Approximating Extent Measures of Points

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Extent Measures

- Given a point set $P \subseteq \mathbb{R}^d$

- **Extent measures** – statistics about $P$ or its enclosing shape

- Some examples
  - $k$-th largest distance
  - Min. volume bounding box
  - Min. width bounding slab
  - Min. enclosing sphere
  - Min. enclosing cylinder
  - Min. width enclosing spherical shell
  - Min. width enclosing cylindrical shell
Main Result

- Technique for \( \varepsilon \)-approximating (a large class of) extent measures

- Compute a subset of the input \( Q \subseteq P \) (coreset) which
  - Preserves the solution to \( \varepsilon \)-accuracy
  - Small size (does not depend on \( |P| \), only on \( \varepsilon \))

- General properties
  - Strong LTAS, running time \( O \left( n + \left( \frac{1}{\varepsilon} \right)^{O(1)} \right) \)
  - Coreset size \( O \left( \left( \frac{1}{\varepsilon} \right)^{O(1)} \right) \)
  - Exponents depend on \( d \)

- Simple to implement, and some improvements
  - Min. enclosing spherical shell exponent down from \( O(d^2) \) [Chan 02] to \( O(d) \)
Key Definitions

- Lead to two main approximation primitives

- **Directional width** of a point set $P$ in the direction $u \in \mathbb{R}^{d-1}$
  \[ w(u, P) = \max_{p \in P} \langle [u, 1], p \rangle - \min_{p \in P} \langle [u, 1], p \rangle \]

- **Extent** of a set $F$ of $(d - 1)$-variate functions at $x \in \mathbb{R}^{d-1}$
  \[ e(x, F) = \max_{f \in F} f(x) - \min_{f \in F} f(x) \]
Optimization Primitives

- \( Q \subseteq P \) is an \( \epsilon \)-approximation for \( P \) on \( \Delta \subseteq \mathbb{R}^{d-1} \) if for all \( u \in \Delta \)
  \[
  (1 - \epsilon)w(u, P) \leq w(u, Q) \leq w(u, P)
  \]

- \( G \subseteq F \) is an \( \epsilon \)-approximation for \( F \) on \( \Delta \subseteq \mathbb{R}^{d-1} \) if for all \( x \in \Delta \)
  \[
  (1 - \epsilon)e(x, F) \leq w(u, G) \leq e(x, F)
  \]

- **Note:** Always pick a subset of the input – **coreset**
Classes of Extent Measures

- **Faithful**
  - Approximated through directional width
  - “Convex” measures (bounding shapes)

- **Other**
  - Approximated through extent
  - “Concave” measures ("shells")
Overview

(A) **Strong LTASs for directional width**
- Reduction to “fat” point sets
- Algorithm 1: Grid
- Algorithm 2: Polytope
- Algorithm 3: Decomposition

**Strong LTASs for extent**
- Linear functions (hyperplanes)
- Polynomial functions
- $r$-th roots of polynomials

(B) **Dynamic updates**
Applications to specific extent measures
Reduction to “Fat” Point Sets

- A point set \( P \in \mathbb{R}^d \) is \( \alpha\)-fat if there exists a translation \( t \) such that
  \[
  \alpha \mathbb{C} \subseteq \text{CH}(P) + t \subseteq \mathbb{C} = [-1, 1]^d
  \]

- Sufficient to consider computing coresets for \( \alpha\)-fat point sets

- **Step 1:**
  Every point set can be made \( \alpha \)-fat by applying a linear transformation, where \( \alpha = \alpha(d) \)

- **Step 2:**
  Every linear transformation preserves the approximation ratio of an arbitrary coreset
  - The size and construction time are clearly preserved
Step 1: There Exists a “Fattening Transform”

