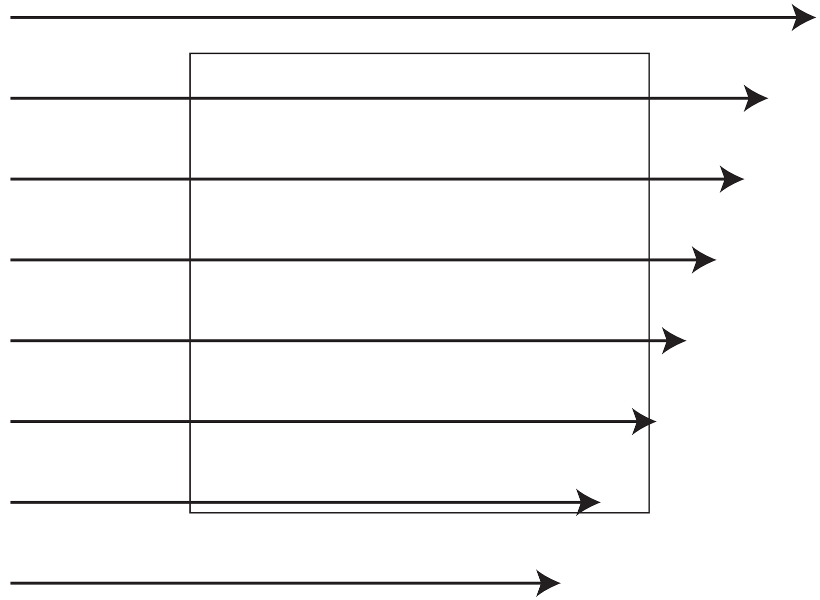

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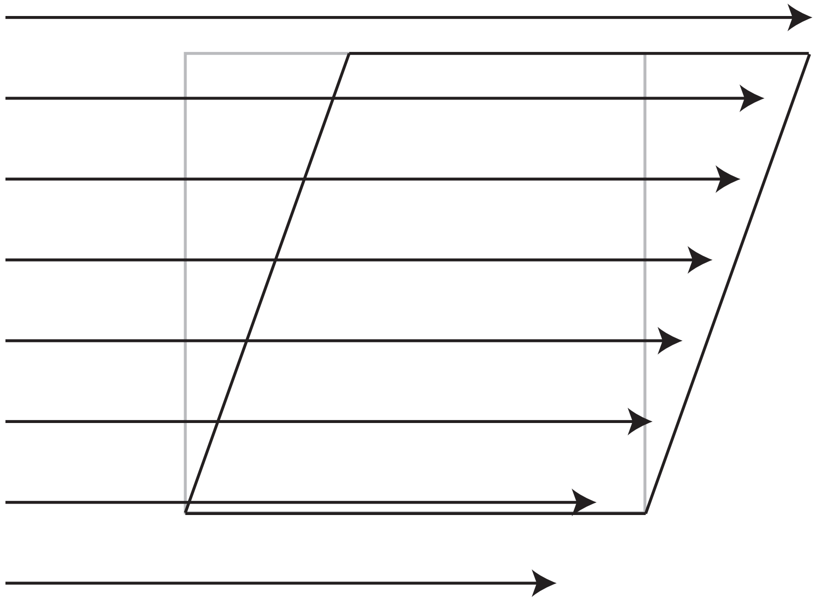

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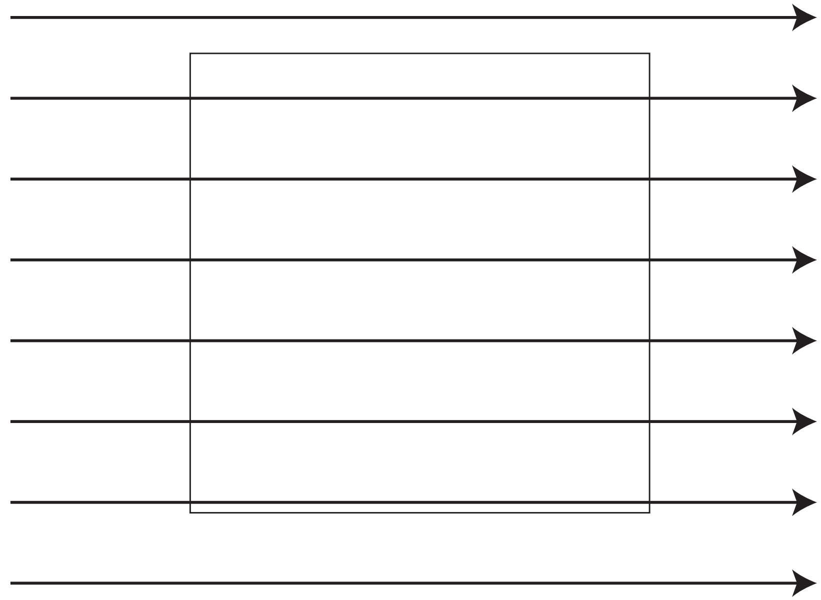

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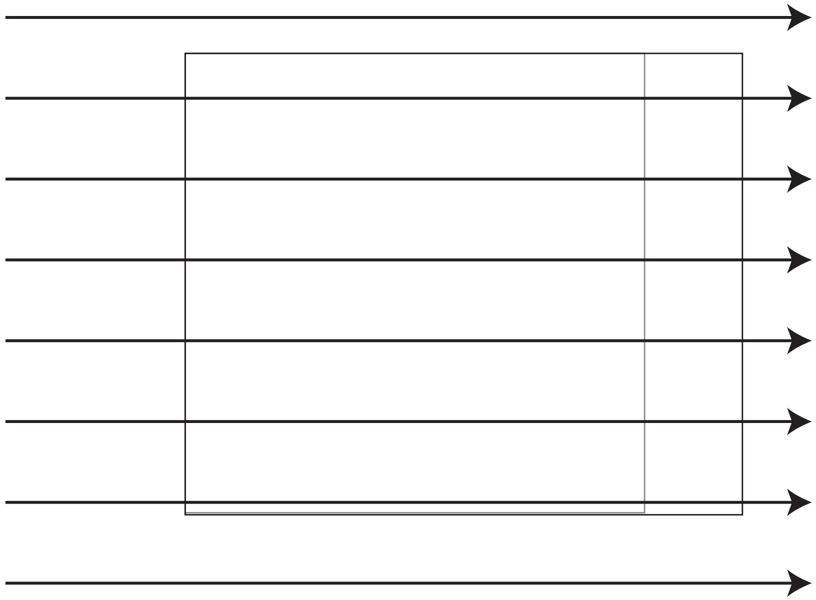

