Using matching distance in Size Theory: a survey

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Abstract

In this survey we illustrate how the matching distance between reduced size functions can be applied for shape comparison.

We assume that each shape can be thought of as a compact connected manifold with a real continuous function defined on it, that is a pair $(\mathcal{M}, \varphi : \mathcal{M} \to \mathbb{R})$, called *size pair*. In some sense, the function φ focuses on the properties and the invariance of the problem at hand. In this context, matching two size pairs $(\mathcal{M}, \varphi)$ and $(\mathcal{N}, \psi)$ means looking for a homeomorphism between $\mathcal{M}$ and $\mathcal{N}$ that minimizes the difference of values taken by φ and ψ on the two manifolds. Measuring the dissimilarity between two shapes amounts to the difficult task of computing the value $\delta = \inf_f \max_{P \in \mathcal{M}} |\varphi(P) - \psi(f(P))|$ where f varies among all the homeomorphisms from $\mathcal{M}$ to $\mathcal{N}$.

From another point of view, shapes can be described by reduced size functions associated with size pairs. The matching distance between reduced size functions allows for a robust to perturbations comparison of shapes.

The link between reduced size functions and the dissimilarity measure δ is established by a theorem stating that the matching distance provides an easily computable lower bound for δ.

Throughout this paper we illustrate this approach to shape comparison by means of examples and experiments.

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1 Introduction

Shape matching plays an important role in a number of Computer Vision problems, such as, e.g., shape retrieval, shape recognition and shape classification. Various techniques have been proposed to deal with the shape matching problem (see, e.g., (Veltkamp and Hagedoorn 2001)). A possible approach to this subject is to compare shapes by solving some minimization problem (see, e.g., (Hancock and Pelillo 1999)). This research line includes the natural pseudo-distance.

We assume that shapes can be described by pairs $(\mathcal{M}, \varphi)$, where $\mathcal{M}$ is a compact connected manifold, and $\varphi : \mathcal{M} \to \mathbb{R}$ is a continuous function focusing on the properties and the invariance of the matching problem at hand. In our setting, comparing two shapes represented by $(\mathcal{M}, \varphi)$ and $(\mathcal{N}, \psi)$, with $\mathcal{M}$ and $\mathcal{N}$ homeomorphic, means considering all the possible homeomorphisms $f : \mathcal{M} \to \mathcal{N}$ and computing the number $\inf_f \max_{P\in\mathcal{M}} |\varphi(P) - \psi(f(P))|$. The latter is a measure of the dissimilarity between the shapes represented by $(\mathcal{M}, \varphi)$ and $(\mathcal{N}, \psi)$, called *natural pseudo-distance* (see, e.g. (Frosini and Landi 1999), (Donatini and Frosini 2004*b*)).

In order to compute this dissimilarity measure, one must deal with an optimization problem and look for the transformation that minimizes the difference between two pairs $(\mathcal{M}, \varphi)$, $(\mathcal{N}, \psi)$ (in case it exists). This optimization problem is intrinsically difficult. In order to obviate this difficulty we rather estimate the dissimilarity by looking for a lower bound for the natural pseudo-distance.

A result recently proved in (d'Amico *et al.* 2003) states that a lower bound for the natural pseudo-distance is provided by a suitable matching distance between *reduced size functions*. These are (easily computable) functions, defined to describe shapes: the reduced size function $\ell^*_{(\mathcal{M},\varphi)} : \{(x, y) \in \mathbb{R}^2 : x < y\} \to \mathbb{N}$ is defined by setting $\ell^*_{(\mathcal{M},\varphi)}(x, y)$ equal to the number of connected components of the lower level set $\{P \in \mathcal{M} : \varphi(P) \le y\}$ which contain at least a point of $\{P \in \mathcal{M} : \varphi(P) \le x\}$.

A matching distance d_{match} between reduced size functions can be easily introduced. When $\mathcal{M}$ and $\mathcal{N}$ are homeomorphic, the following inequality holds:

$$\inf_{f:\mathcal{M}\to\mathcal{N}} \max_{P\in\mathcal{M}} |\varphi(P) - \psi(f(P))| \ge d_{match}(\ell^*_{(\mathcal{M},\varphi)}, \ell^*_{(\mathcal{N},\psi)}),$$

where f varies among all possible homeomorphisms. This yields an easily computable lower bound for the dissimilarity measure problem. This and other related results are examined in detail in (d'Amico *et al.* 2005).

This paper is devoted to illustrate all the previous concepts and related properties, and to point out the usefulness of this approach to shape comparison.

In Section 2 we shall recall the definitions of natural pseudo-distance between size pairs and of reduced size function. In Section 3 the definition of matching distance between reduced size functions will be given together with some theoretical results, and exemplified. Section 4 and Section 5 will be devoted to experiments and conclusions, respectively.

2 Natural pseudo-distance and reduced size functions

We begin with the definition of a pseudo-distance that allows us to measure the extent to which two shapes are similar to each other.

We stress the fact that when we think to the concept of shape, we have in mind a compact connected n-manifold $\mathcal{M}$ with a continuous real-valued function φ defined on it (no assumption is made about the regularity of $\mathcal{M}$). The manifold represents the object whose shape we are interested in (e.g., a silhouette), whereas the continuous function is chosen arbitrarily, usually according to the properties and the invariance of interest for the problem at hand (see, e.g., (Kaczynski *et al.* 2004), (Verri and Uras 1994), (Landi and Frosini 2002)). The pair $(\mathcal{M}, \varphi)$ is called an n-dimensional *size pair*.

Hereafter, $\mathcal{M}$ and $\mathcal{N}$ will denote compact connected n-manifolds, and $\varphi : \mathcal{M} \to \mathbb{R}$, $\psi : \mathcal{N} \to \mathbb{R}$ will be continuous functions, called *measuring functions.*

We point out that there is no limitation on the dimension of $\mathcal{M}$. Therefore, although so far most of the experiments in this field have been carried out for 2-D objects, the theory holds in general for any dimension.

The assumption on the connectedness of $\mathcal{M}$ can easily be weakened to any finite number of connected components, without much affecting the following results. More serious problems would derive from considering an infinite number of connected components.

Definition 2.1 *Let $(\mathcal{M}, \varphi)$, $(\mathcal{N}, \psi)$ be two size pairs and let $H(\mathcal{M}, \mathcal{N})$ be the set of all the homeomorphisms from $\mathcal{M}$ onto $\mathcal{N}$. If $H(\mathcal{M}, \mathcal{N}) \neq \emptyset$, let us consider the function Θ that takes each homeomorphism $f \in H(\mathcal{M}, \mathcal{N})$ to the real number $\Theta(f) = \max_{P \in \mathcal{M}} |\varphi(P) - \psi(f(P))|$. We shall call Θ the* natural measure *in $H(\mathcal{M}, \mathcal{N})$ with respect to the measuring functions φ and ψ.*

In plain words Θ measures how much f changes the values taken by the measuring functions.