- [Barequet, Har-Peled 01]: Let $P \subseteq \mathbb{R}^d$ be of size $n$. Can compute in $O(n)$ time a box $B$ and a vector $t \in \mathbb{R}^d$ such that

$$\alpha B \subseteq \mathcal{CH}(P + t) \subseteq B$$

- Recall:
  $$\alpha = 1/10^? \text{ for } d = 3$$

- Choose $T$ so that
  $$T(B) = \mathbb{C}$$
Step 2: Invariance Under Linear Transforms

- **Lemma:**
  Let $T(x) = Mx + b$ be a non-degenerate linear transform.

  \[ Q \subseteq P \text{ is } \varepsilon\text{-approximates } P \text{ over } \Delta \subseteq \mathbb{R}^{d-1} \]

  if and only if

  \[ T(Q) \text{ is } \varepsilon\text{-approximates } T(P) \text{ over } \{v \mid [v, 1] = M^T[u, 1], \ u \in \Delta\} \]

- **Proof:** Easy by definition
  - Note: can assume $T(x) = Mx$
  - Also, $\langle [u, 1], Mp \rangle = \langle M^T[u, 1], p \rangle$
The Case of $\alpha$-fat Point Sets

- From now on assume $\alpha \mathcal{C} \subset \mathcal{CH}(P) \subset \mathcal{C}$ where $\alpha$ depends only on $d$, not on $n$
The Case of $\alpha$-fat Point Sets

- From now on assume $\alpha \subset C \subseteq CH(P) \subseteq \mathbb{C}$ where $\alpha$ depends only on $d$, not on $n$

- **Lemma** [Rough approximation]: $w(x, P) \geq 2\alpha \|x\|$

  \[ \forall x \in \mathbb{R}^d: \max_{p \in P} \langle x, p \rangle = \|x\| \cdot \max_{p \in P} \left\langle \frac{x}{\|x\|}, p \right\rangle \geq \|x\|\alpha \]

  For $x_d = 1$:
  \[ \max_{p \in P} \langle x, p \rangle \geq \alpha \|x\| \quad \text{min}_{p \in P} \langle x, p \rangle \leq -\alpha \|x\| \]
The Case of $\alpha$-fat Point Sets

- From now on assume $\alpha \mathbb{C} \subseteq \mathcal{H}(P) \subseteq \mathbb{C}$ where $\alpha$ depends only on $d$, not on $n$

- Lemma [Rough approximation]: $w(x, P) \geq 2\alpha \|x\|$  
  $$\forall x \in \mathbb{R}^d: \max_{p \in P} \langle x, p \rangle = \|x\| \cdot \max_{p \in P} \langle \frac{x}{\|x\|}, p \rangle \geq \|x\| \alpha$$

  For $x_d = 1$:  
  $$\max_{p \in P} \langle x, p \rangle \geq \alpha \|x\| \quad \min_{p \in P} \langle x, p \rangle \leq -\alpha \|x\|$$

- Lemma [Hausdorff dist.]: If $\max_{p \in P} \min_{q \in Q} \|p - q\| \leq \epsilon \alpha$ then $Q$ is an $\epsilon$-approximation for $P$
The Case of $\alpha$-fat Point Sets

- From now on assume $\alpha C \subseteq C\mathcal{H}(P) \subseteq C$ where $\alpha$ depends only on $d$, not on $n$
- **Lemma** [Rough approximation]: $w(x, P) \geq 2\alpha \|x\|$

  $\forall x \in \mathbb{R}^d$: $\max_{p \in P} \langle x, p \rangle = \|x\| \cdot \max_{p \in P} \left( \frac{x}{\|x\|}, p \right) \geq \|x\| \alpha$

  For $x_d = 1$: $\max_{p \in P} \langle x, p \rangle \geq \alpha \|x\|$, $\min_{p \in P} \langle x, p \rangle \leq -\alpha \|x\|$ 

