Shape Matching: A Metric Geometry Approach

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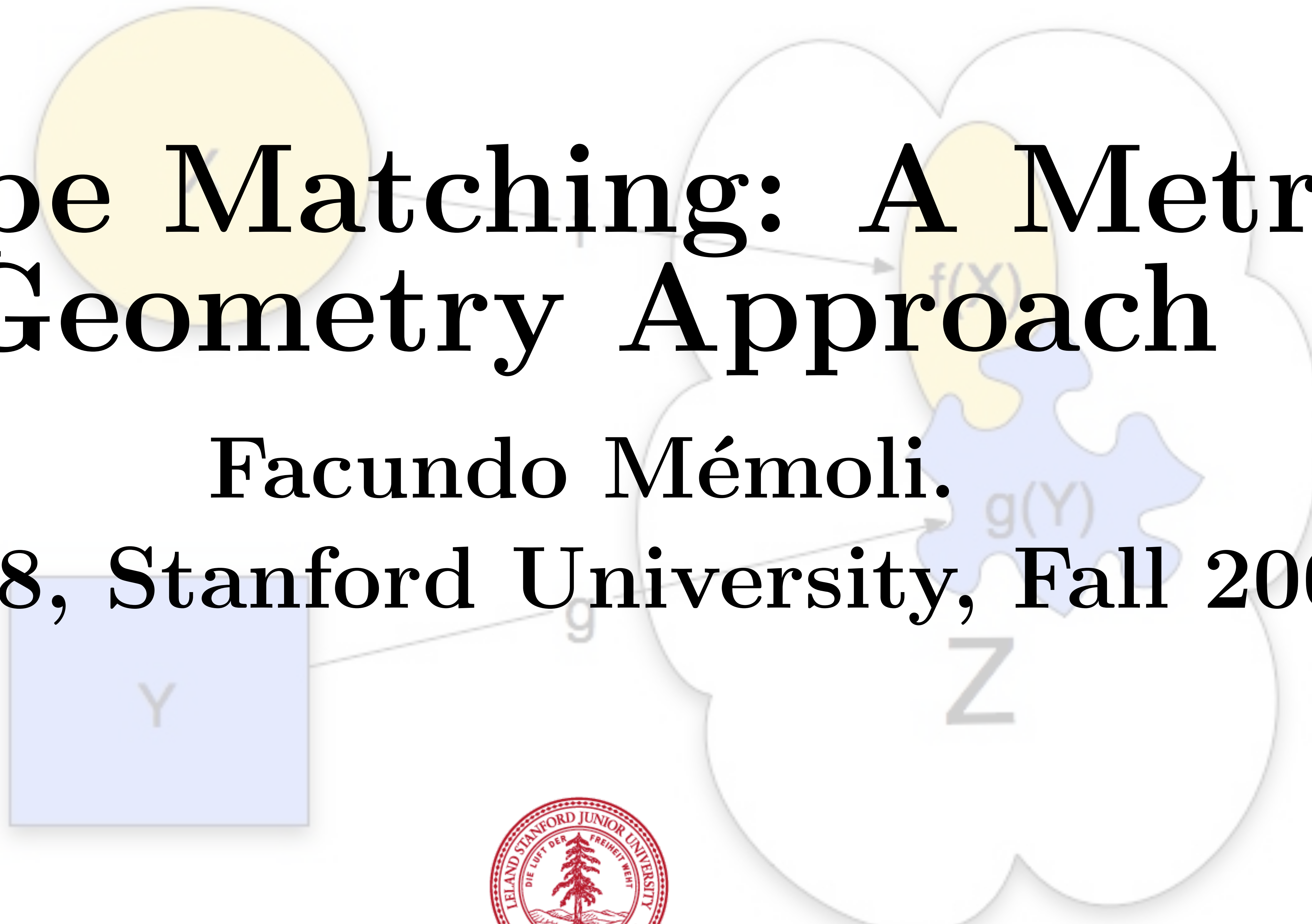
CS 468, Stanford University, Fall 2008.