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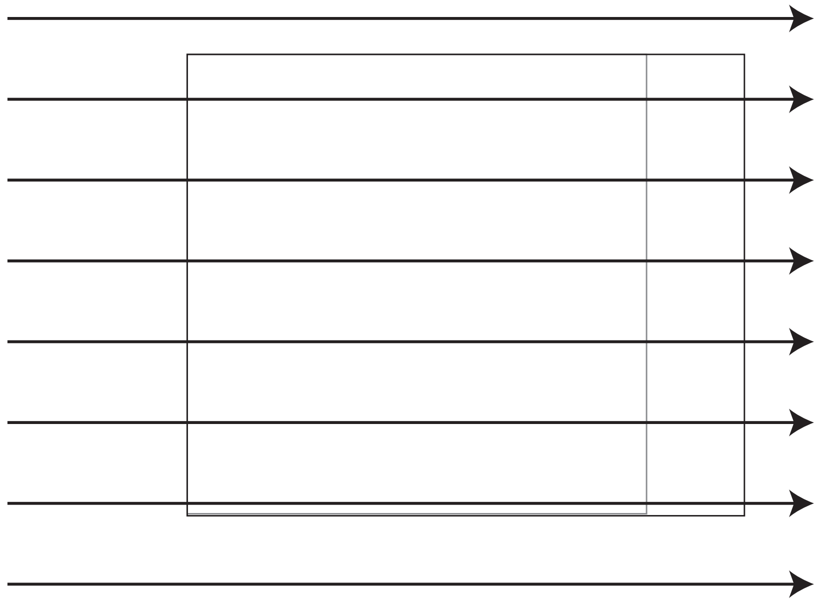
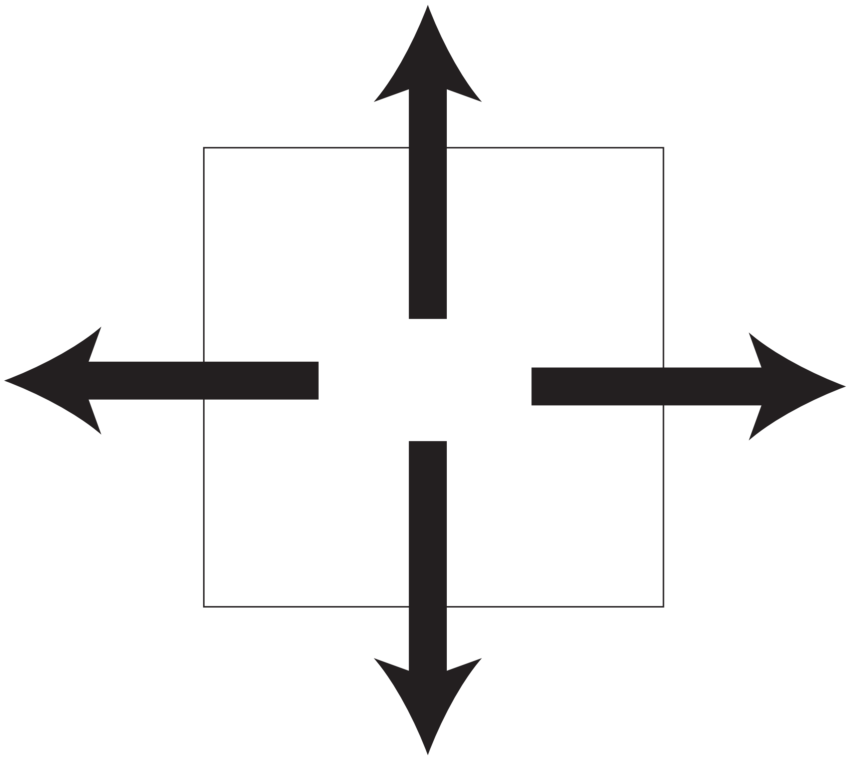


- In 3-D:

$$F_x = \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Conservation of Mass

- Incompressible fluid
- Divergence should be zero



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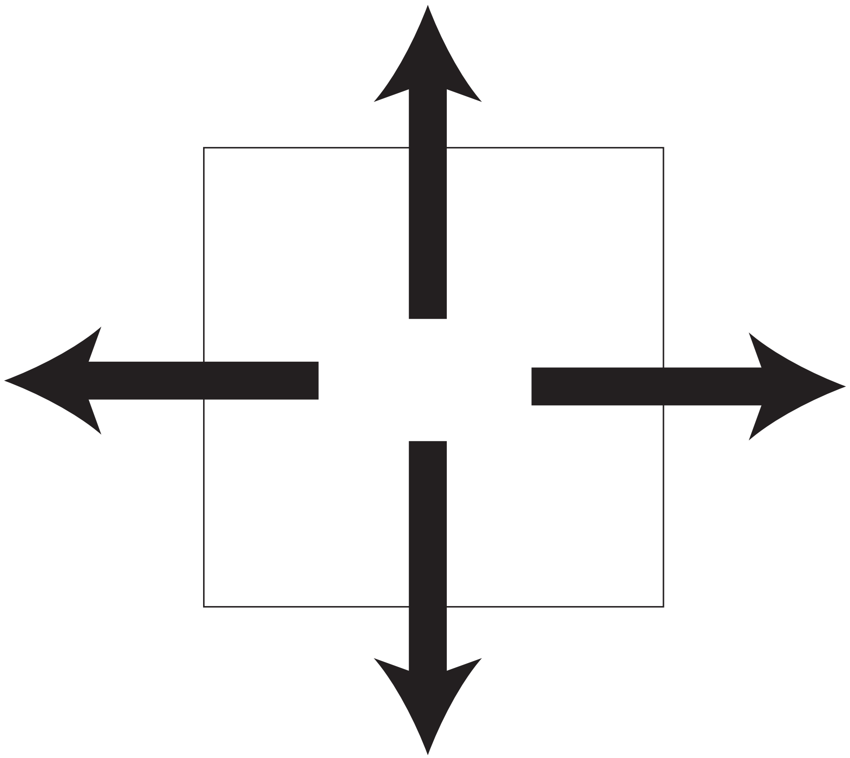
$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$



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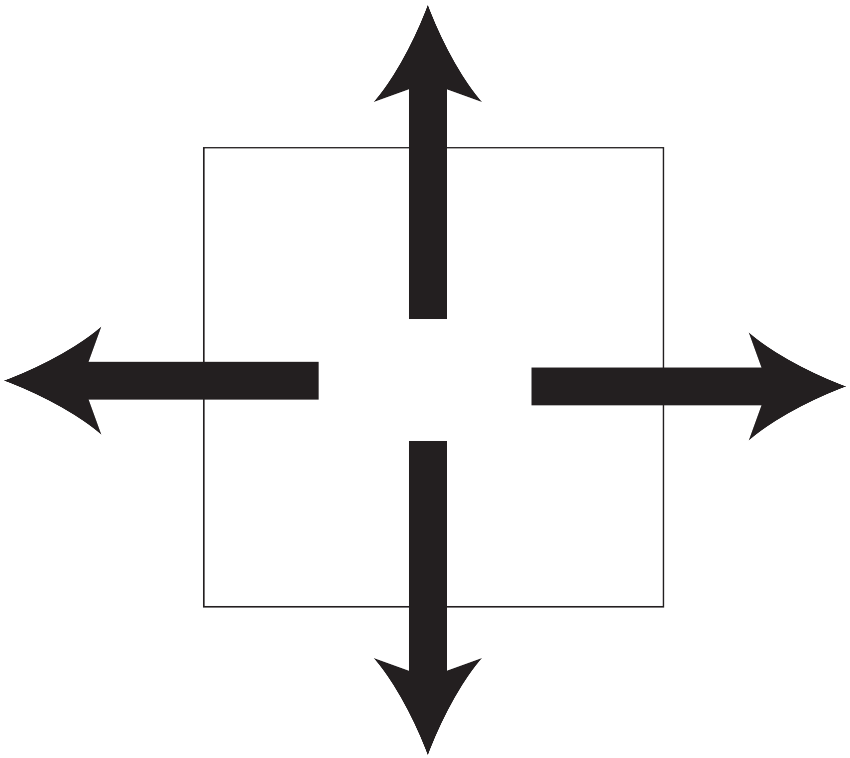
$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$