Definition 2.2 *We define the* natural pseudo-distance *between $(\mathcal{M}, \varphi)$ and $(\mathcal{N}, \psi)$ as $\inf_{f \in H(\mathcal{M}, \mathcal{N})} \Theta(f)$ if $H(\mathcal{M}, \mathcal{N}) \neq \emptyset$ and $+\infty$ otherwise.*

It is not difficult to see that Def. 2.2 really gives a pseudo-distance. We point out that it is not a distance, but just a pseudo-distance, since it can be vanishing for size pairs which are not equal. However it is symmetric, satisfies the triangular inequality and vanishes for equal size pairs.

As an example of the natural pseudo-distance between two size pairs, consider the two tori $\mathcal{T}, \mathcal{T}' \subset \mathbb{R}^3$ of Figure 1, generated by the rotation around the y-axis of the circles lying in the plane yz and with centres $A = (0,0,3)$ and $B = (0,0,4)$, and radii 2 and 1, respectively (see, e.g., (Donatini and Frosini 2004b)). As measuring function φ (resp. φ') on $\mathcal{T}$ (resp. on $\mathcal{T}'$) we take the restriction to $\mathcal{T}$ (resp. to $\mathcal{T}'$) of the function $\zeta : \mathbb{R}^3 \to \mathbb{R}$, $\zeta(x,y,z) = z$. We point out that, for both $\mathcal{T}$ and $\mathcal{T}'$, the image of the measuring function is the closed interval $[-5,5]$. It is intuitive and not difficult to prove that the natural pseudo-distance between $(\mathcal{T}, \varphi)$ and $(\mathcal{T}', \varphi')$ is 2 (for a proof see (Frosini and Mulazzani 1999)).

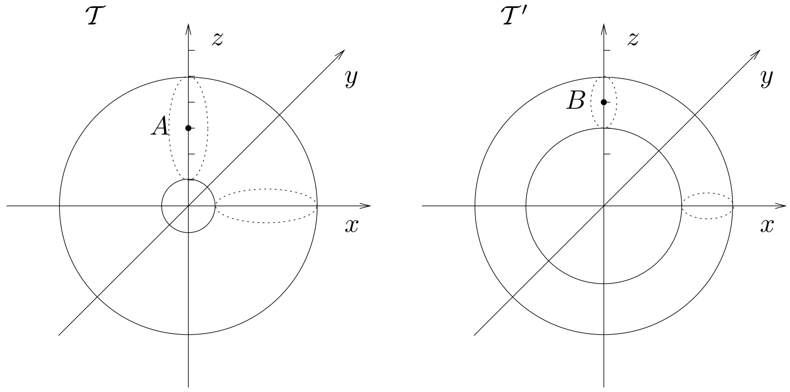

Figure 1: The size pairs $(\mathcal{T}, \varphi)$ and $(\mathcal{T}', \varphi')$, with φ and φ' equal to the height function, have natural pseudo-distance equal to 2.

It must be noted that the computation of the natural pseudo-distance is feasible just in few cases, as it involves the study of all homeomorphisms between two manifolds. On the other hand, using the natural pseudo-distance we can compare compact manifolds with respect to given measuring functions in a very powerful manner, and quantify the difference. Thus we need a tool to easily obtain information about the natural pseudo-distance without computing it directly: the concept of reduced size function is such a tool.

Remark 2.3 *We point out that an alternative definition of dissimilarity measure between size pairs based on the integral of the change of the measuring functions rather than on the* max *may present some drawbacks.*

For example, let us consider the following size pairs $(\mathcal{M}, \varphi)$, $(\mathcal{N}, \psi)$, $(\mathcal{N}, \chi)$, *where* $\mathcal{M}$ *is a circle of radius* 2, $\mathcal{N}$ *is a circle of radius* 1, *and the measuring functions are constant functions given by* $\varphi \equiv 1$, $\psi \equiv 1$, $\chi \equiv 2$. *Let* μ *and* ν *denote the* 1*-dimensional measures induced by the usual embeddings of* $\mathcal{M}$ *and* $\mathcal{N}$ *respectively in the Euclidean plane. By setting, for any homeomorphism* $f : \mathcal{M} \to \mathcal{N}$,

$$\hat{\Theta}(f) = \int_{\mathcal{M}} |\varphi - \psi \circ f| \, d\mu,$$

we have $\hat{\Theta}(f) = 0$. *For any homeomorphism* $g : \mathcal{N} \to \mathcal{N}$, *we have*

$$\hat{\Theta}(g) = \int_{\mathcal{N}} |\psi - \chi \circ g| \, d\nu = 2\pi.$$

On the other hand,

$$\hat{\Theta}(g \circ f) = \int_{\mathcal{M}} |\varphi - \chi \circ g \circ f| \, d\mu = 4\pi.$$

Hence, the inequality $\hat{\Theta}(g \circ f) \leq \hat{\Theta}(f) + \hat{\Theta}(g)$ *does not hold. This fact prevents the function* $\inf_{f \in H(\mathcal{M},\mathcal{N})} \hat{\Theta}(f)$ *from being a pseudo-distance, since we do not get the triangular inequality. Furthermore this function is not symmetric on all couples of size pairs.*

It could be interesting to explore other dissimilarity measures based on some integral of the change of the measuring functions.

In what follows, Δ denotes the diagonal of $\mathbb{R}^2$, that is $\Delta = \{(x, y) \in \mathbb{R}^2 : x = y\}$; moreover, Δ^+ denotes the open half-plane above Δ, that is $\Delta^+ = \{(x, y) \in \mathbb{R}^2 : x < y\}$.

Definition 2.4 *For every size pair* $(\mathcal{M}, \varphi)$, *the reduced size function* $\ell^*_{(\mathcal{M},\varphi)} : \Delta^+ \to \mathbb{N}$ *is defined by setting* $\ell^*_{(\mathcal{M},\varphi)}(x, y)$ *equal to the number of equivalence classes into which the lower level set* $\{P \in \mathcal{M} : \varphi(P) \leq x\}$ *is divided by the equivalence relation of* $\langle \varphi \leq y \rangle$*-connectedness, where* P *and* Q *are* $\langle \varphi \leq y \rangle$*-connected if they belong to the same connected component of the lower level set* $\{P \in \mathcal{M} : \varphi(P) \leq y\}$.