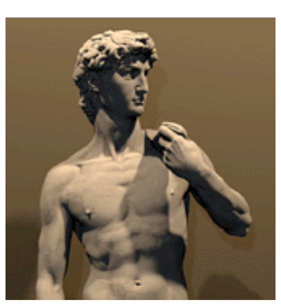
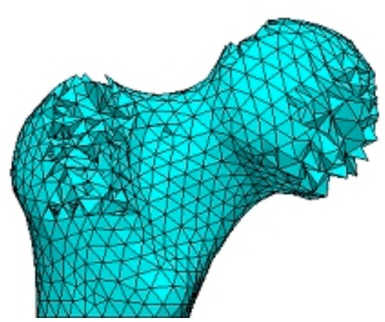
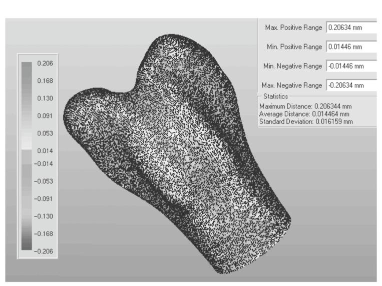
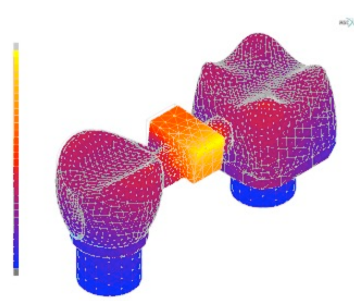
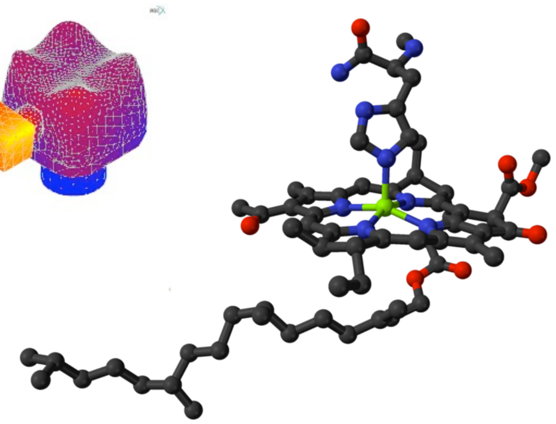
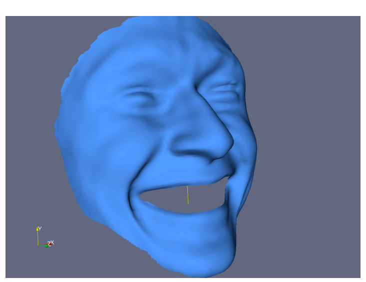


The Problem of Shape/Object Matching

- databases of *objects*
- objects can be many things:
  - proteins
  - molecules
  - 2D objects (imaging)
  - 3D shapes: as obtained via a 3D scanner
  - 3D shapes: modeled with CAD software
  - 3D shapes: coming from design of bone protheses
  - text documents
  - more complicated structures present
    in datasets (things you can't visualize)

3D objects: examples

- cultural heritage (Michelangelo project: `http://www-graphics.stanford.edu/projects/mich/`)
- search of parts in a factory of, say, cars
- face recognition: the face of an individual is a 3D shape...
- proteins: the *shape* of a protein reflects its function.. protein data bank: `http://www.rcsb.org`

Typical situation: classification

- assume you have database $\mathcal{D}$ of objects.
- assume $\mathcal{D}$ is composed by several objects, and that each of these objects belongs to one of n classes $C_1, \ldots, C_n$.
- imagine you are given a new object o, not in your database, and you are asked to determine whether o belongs to one of the classes. If yes, you also need to point to the class.
- One simple procedure is to say that you will assign object o the class of the *closest* object in $\mathcal{D}$:

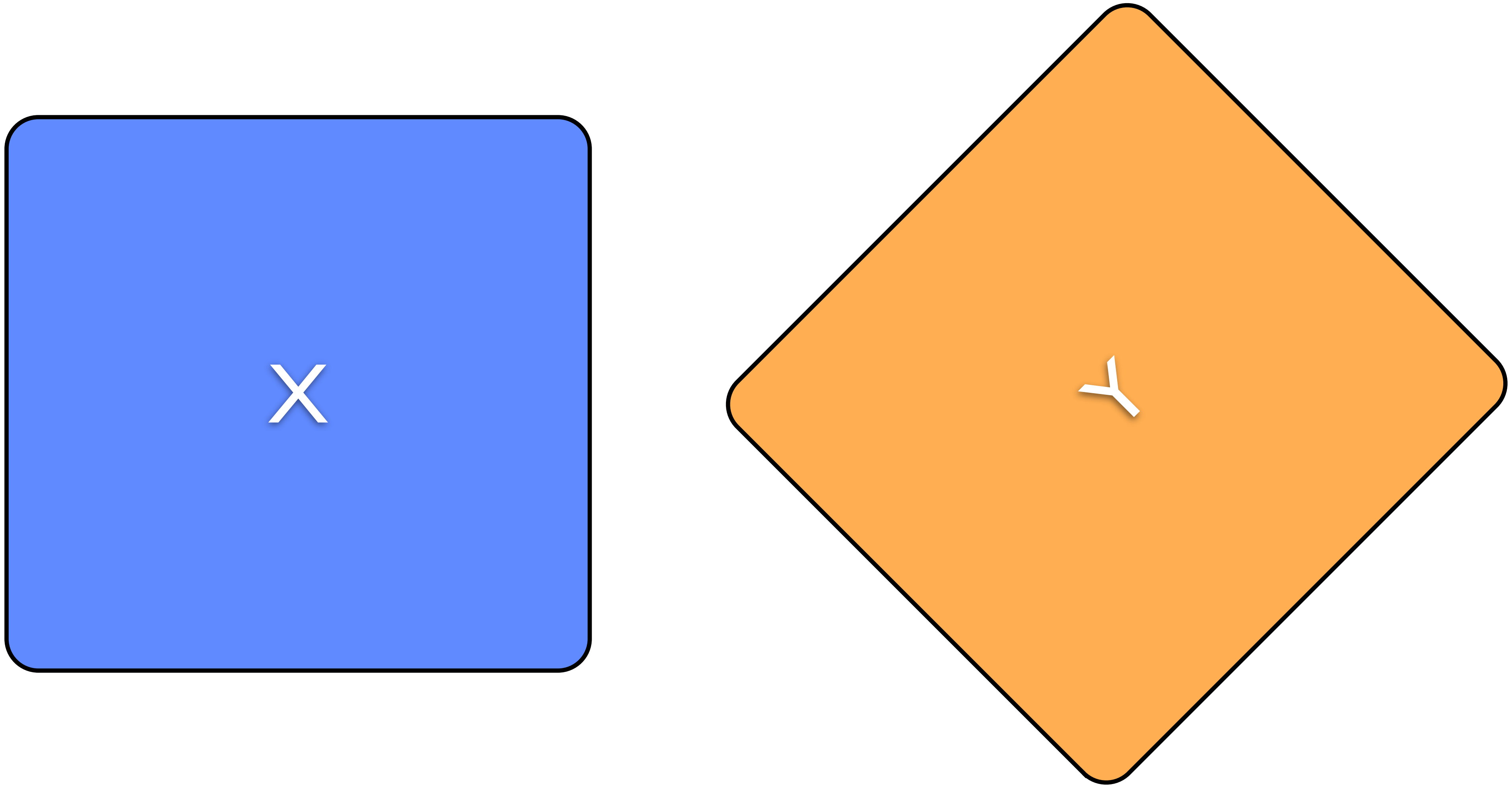
$$\text{class}(o) = \text{class}(z)$$

$$\text{where } z \in \mathcal{D} \text{ minimizes } \textbf{dist}(o, z)$$

- in order to do this, one first needs to define a notion **dist** of *distance* or *dis-similarity between objects.*

Another important point: invariances

Are these two objects the same?