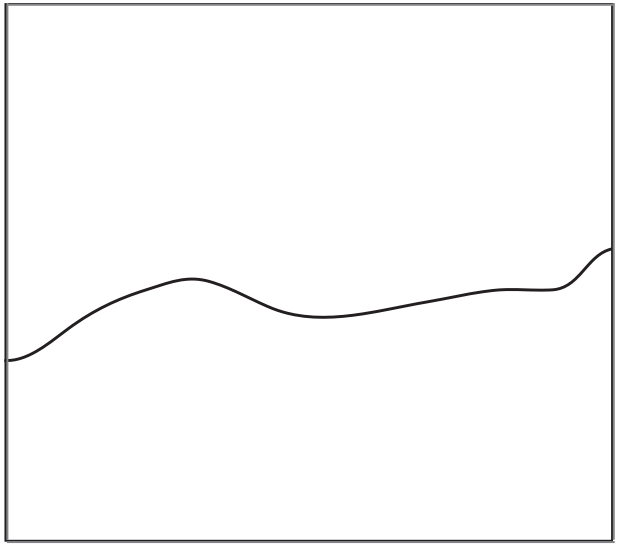
Solving Navier-Stokes

- Specify obstacle/boundaries
- Discretize on a rough grid
- Solve equations
- Interpolate surface of fluid

Discretization

- Rectangular Cartesian grid
- Finite differences
- Velocity defined on cell faces
- Pressure defined in cell
- Stability

$$1 > \max\left[u\frac{\delta t}{\delta x}, v\frac{\delta t}{\delta y}, w\frac{\delta t}{\delta z}\right]$$



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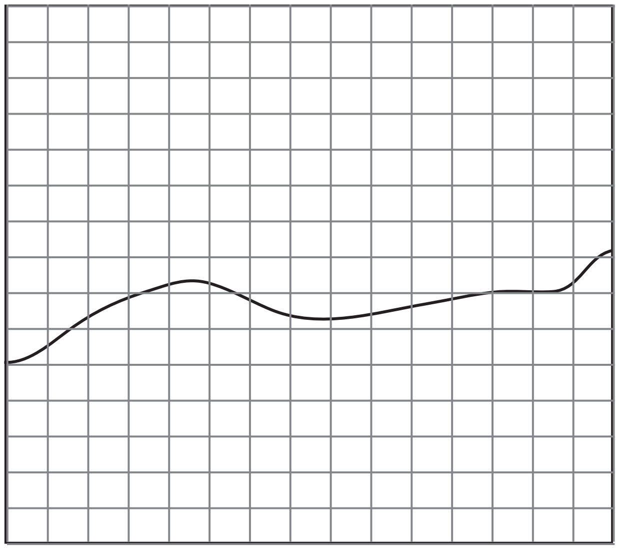

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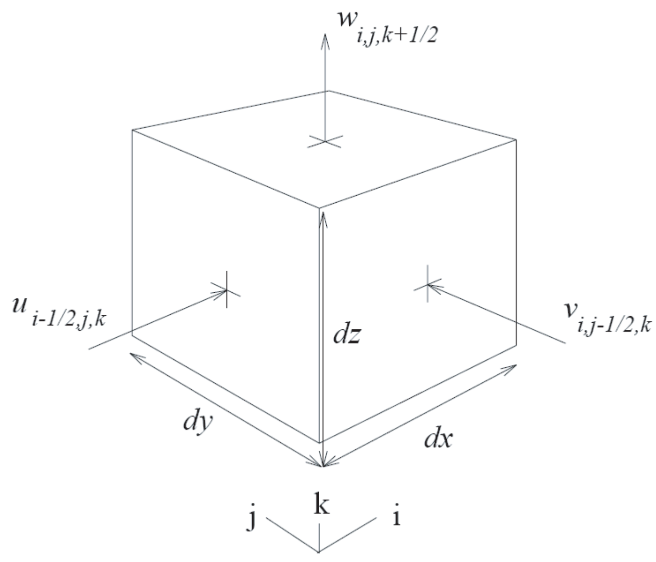

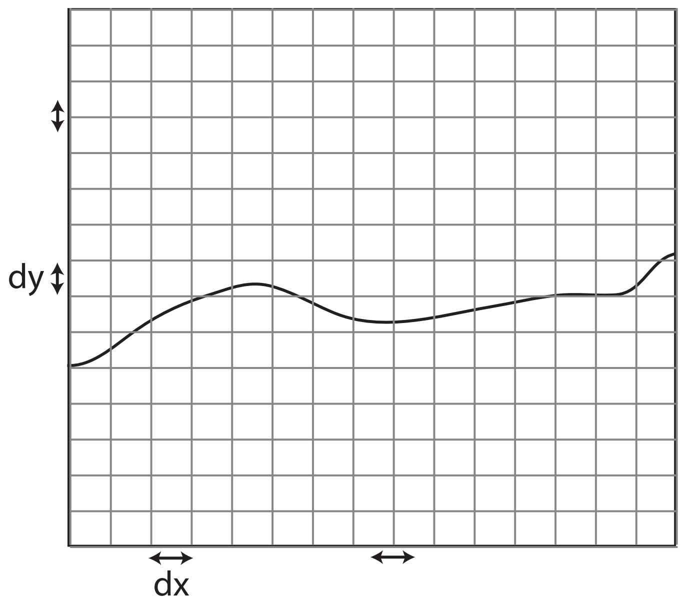

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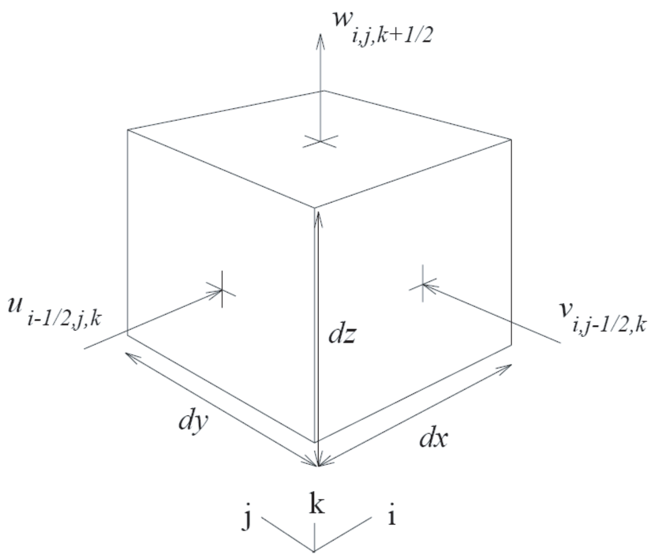
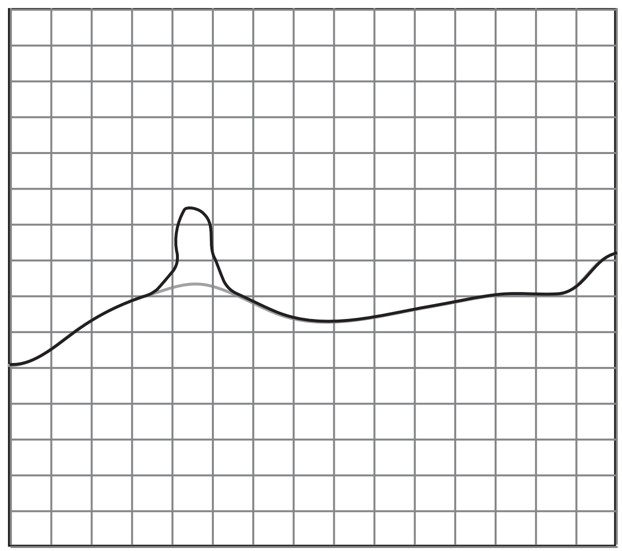