- **Lemma** [Hausdorff dist.]: If $\max_{p \in P} \min_{q \in Q} \|p - q\| \leq \epsilon \alpha$

  then $Q$ is an $\epsilon$-approximation for $P$

  $w(x, P) - w(x, Q) \leq \langle x, p_1 - p_2 \rangle - \langle x, q_1 - q_2 \rangle$
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- From now on assume $\alpha \mathbb{C} \subseteq \mathcal{CH}(P) \subseteq \mathbb{C}$ where $\alpha$ depends only on $d$, not on $n$

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  $\leq |\langle x, p_1 - p_2 \rangle| - |\langle x, q_1 - q_2 \rangle|$
The Case of $\alpha$-fat Point Sets

- From now on assume $\alpha C \subseteq CH(P) \subseteq C$ where $\alpha$ depends only on $d$, not on $n$

- **Lemma** [Rough approximation]: $w(x, P) \geq 2\alpha \|x\|$

  $\forall x \in \mathbb{R}^d : \max_{p \in P} \langle x, p \rangle = \|x\| \cdot \max_{p \in P} \langle \frac{x}{\|x\|}, p \rangle \geq \|x\|\alpha$

For $x_d = 1$:

\[ \max_{p \in P} \langle x, p \rangle \geq \alpha \|x\| ] \quad \min_{p \in P} \langle x, p \rangle \leq -\alpha \|x\| \]

- **Lemma** [Hausdorff dist.]: If $\max_{p \in P} \min_{q \in Q} \|p - q\| \leq \epsilon \alpha$ then $Q$ is an $\epsilon$-approximation for $P$

\[ w(x, P) - w(x, Q) \leq \langle x, p_1 - p_2 \rangle - \langle x, q_1 - q_2 \rangle \]

\[ \leq |\langle x, p_1 - p_2 \rangle| - |\langle x, q_1 - q_2 \rangle| \]

\[ \leq |\langle x, (p_1 - q_1) - (p_2 - q_2) \rangle| \leq \|x\| \cdot 2\alpha \epsilon \]
Algorithm 1: Grid

- Grid of cell size $\frac{\epsilon \alpha}{2\sqrt{d}}$
Algorithm 1: Grid

- Grid of cell size $\frac{e\alpha}{2\sqrt{d}}$
- Clear all “internal” cells
  - Convex hull moves by at most one cell diameter $e\alpha/2$
Algorithm 1: Grid

- Grid of cell size \( \frac{\epsilon \alpha}{2\sqrt{d}} \)

- Clear all “internal” cells
  - Convex hull moves by at most one cell diameter \( \epsilon \alpha / 2 \)

- Eliminate duplicates in the “boundary” cells
  - Another shift of at most \( \epsilon \alpha / 2 \)
Algorithm 1: Grid

- Grid of cell size $\frac{\epsilon \alpha}{2\sqrt{d}}$

- Clear all “internal” cells
  - Convex hull moves by at most one cell diameter $\epsilon \alpha/2$

- Eliminate duplicates in the “boundary” cells
  - Another shift of at most $\epsilon \alpha/2$

- Directional width changes at most $\epsilon \alpha$
  - Determined by convex hull
Algorithm 1: Analysis

- Coreset size $O(1/(\alpha \epsilon)^{d-1})$
  - One point per “boundary” grid cell
  - Cell size $O(1/(\alpha \epsilon))$

- Running time $O(n + 1/(\alpha \epsilon)^{d-1})$

- $O$-notation hides $\sqrt{d}$
- $\alpha$ has a bad dependence on $d$
Algorithm 2: Polytope

- Run Algorithm 1, return \((\epsilon/2)\)-approximation \(P_1\)
- Apply [Dudley 1974] to \(P_1\), lose another \(\epsilon/2\)
  - Sample the sphere of radius \(\sqrt{d} + 1\)
  - Closest point routine [Gärtner 1995], runs in linear time, returns all (at most \(d\)) closest points
- Return the set of closest points \(P_2\)
Algorithm 2: Analysis