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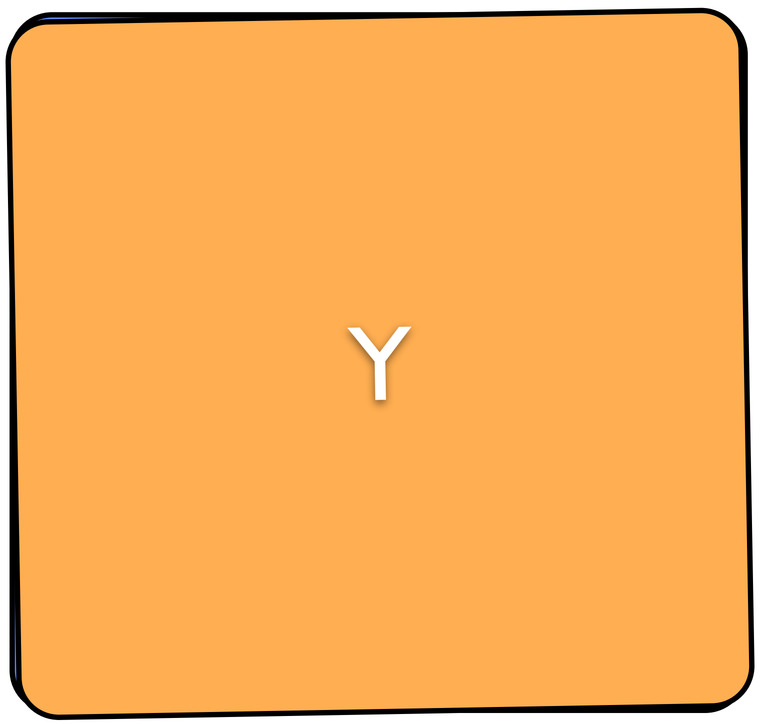
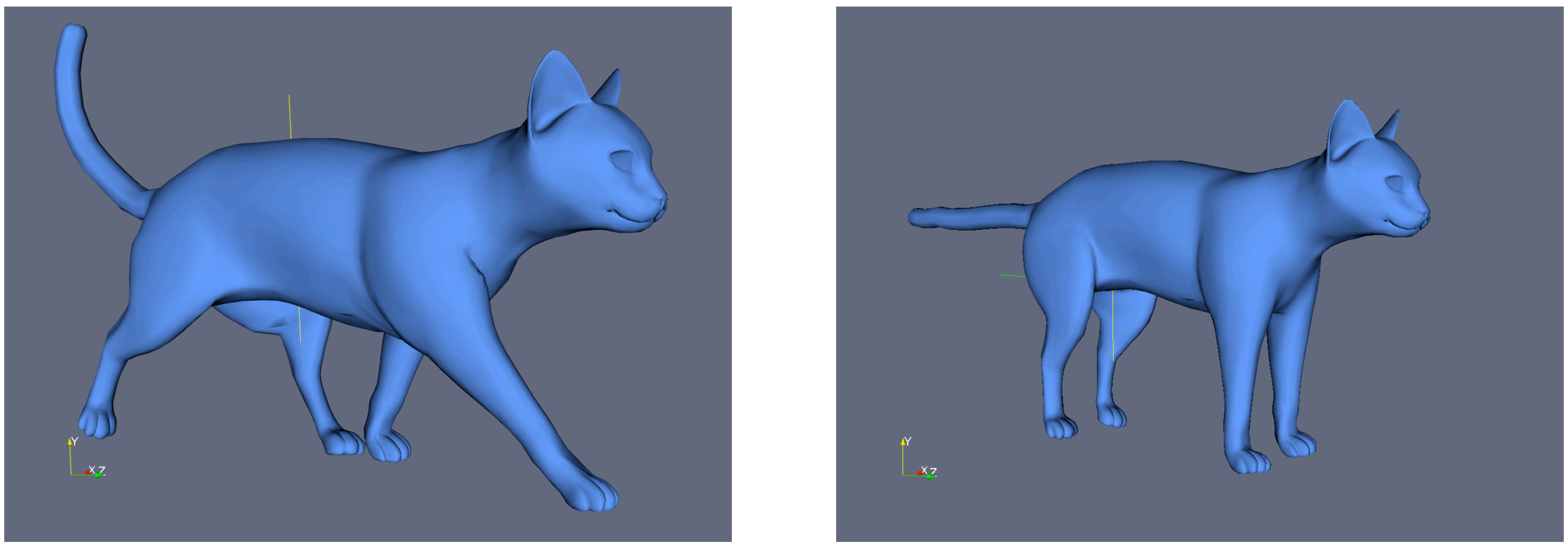