Types of Cells

- *Empty*
  - A cell containing no particles
- *Surface*
  - A cell containing at least 1 particle that is adajacent to an *Empty* cell
- *Full*
  - A cell containing at least 1 particle and not a *Surface* cell

Calculating Velocities

- Velocities from momentum equations

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial t} + g_x - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

- Pressure from continuity equations

$$D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

- δp is proportional to D

Consistency

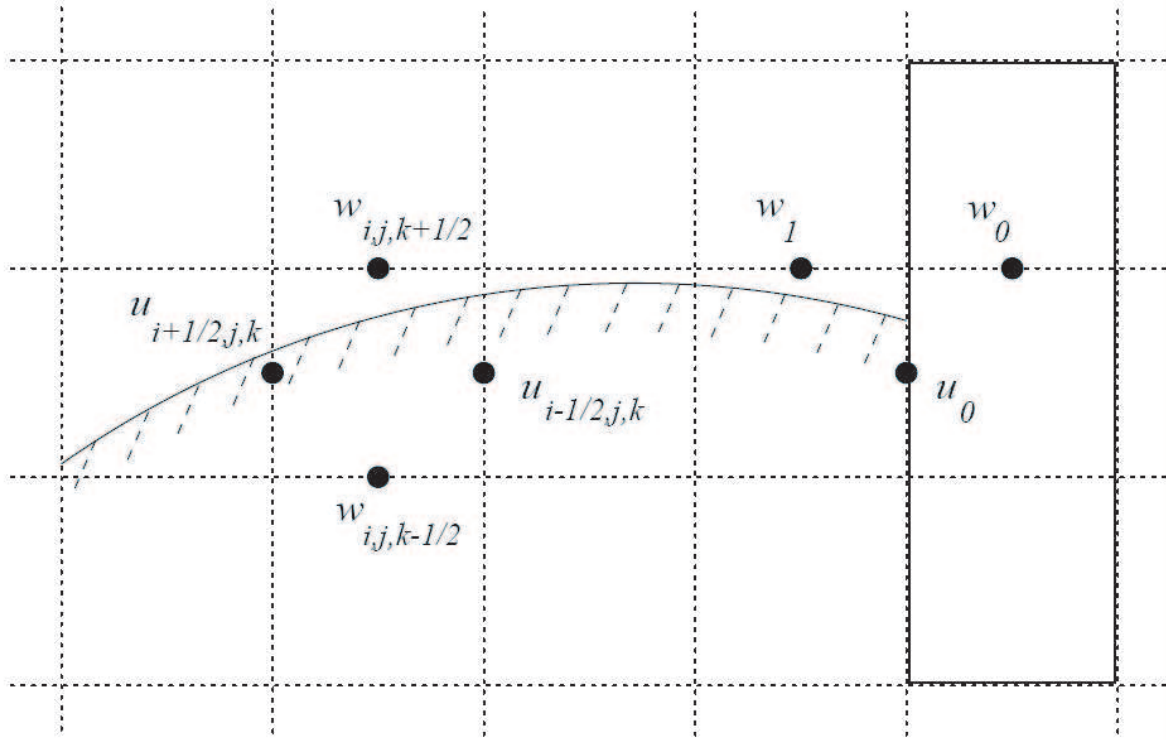
- Velocity and pressure are updated
- Iterate until $D < \epsilon$

$$
\begin{aligned}
\delta p &= \beta D \\
u_{i+1/2,j,k} &= u_{i+1/2,j,k} + (\delta t/\delta x)\delta p \\
u_{i-1/2,j,k} &= u_{i-1/2,j,k} - (\delta t/\delta x)\delta p \\
v_{i,j+1/2,k} &= u_{i,j+1/2,k} + (\delta t/\delta y)\delta p \\
v_{i,j-1/2,k} &= u_{i,j-1/2,k} - (\delta t/\delta y)\delta p \\
w_{i,j,k+1/2} &= u_{i,j,k+1/2} + (\delta t/\delta z)\delta p \\
w_{i,j,k-1/2} &= u_{i,j,k-1/2} - (\delta t/\delta z)\delta p \\
\tilde{p}_{i,j,k} &= p_{i,j,k} + \delta p
\end{aligned}
$$

Boundary Conditions

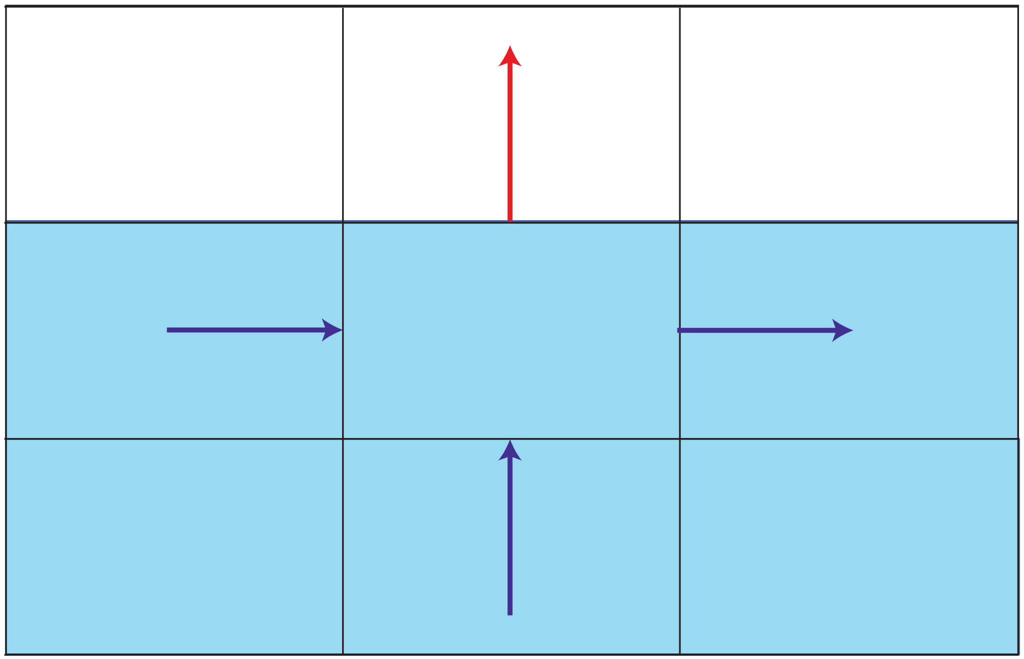
- Stationary Obstacles
  - Non-slip
  - Free-slip
- Infow and outflow
- Free Surfaces

$$
\begin{aligned}
u_0 &= 0 \\
p_0 &= p_1 \\
w_0 &= -w_1 \\
w_0 &= w_1
\end{aligned}
$$



Free Surface

- Free faces set so divergence zero
- Surface pressure set to atmospheric pressure
- Cases
- 3-D: 64 configurations