- **Correctness**
  - **Fact 1**: Dudley polytope is also valid for $P_2$
  - **Fact 2**: $\mathcal{CH}(P_2) \subseteq \mathcal{CH}(P_1) \subseteq \text{DUDLEY}$

- **Want** $(\alpha \epsilon / 2)$-approximation of the convex hull
  - Yields $\epsilon / 2$ for directional width

- **Required number of samples**
  \[ O((\sqrt{\alpha \epsilon / 2})^{d-1}) = O((\alpha \epsilon)^{(d-1)/2}) \]

- **This is also (roughly) the size of the result** $P_2$
  - At most $d$ per sample
  - Compare to Algorithm 1: $O((\alpha \epsilon)^{d-1})$
Algorithm 2: Analysis

- Find closest points for $O((\alpha\epsilon)^{(d-1)/2})$ samples

- The closest point routine [Gärtner 1995]
  - Linear in the number of points
  - In this case, $|P_1| = O((\alpha\epsilon)^{d-1})$
  - Total, $O((\alpha\epsilon)^{3(d-1)/2})$

- Including the call to Algorithm 1, we get $O(n + (\alpha\epsilon)^{3(d-1)/2})$
  - Compare to Algorithm 1: $O(n + (\alpha\epsilon)^{d-1})$
Algorithm 3: Arrangement

- Run Algorithm 2, get an \((\epsilon/2)\)-approximation \(P_2\)
- Decompose a set of directions \(\mathbb{R}^{d-1}\) using an arrangement of \((d-2)\)-flats

\[ u, v \in \mathbb{R}^{d-1} \text{ in the same cell} \Rightarrow \left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\| \leq \frac{\alpha \epsilon}{4\sqrt{d}} \]

- Identify diametrically opposite cells (induced partition of \(\mathbb{P}^{d-1}\), the set of unoriented directions)

- Return \(P_3\), consisting of two points per cell
  - Extremal points in one arbitrary direction within the cell
  - Lose another \(\epsilon/2\)

\[
\langle \text{ANGLE ERROR, EXTREME POINT ERROR} \rangle \leq \left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\| \cdot \|p - q\| \leq \frac{\alpha \epsilon}{4\sqrt{d}} \cdot 2\sqrt{d} = \frac{\alpha \epsilon}{2}
\]
Algorithm 3: Computing the Arrangement
Algorithm 3: Computing the Arrangement

- $(d - 1)$-dimensional grid within each of the $d$ facets of $\mathbb{C}$
  - Cell size $\frac{\alpha \epsilon}{4\sqrt{d}}$
  - Results in a set of $(d - 2)$-flats

- Affine hull with the origin, intersect with $\mathbb{P}^{d-1}$
Algorithm 3: Computing the Arrangement

- $(d - 1)$-dimensional grid within each of the $d$ facets of $\mathbb{C}$
  - Cell size $\frac{\alpha \epsilon}{4\sqrt{d}}$
  - Results in a set of $(d - 2)$-flats

- Affine hull with the origin, intersect with $\mathbb{P}^{d-1}$

- Complexity of the arrangement
  - Number of hyperplanes $d(d - 1) \frac{4\sqrt{d}}{\alpha \epsilon}$
  - Complexity $O(1/(\alpha \epsilon)^{d-1})$
Algorithm 3: Analysis

- Coreset size $|P_3| = O(1/(\alpha \varepsilon)^{d-1})$
  - Same as the arrangement complexity

- Extremal points are computed by linear search through $P_2$
  - $|P_2| = O(1/(\alpha \varepsilon)^{(d-1)/2})$ time per cell
  - $O(1/(\alpha \varepsilon)^{d-1})$ cells
  - $O(1/(\alpha \varepsilon)^{3(d-1)/2})$ total
  - Combined with Algorithm 2, $O(n + 1/(\alpha \varepsilon)^{3(d-1)/2})$