this is called invariance to *rigid transformations*

Another important points: invariances

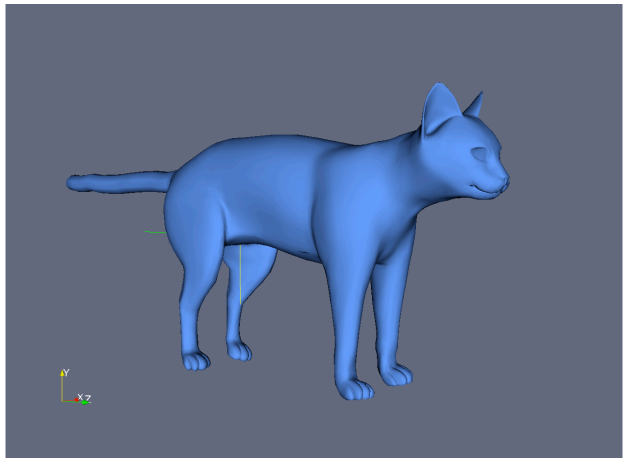
what about these two?



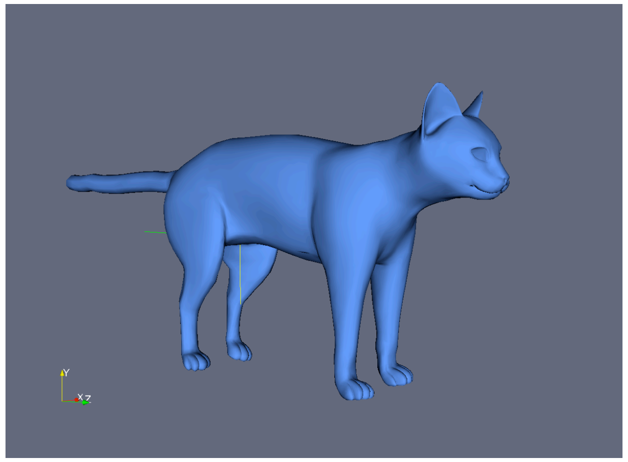
roughly speaking, this corresponds to invariance to *bending transformations*..

invariances...

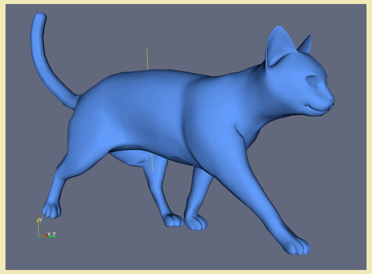
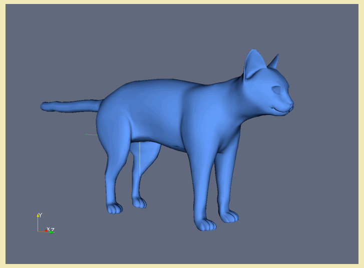
The measure of dis-similarity **dist** must capture the type of invariance you want to encode in your classification system.