As an example of reduced size function, consider the size pair $(\mathcal{M}, \varphi)$ where $\mathcal{M}$ is the curve depicted in Fig. 2(left), and φ is the measuring function that takes each point of $\mathcal{M}$ to its distance from the barycenter of $\mathcal{M}$. In Fig. 2(right), we show the reduced size function $\ell^*_{(\mathcal{M},\varphi)} : \Delta^+ \to \mathbb{N}$. The number given in every region of the domain is the constant value taken by the reduced size function in that part of the domain. For instance, for $b \leq x < c$ the set $\{P \in \mathcal{M} : \varphi(P) \leq x\}$ has three connected components, two of which are contained in the same connected component of $\{P \in \mathcal{M} : \varphi(P) \leq y\}$ when $c \leq y < d$. Therefore, $\ell^*_{(\mathcal{M},\varphi)}(x, y) = 2$ for $b \leq x < c$ and $c \leq y < d$. When $b \leq x < c$ and $y \geq d$ all of the three connected components of $\{P \in \mathcal{M} : \varphi(P) \leq x\}$ belong to the same connected component of $\{P \in \mathcal{M} : \varphi(P) \leq y\}$, implying that in this case $\ell^*_{(\mathcal{M},\varphi)}(x, y) = 1$. For $a \leq x < b$ the set $\{P \in \mathcal{M} : \varphi(P) \leq x\}$ has only two connected components, which are contained in different connected components of $\{P \in \mathcal{M} : \varphi(P) \leq y\}$ when $x < y < d$. Therefore, $\ell^*_{(\mathcal{M},\varphi)}(x, y) = 2$ for $a \leq x < b$ and $x < y < d$.

The reason for introducing reduced size functions instead of working with classical size functions ((Verri *et al.* 1993), (Frosini and Landi 1999)) is that, while maintaining all the fundamental properties of size functions, reduced size

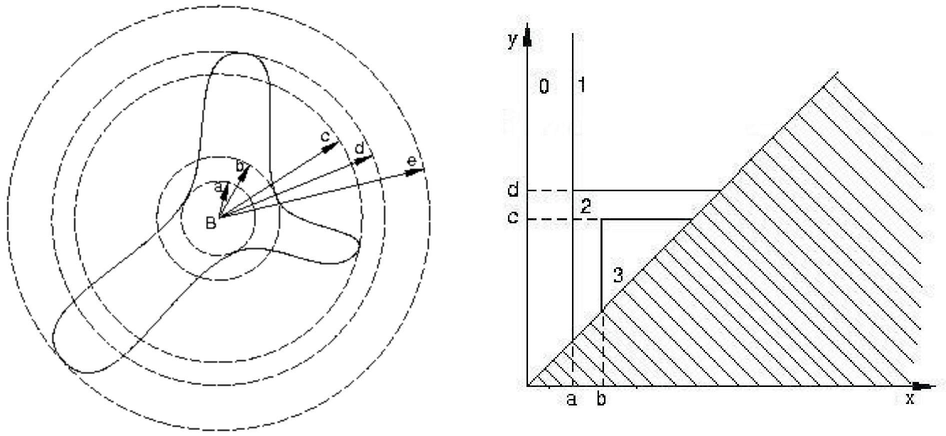

Figure 2: Left: A size pair $(\mathcal{M}, \varphi)$, where $\mathcal{M}$ is the curve depicted by a solid line, and φ is the distance from the barycenter of the curve. Right: the corresponding reduced size function.

functions allow us to avoid some technicalities and simplify a number of proofs. From the mathematical viewpoint, the difference between the two concepts lies in the use of connectedness instead of arcwise connectedness and the restriction of the domain from $\mathbb{R}^2$ to Δ^+.

It is important to remark that classical and reduced size functions are easily computable, (cf. (Frosini 1992), (Frosini and Pittore 1999) and (d'Amico 2000)).

3 Estimating the natural pseudo-distance via the matching distance

3.1 Matching distance for reduced size functions

Size functions and reduced size functions are useful for comparison of shapes even independently of the natural pseudo-distance. Indeed they can be considered as shape descriptors (see, e.g., (Dibos *et al.* 2004), (Donatini *et al.* 1998), (Collina *et al.* 1998), (Ferri *et al.* 1998), (Handouyaya *et al.* 1999), (Ferri *et al.* 1994)). Therefore, they allow to translate the problem of comparing shapes to the problem of comparing functions, that is a much simpler task. In order to perform the comparison between reduced size functions we can measure the cost necessary to deform reduced size functions into each other. Minimizing such a cost allows for a measure of the similarity between shapes.

In order to do so, reduced size functions are firstly transformed into simpler objects, precisely into sequences of points. This representation by means of sequences of points contains the same amount of information about the shape under study as the original reduced size function does but it is much easier to handle. Then we define a suitable matching distance between these sequences of points. This way we can measure the extent to which two shapes resemble each other by computing this matching distance.

We begin by describing how reduced size functions can be transformed into sequences of points.

We introduce cornerpoints, that are particular points in $\mathbb{R}\times(\mathbb{R}\cup\{\infty\})$ with reference to a reduced size function. The reader is referred to (Frosini and Landi 2001) for more details concerning cornerpoints.

Definition 3.1 *For every point $p=(x,y)\in\Delta^+$, let us define the number $\mu(p)$ as the minimum, over all the positive real numbers ϵ with $x+\epsilon<y-\epsilon$, of*

$$\ell^*_{(\mathcal{M},\varphi)}(x+\epsilon,y-\epsilon)-\ell^*_{(\mathcal{M},\varphi)}(x-\epsilon,y-\epsilon)-\ell^*_{(\mathcal{M},\varphi)}(x+\epsilon,y+\epsilon)+\ell^*_{(\mathcal{M},\varphi)}(x-\epsilon,y+\epsilon).$$

The finite number $\mu(p)$ will be called multiplicity of p *for $\ell^*_{(\mathcal{M},\varphi)}$. Moreover, we shall call* proper cornerpoint *for $\ell^*_{(\mathcal{M},\varphi)}$ any point $p\in\Delta^+$ such that the number $\mu(p)$ is strictly positive.*

Definition 3.2 *For every vertical line r, with equation $x=k$, let us define the number $\mu(r)$ as the minimum, over all the positive real numbers ϵ with $k+\epsilon<1/\epsilon$, of*

$$\ell^*_{(\mathcal{M},\varphi)}(k+\epsilon,1/\epsilon)-\ell^*_{(\mathcal{M},\varphi)}(k-\epsilon,1/\epsilon).$$

When this finite number, called multiplicity of r *for $\ell^*_{(\mathcal{M},\varphi)}$, is strictly positive, we call the line r a* cornerpoint at infinity *for the reduced size function, and we identify r with the pair (k,∞).*

Under our assumption on the connectedness of $\mathcal{M}$, $\mu(r)$ can take just the values 0 and 1, but the definition can be easily extended to disconnected manifolds so that $\mu(r)$ can equal any natural number.

