Shape Matching: A Metric Geometry Approach
CS468
Assignment 3

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1 Description

The idea is to become more familiar with the invariants known as *shape distributions* [**osada**], *shape contexts* [**BM,BM-1**] and those proposed by Hamza and Krim [**HK**].

A second goal is that you use these invariants to compute the lower bounds for $\mathbf{D}_p$ discussed in [**M07**].

The zipfile `some-functions.zip` contains a few `matlab` functions and a script `test.m` that you'll find useful as a template for your computations.

1.1 Some test spaces

In the package you'll find functions that produce samples from spheres S^{d-1} of different dimensions, endowed with either geodesic or Euclidean metrics. Also, it is of interest to compare these to spaces $I^k := [0,1]^k$ for $k = 1, 2, \ldots$. In order to make the comparison more interesting, you should normalize all distance matrices so that diameter equals 1 for all the spaces you generate.

In particular, you should check that

- For $X = S^d$ with geodesic metric, $s_{X,1}$ and $\mathbf{diam}_1(X)$ are independent of d. Compute these for samples of $n = 2000$ points on S^1, S^2, S^3.

1.2 The invariants

In all considerations below, (X, d_X, μ_X) is a finite mm-space. You need to write `matlab` functions for computing

- the p-diameter of a mm-space. Remember that

$$\mathbf{diam}_p\,(X) = \left(\sum_{x,x'} d_X^p(x,x')\mu_X(x)\mu_X(x')\right)^{1/p}.$$

- Hamza-Krim invariant. This invariant is a.k.a. *eccentricity* (see paper by Peyre et al.). For $p \geqslant 1$,

$$s_{X,p}(x) = \left(\sum_{x'} d_X^p(x,x')\mu_X(x')\right)^{1/p}$$

- Shape distributions: Compute the shape distributions signature F_X of X. Recall ([**M07**], Proposition 7) that

$$F_X(t) = \mu_X \otimes \mu_X \left((x, x') \,\text{s.t.}\, d_X(x, x') \leqslant t\right)$$

  Note: the only challenge here is understanding what's the meaning of the formula above.

- (geodesic) Shape contexts: Compute the geodesic shape context signature C_X of X:

$$C_X(x, t) = \mu_X(x' \,\text{s.t.}\, d_X(x, x') \leqslant t)$$

1.3 Lower bounds

- Write a `matlab` function that takes as input two mm-spaces X and Y and $p \in [1, \infty]$ and returns the lower bound for $\mathbf{D}_p$ given by equation (21) in [**M07**]:

$$\frac{1}{2}|\mathbf{diam}_p(X) - \mathbf{diam}_p(Y)|.$$

  We call this lower bound, ZLB_p (zero-lower bound).

- Let $p = 1$. Write a `matlab` function that takes as input two finite *mm*-spaces X and Y and computes SLB_1 ([**M07**] Proposition 7).

- Write a `matlab` function that takes as input two finite *mm*-spaces X and Y and $p \in [0, \infty)$ and computes FLB_p (§6.1 [**M07**]). **Note:** For this you will have to solve a simple LOP. Remember that `matlab`'s optimization toolbox knows how to do this.

2 Comparison

Consider the following 9 mm-spaces (formed by 2000 random samples each, use the functions I provided):

1. $(S^k, \text{Euclidean}, \text{uniform})$ for $k = 1, 2, 3$.
2. $(S^k, \text{geodesic}, \text{uniform})$ for $k = 1, 2, 3$.
3. $(I^k, \text{Euclidean}, \text{uniform})$ for $k = 1, 2, 3$.

with normalized distance matrices. Let $\mathcal{S}$ denote the set of these nine mm-spaces.

You need to compute,

- ZLB_p for all pairs of spaces in $\mathcal{S}$. Display result as a matrix. Do this for several choices of p to see if you obtain better discrimination. Recall that $p = 1$ is likely to give bad discrimination in the case of spheres with geodesic distance.
- Obtain a matrix with all the pair-wise comparisons using SLB_1.
- Obtain a matrix with all the pair-wise comparisons using FLB_p for $p = 1$ and $p = 2$.

Question: Which of the methods tested gives the best discrimination? How would you quantify this?

3 Details

- I estimate this should take no more than 4 hrs. Due date is **Nov. 10th.**
- What I expect to get back from you: all requisite functions and a `matlab` script that makes the computations and displays the results, with graphs etc and the final tables of comparisons according to Section 2.