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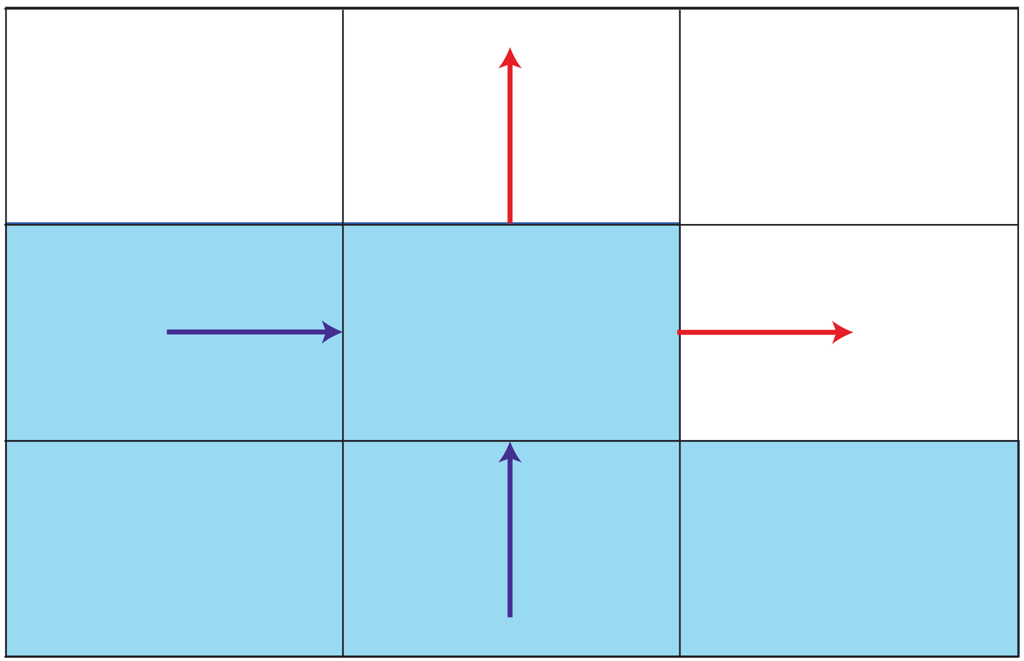

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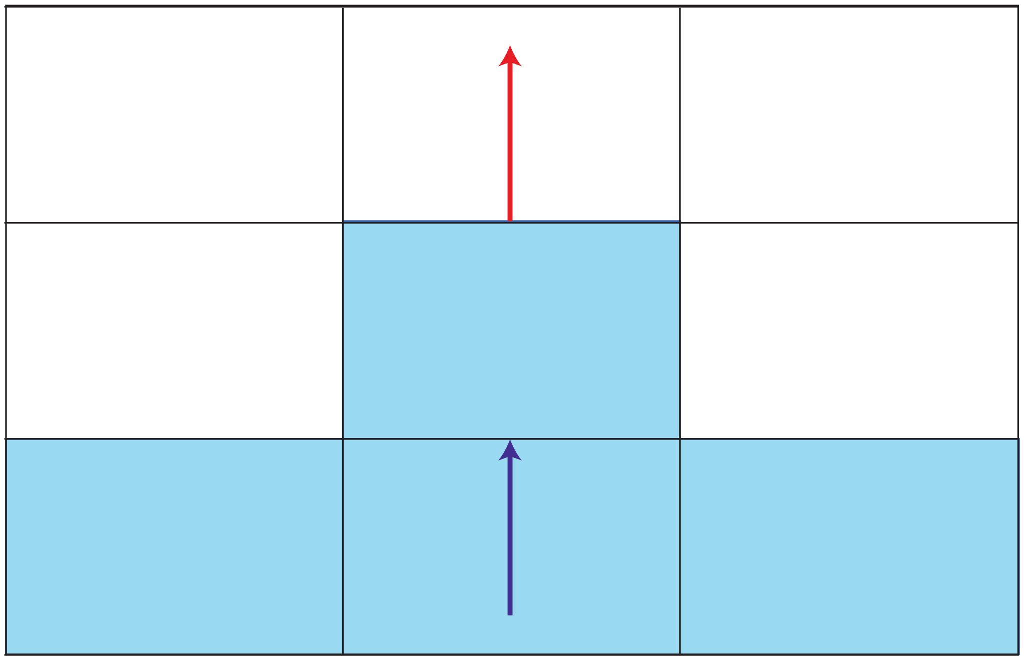


Recap

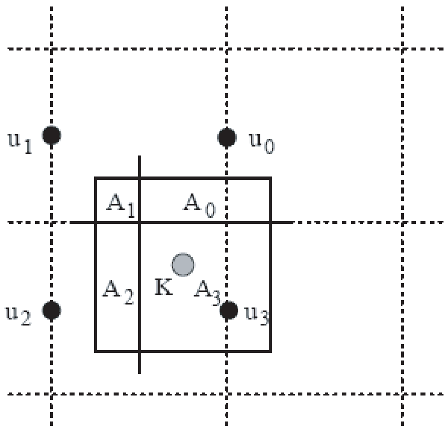
1. Set boundary conditions
2. Classify cells
3. Calculate velocities for *Full* cells
4. Calculate pressures for *Full* cells
5. Update velocities for *Surface* cells
6. Iterate

Problem

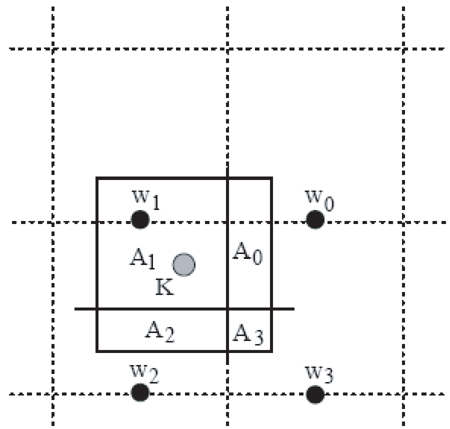
- Stability condition
- Limited spatial resolution
- Computational complexity
- Main contribution
  - Interpolating methods for displaying the surface
- 3 methods for tracking fluid positions

Marker Particles

- Convect massless particles with local fluid velocity
- Use weighted average of 4 nearest velocities
  - Weighting based on area
  - Multiplied by timestep
- Labeling done based on particles
  - 1 particle test