The open (resp. closed) half-plane Δ^+ (resp. $\bar{\Delta}^+$) extended by the points at infinity of the kind (k,∞), with $|k|<\infty$, will be denoted by Δ^* (resp. $\bar{\Delta}^*$).

As an example of cornerpoints in reduced size functions, in Fig. 3 we see that the proper cornerpoints are the points A, B and C (with multiplicity 3, 2 and 1, respectively). The line m is the only cornerpoint at infinity.

In the framework of Size Theory, cornerpoints and their multiplicities are fundamental features, since they completely determine reduced size functions. Indeed, the following representation theorem can be proved (cf. (Frosini and Landi 2001) and (d'Amico *et al.* 2003)):

Theorem 3.3 *For every $(\bar{x},\bar{y})\in\Delta^+$ we have*

$$\ell^*_{(\mathcal{M},\varphi)}(\bar{x},\bar{y})=\sum_{\substack{(x,y)\in\Delta^*\\ x\le\bar{x},\,y>\bar{y}}}\mu\big((x,y)\big).$$

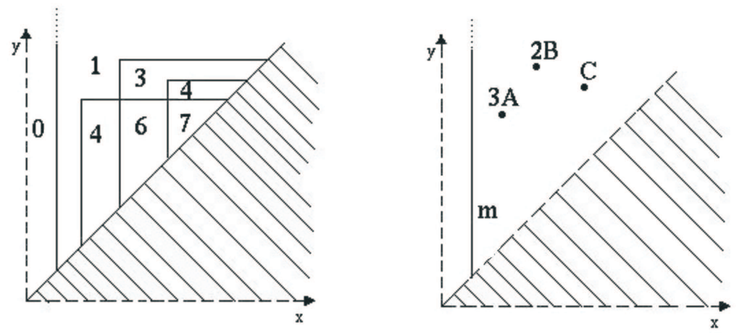

Figure 3: Cornerpoints of a reduced size function: A, B and C are the only proper cornerpoints, and have multiplicity equal to 3 (A), 2 (B), and 1 (C). The line m is the only cornerpoint at infinity.

This can be verified in the example of Fig. 3. For instance, let us take a point P in the region of the domain where the reduced size function takes value equal to 6. According to the above theorem, the value of the reduced size function at P must be equal to $\mu(m) + \mu(A) + \mu(B) = 1 + 3 + 2 = 6$.

We underline that even an infinity, though countable, of cornerpoints may occur in a reduced size function. Nevertheless, these cornerpoints necessarily accumulate onto the diagonal Δ. On the contrary, the connectedness of $\mathcal{M}$ implies that each reduced size function has exactly one cornerpoint at infinity.

Moreover, it can be observed that, roughly speaking, the further a cornerpoint is to Δ, the coarser is the shape feature that generates it; the closer a cornerpoint is to Δ, the finer is the shape detail it represents.

These observations provide the rationale for introducing the following notions.

Definition 3.4 *Let ℓ^* be a reduced size function. We shall call* representative sequence for ℓ^* *any sequence of points $a : \mathbb{N} \to \bar{\Delta}^*$, (briefly denoted by (a_i)), with the following properties:*

1. *a_0 is the cornerpoint at infinity for ℓ^*;*
2. *For each $i > 0$, either a_i is a proper cornerpoint for ℓ^*, or a_i belongs to Δ;*
3. *If p is a proper cornerpoint for ℓ^* with multiplicity $\mu(p)$, then the cardinality of the set $\{i \in \mathbb{N} : a_i = p\}$ is equal to $\mu(p)$;*
4. *The set of indexes for which a_i belongs to Δ is countably infinite.*

In the example of Fig. 3, one obtains a representative sequence by taking, for instance, $a_0 = m$, $a_1 = A$, $a_2 = A$, $a_3 = A$, $a_4 = B$, $a_5 = B$, $a_6 = C$, and $a_i \in \Delta$ for every $i > 6$.

We now define a pseudo-metric in $\bar{\Delta}^*$ that will give rise to a distance between reduced size functions, based on the matching of two representative sequences.

Definition 3.5 *Throughout the rest of the paper, d will denote the pseudo-distance on $\bar{\Delta}^*$ defined by setting, for any (x, y), (x', y') in $\bar{\Delta}^*$,*

$$d\left((x,y),(x',y')\right) = \min\left\{\max\left\{|x-x'|,|y-y'|\right\},\max\left\{\frac{y-x}{2},\frac{y'-x'}{2}\right\}\right\},$$

with the convention about ∞ that $\infty - y = y - \infty = \infty$ for $y \neq \infty$, $\infty - \infty = 0$, $\frac{\infty}{2} = \infty$, $|\infty| = \infty$, $\min\{\infty, c\} = c$, $\max\{\infty, c\} = \infty$.

In other words, the pseudo-distance d between two points p and p' above the diagonal measures the smaller between the cost of moving p to p', and the cost of moving p and p' onto the diagonal, where costs are computed by using the distance induced by the max-norm. The pseudo-distance d between two points p and p' on the diagonal is always 0. The pseudo-distance d between two points p and p', with p above the diagonal and p' on the diagonal is equal to the distance, induced by the max-norm, between p and the diagonal. Points at infinity have a finite distance only to other points at infinity and their distance depends on their abscissas.

Definition 3.6 *If (a_i) and (b_i) are two representative sequences for ℓ_1^* and ℓ_2^* respectively, then the* matching distance *between ℓ_1^* and ℓ_2^* is the number*

$$d_{match}(\ell_1^*, \ell_2^*) := \inf_\sigma \sup_i d(a_i, b_{\sigma(i)}),$$

where i varies in $\mathbb{N}$ and σ varies among all the bijections from $\mathbb{N}$ to $\mathbb{N}$.

It is easy to see that this definition is independent from the choice of the representative sequences of points for the reduced size functions ℓ_1^* and ℓ_2^*.

Moreover, the inf and the sup in the definition of matching distance are actually attained, that is to say $d_{match}(\ell_1^*, \ell_2^*) = \min_\sigma \max_i d(a_i, b_{\sigma(i)})$ (cf. (d'Amico *et al.* 2003), (d'Amico *et al.* 2005)).

We point out that d_{match} is actually a distance and not just a pseudo-distance between reduced size functions.