- Note $|P_3| > |P_2|$, a point may be extremal in multiple cells
Approximating \((d - 1)\)-variate Functions

- Recall: \(\epsilon\)-approximation in terms of extent
Approximating \((d - 1)\)-variate Functions

- Recall: \(\epsilon\)-approximation in terms of extent

1. Linear functions (hyperplanes), using duality
2. General polynomials, using linearization
3. \(r\)-th roots of polynomials, \(r \in \mathbb{Z}\)
The hyperplane
\[ h : x_d = a_1 x_1 + a_2 x_2 + \cdots + a_{d-1} x_{d-1} + a_d \]
corresponds to the point
\[ h^* = (a_1, a_2, \ldots, a_d) \in \mathbb{R}^d \]

**Lemma:** Set of hyperplanes \( H = \{h_1, h_2, \ldots, h_n\} \) is \( \epsilon \)-approximated by \( K \subset H \) over \( \Delta \in \mathbb{R}^{d-1} \) iff \( H^* \) is \( \epsilon \)-approximated by \( K^* \) over \( \Delta \)

**Proof:** Directional width \( w(u, H^*) \) is the same as the extent \( e(u, H) \)
- Holds for any \( u \in \mathbb{R}^{d-1} \) and any set of hyperplanes \( H \) in \( \mathbb{R}^d \)
- Important to define \( w(\cdot, \cdot) \) using the \( x_d = 1 \) plane

Immediately implies \( \epsilon \)-approximation algorithms for linear functions in \((d-1)\)-variables
Linearization

- A set of \((d - 1)\)-variate polynomials \(F = \{f_1, f_2, \ldots, f_n\}\)
- Linearization of dimension \(k\)
  \[
  f_i(x) = a_0^{(i)} + a_1^{(i)} \phi_1(x) + \cdots + a_k^{(i)} \phi_k(x) \quad i = 1, 2, \ldots, n
  \]
- Reduces to a set of \(k\)-variate linear functions
  \[
  f_i(y) = a_0^{(i)} + a_1^{(i)} y_1 + \cdots + a_k^{(i)} y_k \quad i = 1, 2, \ldots, n
  \]
  with \(y = \phi(x), \phi: \mathbb{R}^{d-1} \to \mathbb{R}^k\)
- An \(\epsilon\)-approximation for \(\{f_i(y)\}\) over \(\Delta \in \mathbb{R}^k\) implies an \(\epsilon\)-approximation for \(\{f_i(x)\}\) over \(\phi^{-1}(\Delta \cap \phi(\mathbb{R}^{d-1}))\)
Linearization Example

- $f_i(x)$ is the distance of the point $(x_1, x_2) \in \mathbb{R}^2$ to a circle in $\mathbb{R}^2$ with center $(p^{(i)}, q^{(i)})$ and radius $r^{(i)}$

$$
    f_i(x) = (r^{(i)})^2 - (x_1 - p^{(i)})^2 - (x_2 - q^{(i)})^2
$$

- Can be written as

$$
    f_i(x) = (r^{(i)} - p^{(i)} - q^{(i)})^2 + (2p^{(i)})x_1 + (2q^{(i)})x_2 - [x_1^2 + x_2^2]
$$

- A linearization of dimension $k = 3$ with

$$
    a^{(i)} = [(r^{(i)} - p^{(i)} - q^{(i)})^2, 2p^{(i)}, 2q^{(i)}, -1]
$$

$$
    \phi(x) = [x_1, x_2, x_1^2 + x_2^2]
$$

- [Agarwal, Matoušek 1994] Computing linearization of minimum dimension
Polynomials: Algorithms

- For a set $F$ of $n (d - 1)$-variate polynomials admits a linearization of dimension $k$, can compute