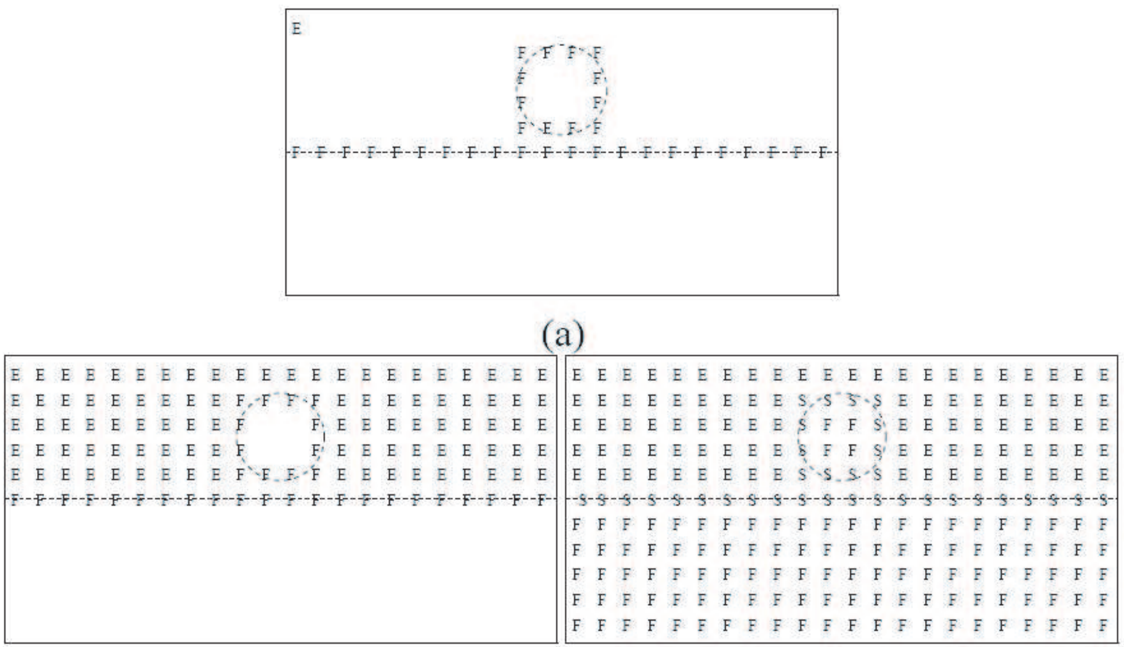
$u_k = A_0 u_0 + A_1 u_1 + A_2 u_2 + A_3 u_3$



$w_k = A_0 w_0 + A_1 w_1 + A_2 w_2 + A_3 u_3$

Free Surface Particles

- Only keep particles at surface
- Rules for adding and removing particles
  - If far apart: Insert a particle
  - If close: remove and connect neighbors
- Region growing algorithm



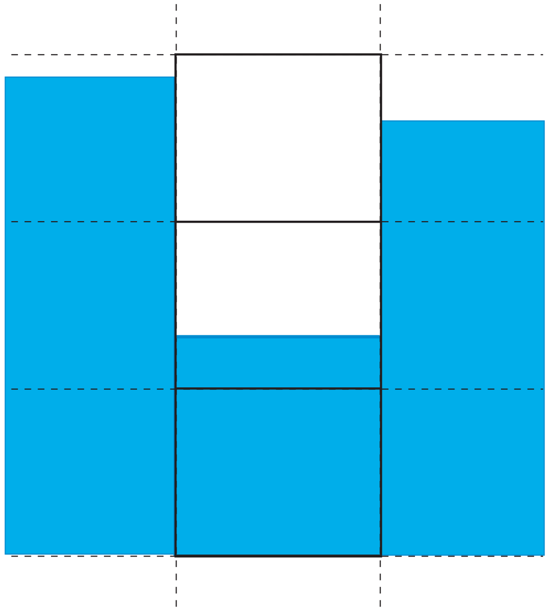
(a) (b) (c)

Height Field

- Scalar function

$$\frac{\partial h}{\partial t} = w - u\frac{\partial h}{\partial x} - v\frac{\partial h}{\partial y}$$

- Cell configuration trivial
- Need other methods
  - Overturning wave
  - Spray
  - Foam



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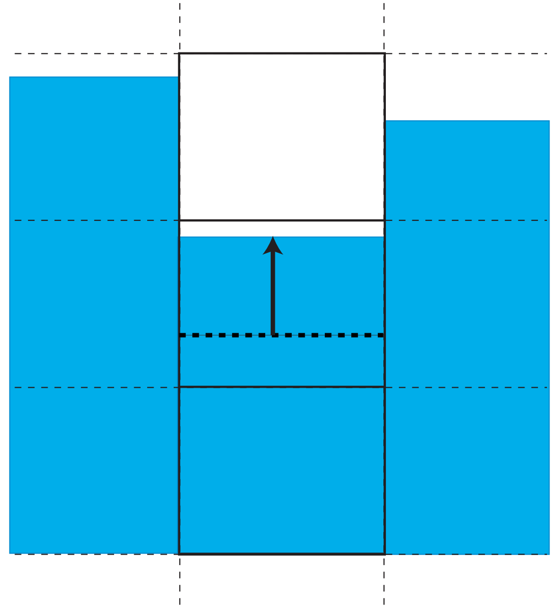


Buoyancy

- Rigid object
- Force formula

$$\begin{aligned} \mathbf{f}_{n_i} &= -\nabla p_i dV_i + m_i \mathbf{g} \\ \mathbf{f}_{fluid} &= \sum_i \mathbf{f}_{n_i} \end{aligned}$$

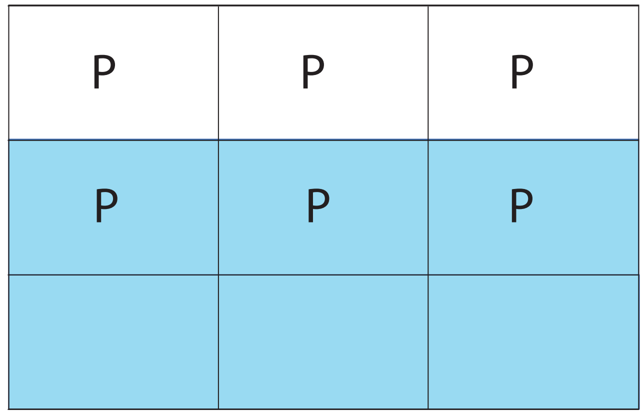
- Lagrange equations of motions

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} = \mathbf{f}_q + \mathbf{g}_q$$