This kind of metric based on the matching between two point sets is known in the literature also with the name of bottleneck distance (see, e.g., (Veltkamp and Hagedoorn 2001)). Algorithms for its computation are discussed for example in (Efrat *et al.* 2001). The computational complexity for the matching distance is polynomial ($O(n^{2.5})$, where n is the maximum number of cornerpoints allowed in the reduced size functions).

Although not thoroughly studied from a theoretical viewpoint until (d'Amico *et al.* 2003), applications of the d_{match} distance have already proved to be successful in the implementation of an image retrieval system ((Brucale *et al.* 2002)). Indeed, a discrete counterpart of the size function theory based on graphs has been developed.

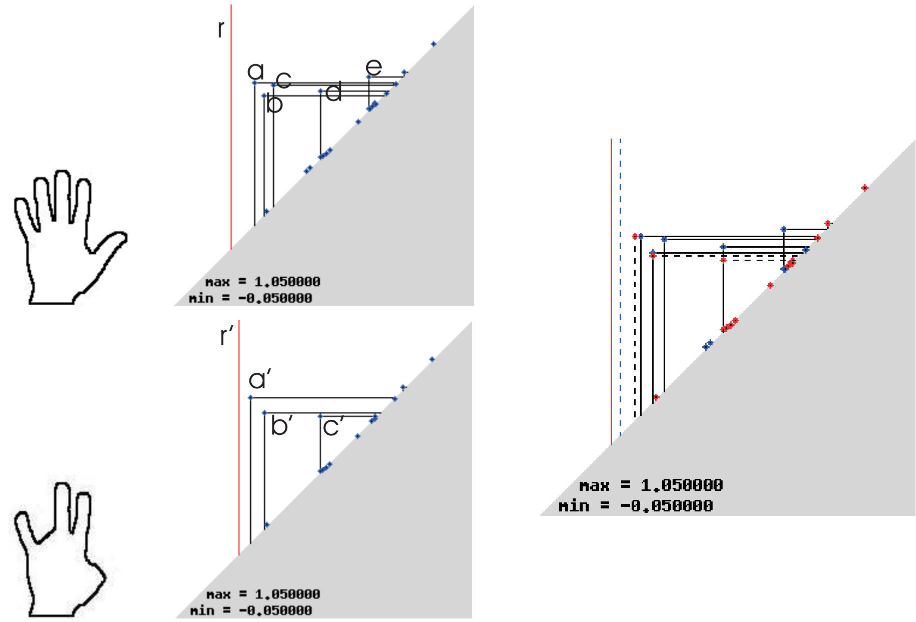

Figure 4: Left: Two curves. Center: Their reduced size functions with respect to the measuring function distance from the center of the image. Right: The superimposition of the two reduced size functions.

Experiments showing the effective capability of the d_{match} distance in comparing shapes are postponed until Sect. 4. Here we confine ourselves to furnish an example of how the matching distance works. Consider Fig. 4. Given two curves, their reduced size functions with respect to the measuring function distance from the center of the image are calculated. One sees that the top reduced size function has many cornerpoints close to the diagonal in addition to the cornerpoints r, a, b, c, d, e. Analogously, the bottom reduced size function has many cornerpoints close to the diagonal in addition to the cornerpoints r', a', b', c'. Cornerpoints close to the diagonal are generated by noise and discretization. The superimposition of the two reduced size functions shows that an optimal matching is given by $r \to r'$, $a \to a'$, $b \to b'$, $c \to c'$, $d \to \Delta$, $e \to \Delta$, and all the other cornerpoints sent to Δ. Sending cornerpoints to points of Δ corresponds to the annihilation of cornerpoints. Since the matching $c \to c'$ is the one that achieves the maximum cost in the max-norm, the matching distance is equal to the distance between c and c' (with respect to the max-norm).

3.2 A lower bound for the natural pseudo-distance

A useful result holds, concerning an estimate for the natural pseudo-distance given in terms of the matching distance between reduced size functions. It can be seen as a consequence of the stability of the matching distance: small changes of the measuring functions produce small changes in the matching distance between reduced size functions. More precisely, if φ and ψ are two measuring functions on $\mathcal{M}$ whose difference on the points of $\mathcal{M}$ is controlled by ϵ (namely $\max_{P\in\mathcal{M}}|\varphi(P)-\psi(P)|\leq\epsilon$), then the matching distance between $\ell^*_{(\mathcal{M},\varphi)}$ and $\ell^*_{(\mathcal{M},\psi)}$ is also controlled by ϵ (namely $d_{match}(\ell^*_{(\mathcal{M},\varphi)},\ell^*_{(\mathcal{M},\psi)})\leq\epsilon$).

Theorem 3.7 *Let $(\mathcal{M},\varphi)$ be a size pair. For every real number $\epsilon\geq 0$ and for every measuring function $\psi:\mathcal{M}\to\mathbb{R}$ such that $\max_{P\in\mathcal{M}}|\varphi(P)-\psi(P)|\leq\epsilon$, the matching distance between $\ell^*_{(\mathcal{M},\varphi)}$ and $\ell^*_{(\mathcal{M},\psi)}$ is smaller than or equal to ϵ.*

As a consequence of Thm. 3.7, the following result can be deduced, providing a lower bound for the natural pseudo-distance between size pairs (cf. (d'Amico *et al.* 2003)).

Theorem 3.8 *Let $(\mathcal{M},\varphi)$ and $(\mathcal{N},\psi)$ be two size pairs, with $\mathcal{M}$ and $\mathcal{N}$ homeomorphic. Then*

$$\inf_{f:\mathcal{M}\to\mathcal{N}}\max_{P\in\mathcal{M}}|\varphi(P)-\psi(f(P))|\geq d_{match}(\ell^*_{(\mathcal{M},\varphi)},\ell^*_{(\mathcal{N},\psi)}),$$

where f varies among all possible homeomorphisms from $\mathcal{M}$ to $\mathcal{N}$.

3.3 Comparison with an earlier result

Theorem 3.8 is not the only link between reduced size functions and the natural pseudo-distance. Indeed, the following result can been proved (cf. (Donatini and Frosini 2004*a*)):

Theorem 3.9 *If there exist (x,y) and (ξ,η) in Δ^+ such that $\ell^*_{(\mathcal{M},\varphi)}(x,y)>\ell^*_{(\mathcal{N},\psi)}(\xi,\eta)$ then*

$$\inf_h\max_{P\in\mathcal{M}}|\varphi(P)-\psi(h(P))|\geq\min\{\xi-x,y-\eta\},$$

where h varies among all possible homeomorphisms from $\mathcal{M}$ to $\mathcal{N}$.