  - **Algorithm 1**: an $\epsilon$-approximation of size $O(1/\epsilon^k)$, in time $O(n + 1/\epsilon^k)$
  - **Algorithm 2**: an $\epsilon$-approximation of size $O(1/\epsilon^{k/2})$, in time $O(n + 1/\epsilon^{3k/2})$
  - **Algorithm 3**: a set of $O(1/\epsilon)(d - 2)$-dimensional surfaces in $\mathbb{R}^{d-1}$, in time $O(n + 1/\epsilon^{3k/2})$, such that within each cell of their arrangement $F$ is $\epsilon$-approximated by 2 of its elements
  - Complexity of the arrangement $O(1/\epsilon^{d-1})$ [Agarwal, Sharir 00] Cells are of complexity $O(1)$
  - **Algorithms 2 and 3**: an $\epsilon$-approximation of size $O(1/\epsilon^{\min\{k/2, d-1\}})$, in time $O(n + 1/\epsilon^{3k/2})$
Roots of Polynomials

- Want $\epsilon$-approximation for $F = \{(f_1)^{1/r}, (f_2)^{1/r}, \ldots, (f_n)^{1/r}\}$ where $r$ is integer
- Cannot linearize directly
- Special cases can be handled [Chan 02]
- If $a, b, A, B \geq 0$ and $[A, B] \subseteq [a, b]$ then
  \[ B - A \geq (1 - \delta)(b - a) \quad \Rightarrow \quad B^{1/r} - A^{1/r} \geq (1 - \epsilon)(b^{1/r} - a^{1/r}) \]
  for $\delta = \left(\frac{\epsilon}{2(r-1)}\right)^r$
- It suffices to compute $O(\epsilon^r)$ approximation to $\{f_1, f_2, \ldots, f_n\}$
Roots of Polynomials: Algorithms

- Set of $F$, $|F| = n$, contains $r$-th roots of $(d - 1)$-variate non-negative polynomials that admit a linearization of dimension $k$

- **Algorithm 1:** $\epsilon$-approximation in time $O(1/\epsilon^{kr})$ and size $O(1/\epsilon^{kr})$

- **Algorithms 2 and 3:** $\epsilon$-approximation in time $O(1/\epsilon^{3kr/2})$ and size $O(1/\epsilon^{r \min\{d-1,k/2\}})$

- **Algorithm 3:** A set of $O(1/\epsilon^{r})$ ($d - 2$)-dimensional surfaces in $\mathbb{R}^{d-1}$, in time $O(1/\epsilon^{3kr/2})$, such that within each cell of their arrangement, $F$ is $\epsilon$-approximated by two of its elements
Classes of Extent Measures

- **Faithful**
  - Approximated through directional width
  - “Convex” measures (bounding shapes)
  - Examples:
    - Diameter
    - Width
    - Radius of the smallest enclosing ball
    - Volume of the minimum bounding box
    - Volume enclosed by the convex hull
    - Surface area of the convex hull

- **Other**
  - Approximated through extent
  - “Concave” measures (“shells”)
  - Examples:
    - Minimum width spherical shell (annulus)
    - Minimum width cylindrical shell
Dynamic Updates

- Based on a **balanced tree** data structure
  - Both insertions and deletions
  - Recompute approximations along the unique path to the root
  - Update time per operation $O((\log^{k+1} n/\epsilon)^k + f(\log n/\epsilon) \log n)$
  - Amortized $O((\log^k n/\epsilon)^k + f(\log n/\epsilon))$

- Based on the **ranked subsets** data structure
  - Insertions only
  - Partition the points into ranked subsets, based on the binary encoding of $n$
  - Data structure size $O(\log^{2k+1} n/\epsilon^k)$
  - Coreset size $O(\log^{2k+1} n/\epsilon^k)$
    amortized insertion time $O((1/\epsilon)^k + f(\epsilon))$
  - Coreset size $O(1/\epsilon^k)$
    amortized insertion time $O(\log^{2k+1} n/\epsilon^k + f(\epsilon))$