- Do not affect water flow
- Collisions treated separately

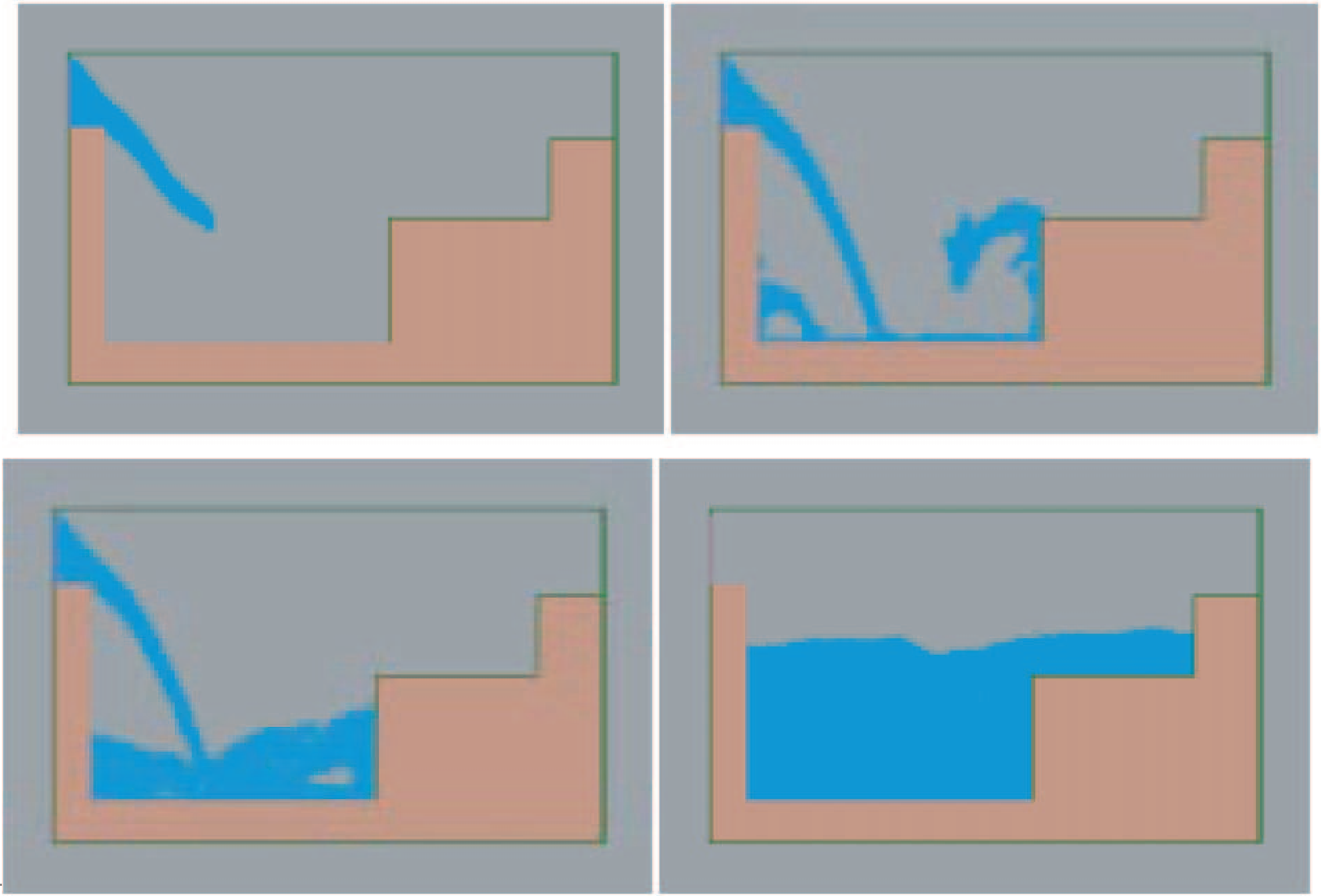
Control

- Inflow and Outflow
- Instead of atmospheric pressure, apply surface pressure history
  - Constant
  - Time-depedent
  - Height of surface



$$p_{applied}(z) = \frac{A + B\cos(Cz - wt)}{\delta t}$$

Results



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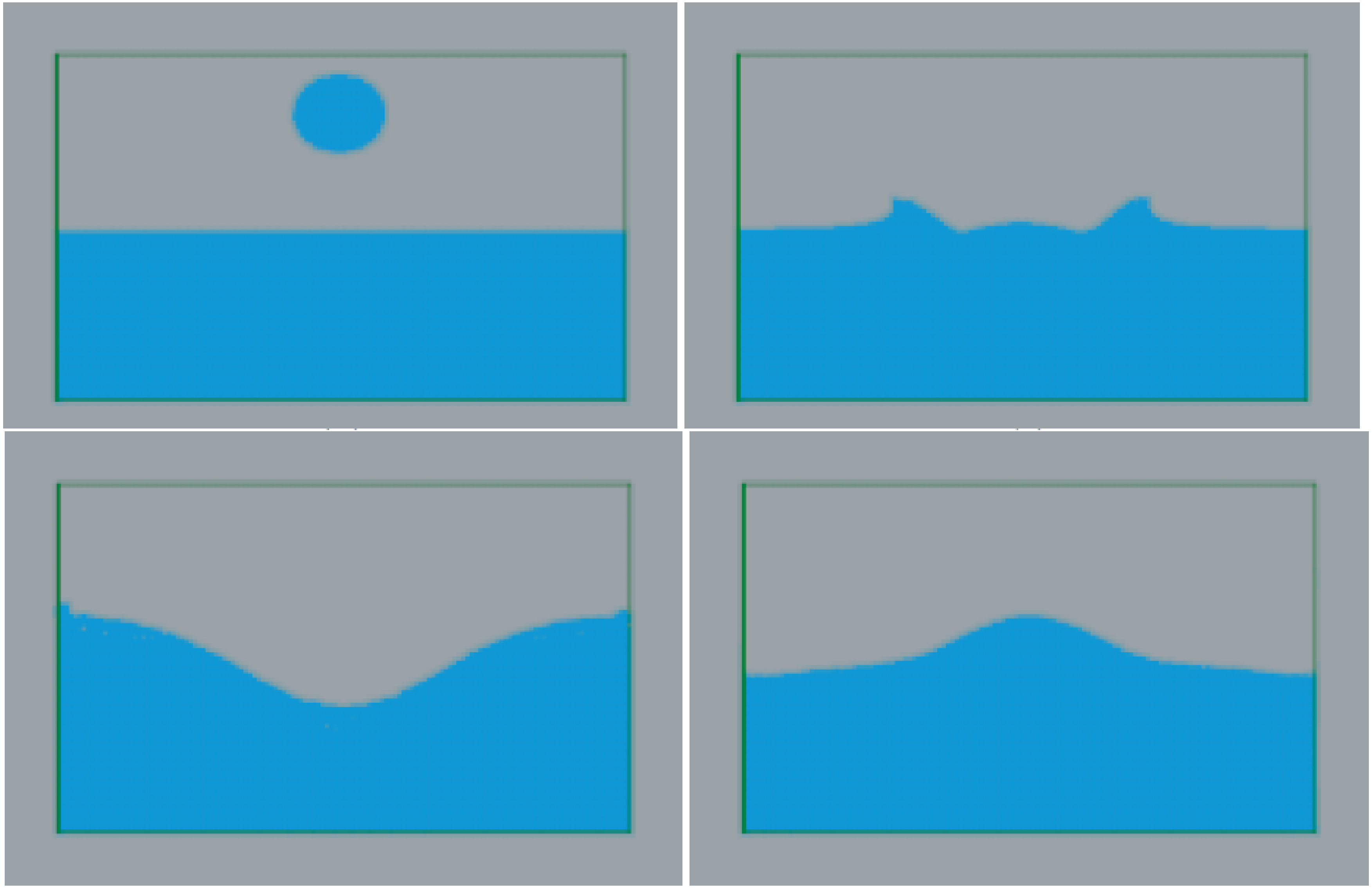