However, the estimate for the natural pseudo-distance stated in Thm. 3.8 improves this earlier result. The following theorem can be proved (cf. (d'Amico *et al.* 2003), (d'Amico *et al.* 2005)):

Theorem 3.10 *Assume that*

$$A=\left\{((x,y),(\xi,\eta))\in\Delta^+\times\Delta^+:\xi\geq x,\eta\leq y,\ell^*_{(\mathcal{M},\varphi)}(x,y)>\ell^*_{(\mathcal{N},\psi)}(\xi,\eta)\right\}$$

is non-empty, and let

$$s=\sup_{((x,y),(\xi,\eta))\in A}\{\min\{\xi-x,y-\eta\}\}$$

(in other words, s is the best lower bound we can get for the natural pseudo-distance $\inf_h\max_{P\in\mathcal{M}}|\varphi(P)-\psi(h(P))|$ by applying Thm. 3.9). Then

$$d_{match}(\ell^*_{(\mathcal{M},\varphi)},\ell^*_{(\mathcal{N},\psi)})\geq s.$$

4 Experiments

We have used the matching distance between reduced size functions to perform some queries in a dataset of 59 images (shown in Fig. 5, by courtesy of Siddiqi and Pelillo, cf. (Pelillo *et al.* 1999)), representing the silhouettes of various objects.

For each image in the dataset the outer boundary is considered. Five reduced size functions have been computed, corresponding to five different measuring functions, that are the distances of a point from the five points of coordinates $(0,0)$, $(0,1)$, $(1,0)$, $(-1,0)$ and $(0,-1)$, respectively. Coordinates are taken in a reference frame with origin in the barycenter B of the curve and axes with fixed direction and unit length equal to the average distance of the curve from B.

Tests have been performed by comparing the family of the five reduced size functions of the query image with those of each image in the dataset, using the matching distance. Then, the final ranking is obtained by summing up these distances.

We report some examples in Fig. 6. The query images are represented in the first column. The first six results are displayed in the next columns, ordered by ranking, and their distance from the query image is displayed.

We point out that the chosen measuring functions are invariant by translations and scale, but not by rotations. For this reason, objects obtained from each other by means of rotations may not give rise to similar reduced size functions, and their matching distance may be great.

The time needed for computing each matching distance in Fig. 6 is below 50 milliseconds on an ordinary PC at the time we are writing. In this case the maximum number of cornerpoints that we consider for each reduced size function is 15.

5 Conclusions

In this survey we have described the approach to comparison of shapes by means of the matching distance between reduced size functions. One of the main properties of this distance, as shown in (d'Amico *et al.* 2003) and (d'Amico *et al.* 2005), is to be robust with respect to small changes of the measuring functions. This stability also allows to obtain a lower bound for the natural pseudo-distance between size pairs, yielding an estimate for their dissimilarity measure. Experiments have been carried out, illustrating the capability of the matching distance to compare shapes.

This approach to shape comparison profits from its modularity. In fact, the reduced size functions inherit the invariance of the measuring functions, and hence changing the invariance group simply means changing the measuring functions, without any other change in the mathematical model.

Moreover, matching distances can be computed in polynomial time when the number of cornerpoints taken into account is bounded.

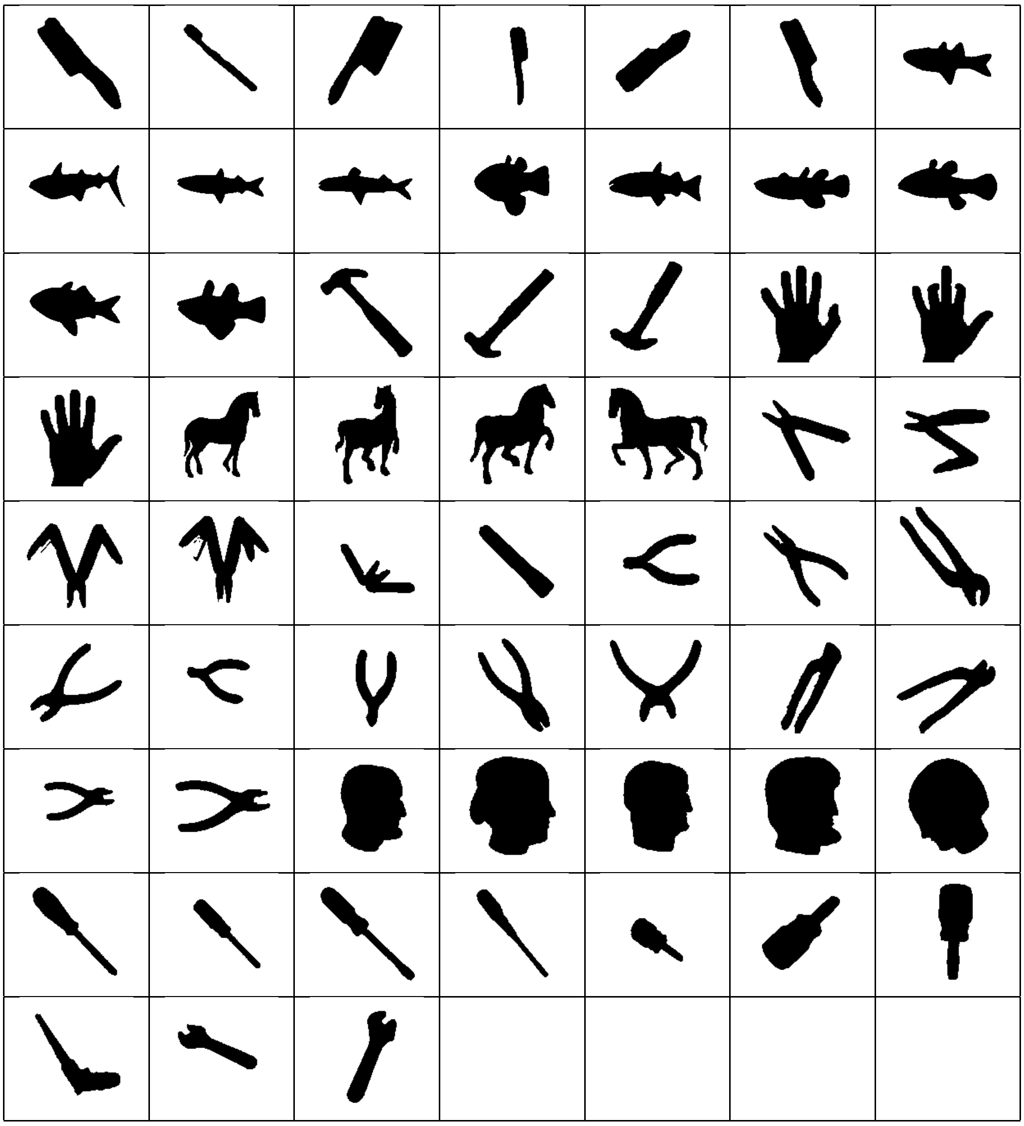

Figure 5: Our dataset.

Query shape	Top 6 matches					
	1	2	3	4	5	6
	0.000	0.204	0.312	0.359	0.368	0.378
	0.000	0.245	0.287	0.291	0.369	0.377
	0.000	0.451	0.617	0.689	0.506	0.515
	0.000	0.307	0.375	0.406	0.472	0.519
	0.000	0.412	0.424	0.448	0.471	0.483
	0.000	0.384	0.450	0.543	0.577	0.589
	0.000	0.406	0.417	0.450	0.550	0.565
	0.000	0.384	0.409	0.469	0.516	0.580
	0.000	0.294	0.616	0.681	0.718	0.729
	0.000	0.598	0.616	0.619	0.623	0.624
	0.000	0.294	0.623	0.673	0.708	0.720
	0.000	0.521	0.576	0.643	0.679	0.697
	0.000	0.505	0.617	0.699	0.773	0.779
	0.000	0.534	0.576	0.617	0.659	0.680
	0.000	0.374	0.626	0.633	0.777	0.829
	0.000	0.520	0.682	0.704	0.719	0.743
	0.000	0.204	0.308	0.323	0.350	0.366
	0.000	0.405	0.766	0.873	0.982	0.984
	0.000	0.374	0.674	0.700	0.767	0.769
	0.000	0.470	0.704	0.744	0.753	0.766
	0.000	0.134	0.766	0.863	0.882	0.887
	0.000	0.188	0.235	0.298	0.313	0.600
	0.000	0.225	0.251	0.275	0.313	0.654
	0.000	0.121	0.132	0.239	0.245	0.366

Figure 6: The result of some queries in our dataset.

The properties and examples shown in this paper provide justification for the use of the proposed theoretical framework for shape matching and comparison.

6 Acknowledgements

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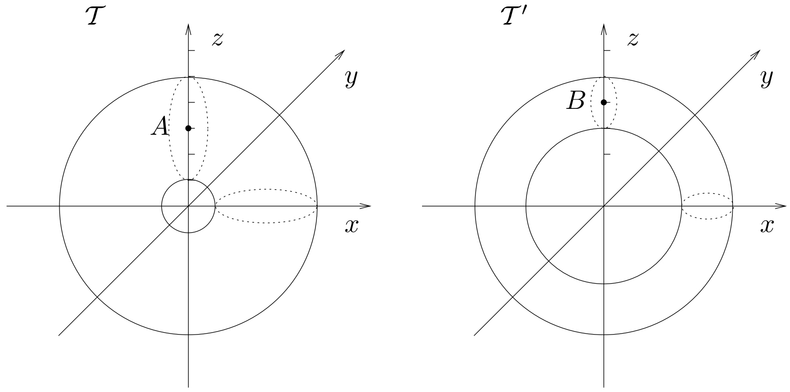