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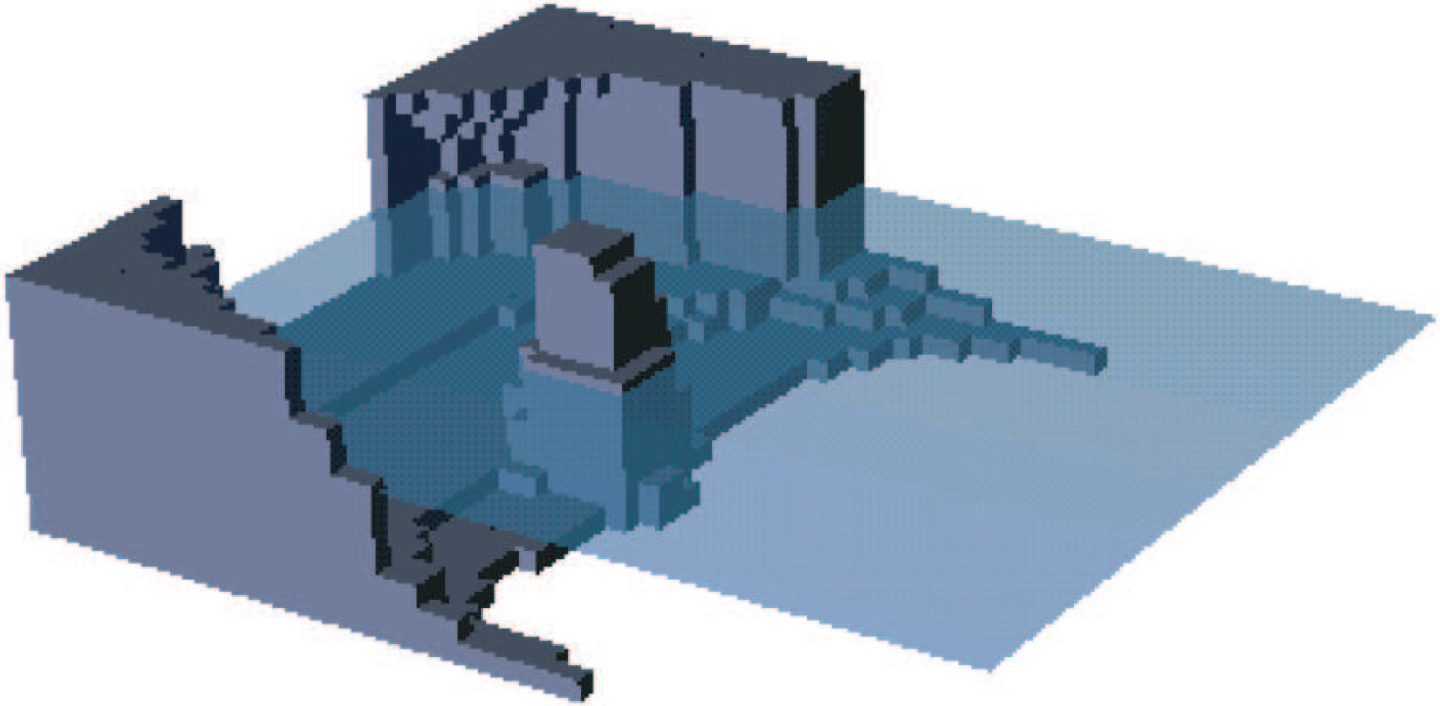

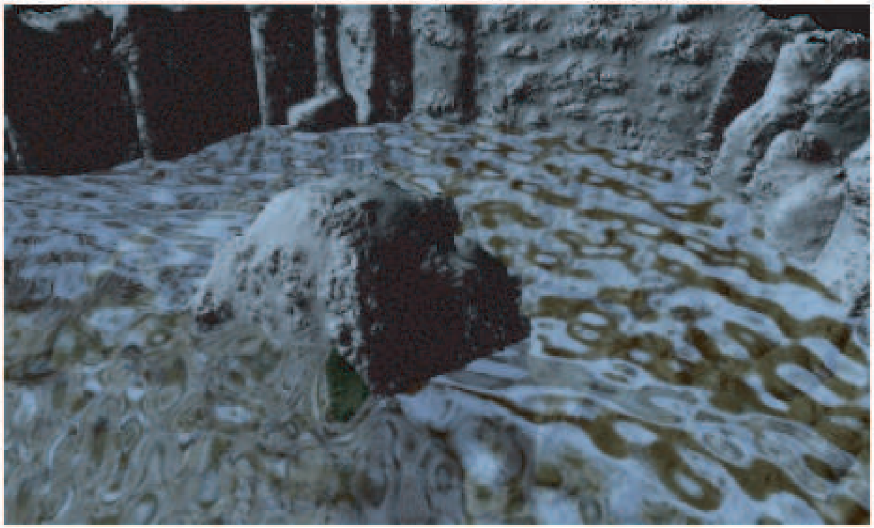

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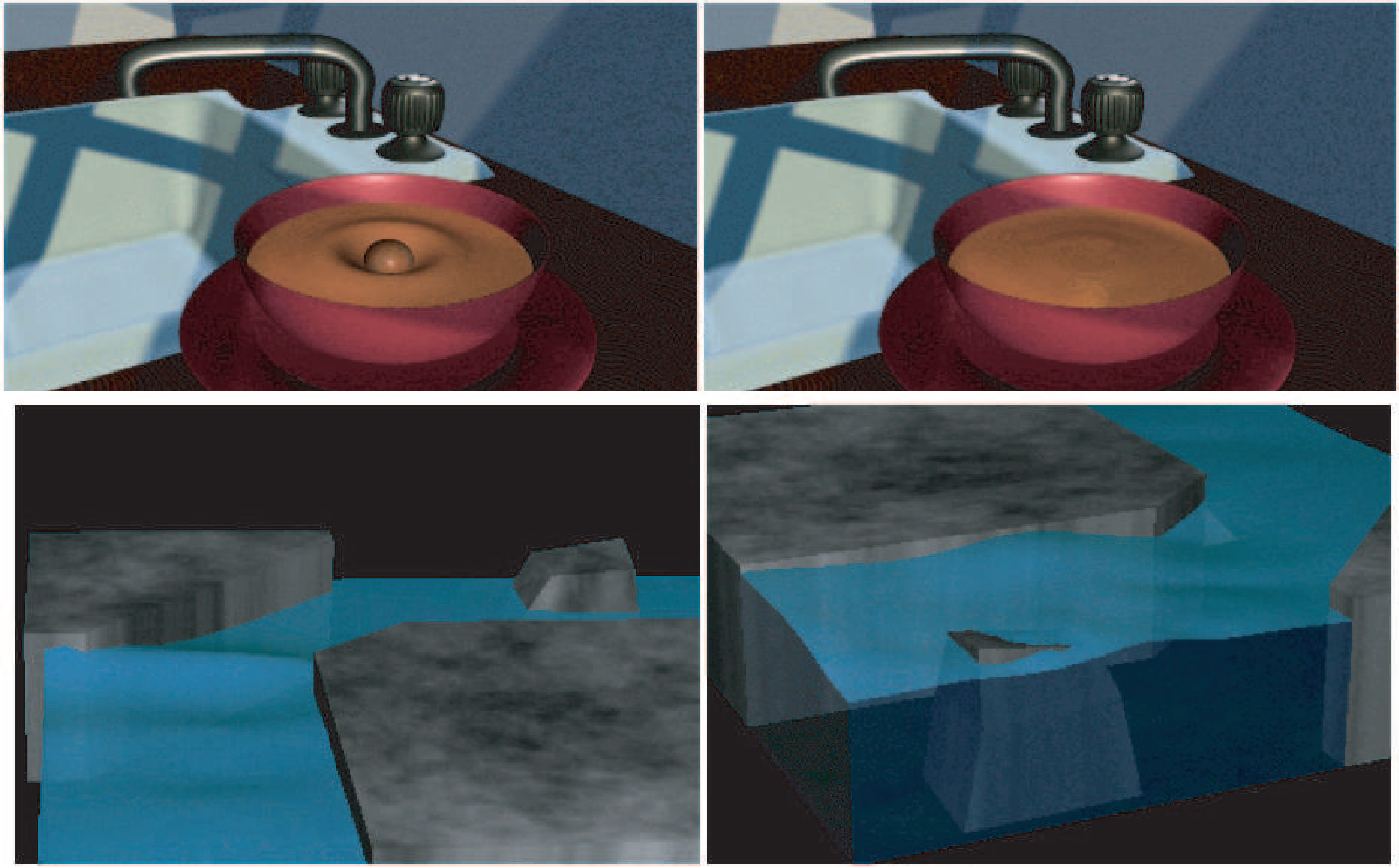

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