Figure 1

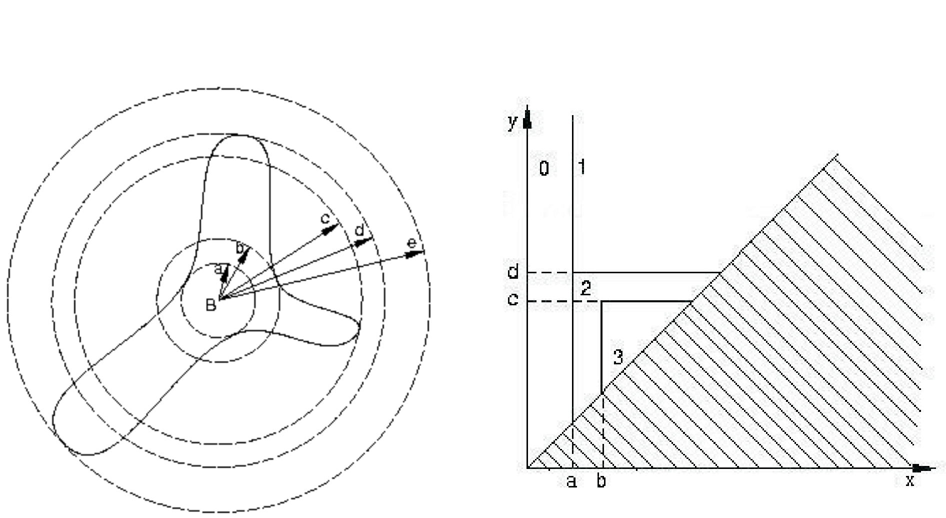


Figure 2

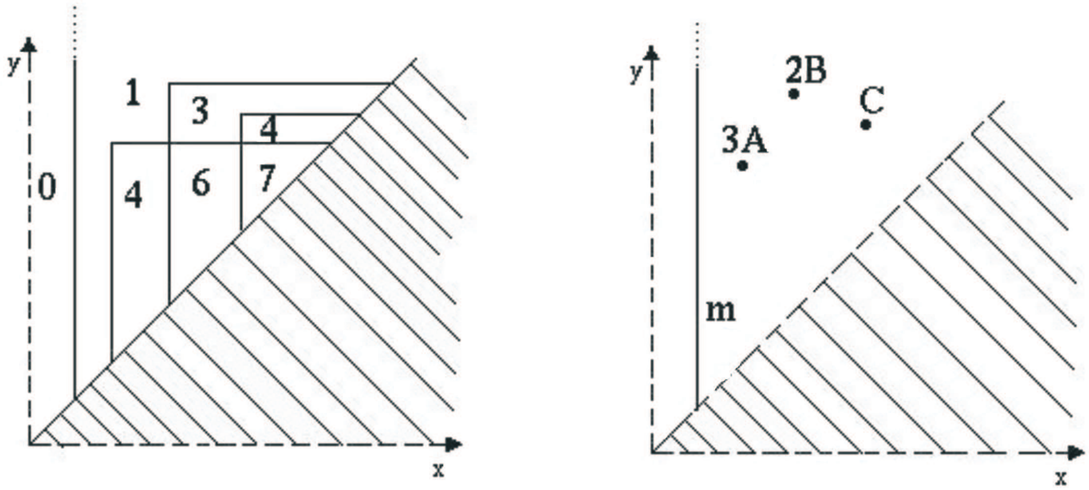
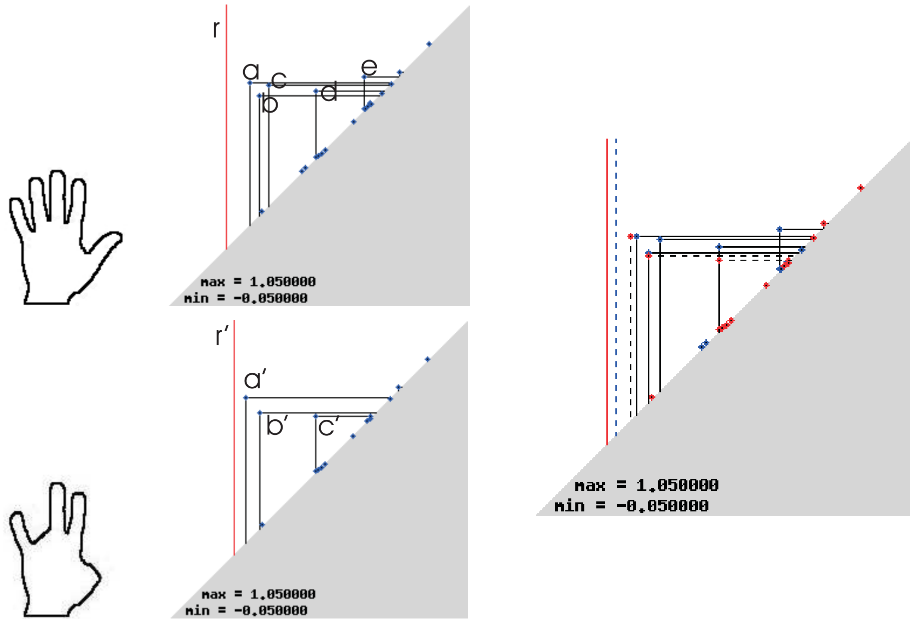


Figure 3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min = -0.050000

Figure 4

Figure 5

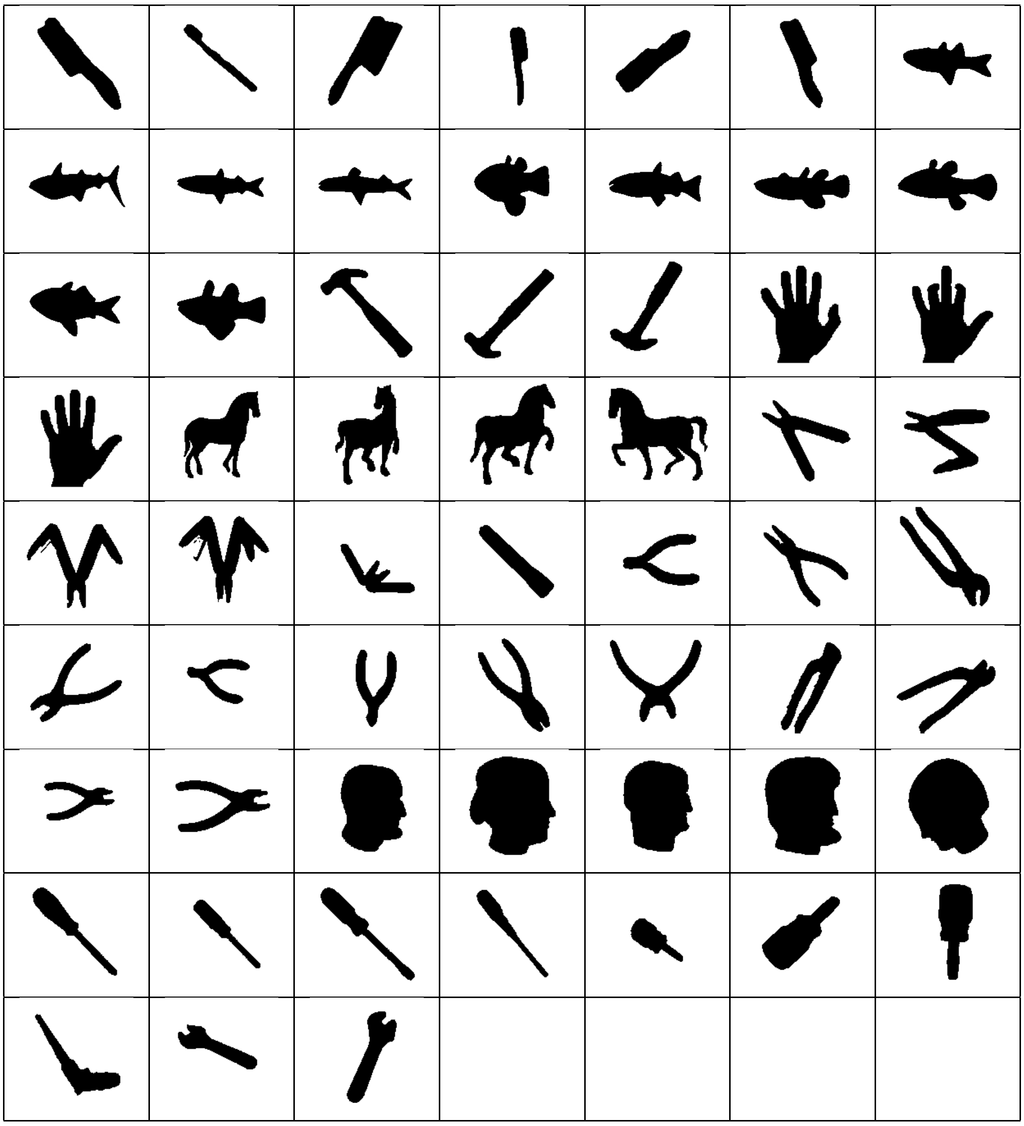

Figure 6

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List of captions

Figure 1: The size pairs $(\mathcal{T}, \varphi)$ and $(\mathcal{T}', \varphi')$, with φ and φ' equal to the height function, have natural pseudo-distance equal to 2.

Figure 2: Left: A size pair $(\mathcal{M}, \varphi)$, where $\mathcal{M}$ is the curve depicted by a solid line, and φ is the distance from the barycenter of the curve. Right: the corresponding reduced size function.

Figure 3: Cornerpoints of a reduced size function: A, B and C are the only proper cornerpoints, and have multiplicity equal to 3 (A), 2 (B), and 1 (C). The line m is the only cornerpoint at infinity.

Figure 4: Left: Two curves. Center: Their reduced size functions with respect to the measuring function distance from the center of the image. Right: The superimposition of the two reduced size functions.

Figure 5: Our dataset.

Figure 6: The result of some queries in our dataset.