$=$



?

dist( , ) $= 0$?

What we proposed in the course:

1. Decide what *invariances* you wish to incorporate. It is OK if you don't have invariances (Hausdorff + Wasserstein)

2. Represent shapes as metric spaces (or mm-spaces):

   - Identify what metric is preserved by the notion of invariance you chose to consider.
   - Endow shapes with that metric
   - Choose weights that are meaningful for your application (if you don't have any reason to choose: then set them to be equal)

3. Define a *metric* on your class of objects.

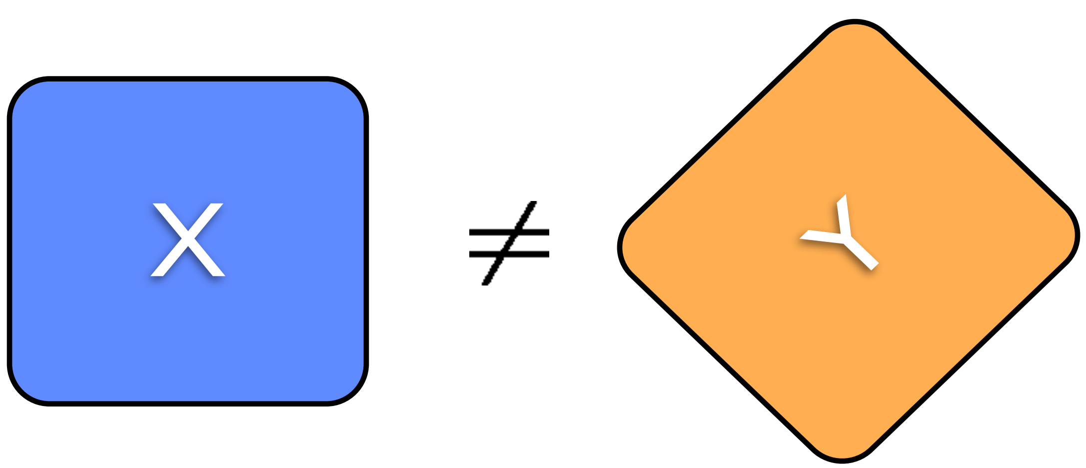
We studied the case of <u>no invariances</u> first.

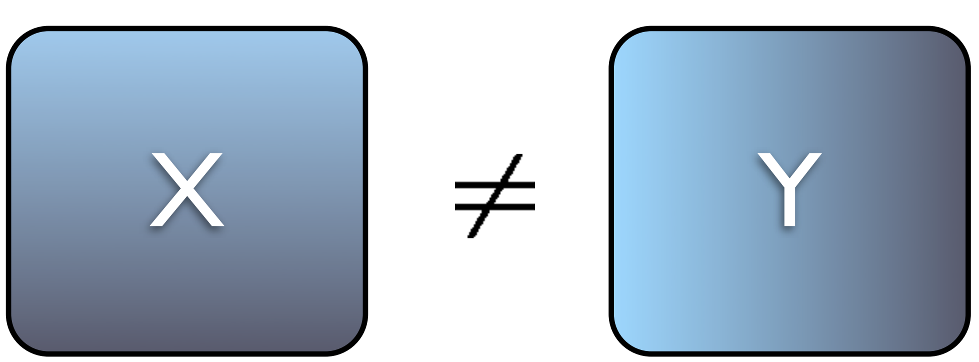
No invariances..

- You start out with a compact metric space (Z, d) (called *ambient space*, typically $Z = \mathbb{R}^k$).
- We saw two constructions:

$$\left(\mathcal{C}(Z), d_{\mathcal{H}}^{Z}\right) \quad \text{and} \quad \left(\mathcal{C}_w(Z), d_{\mathcal{W},p}^{Z}\right)$$

where $\mathcal{C}(Z)$ stands for all compact subsets of Z and $\mathcal{C}_w(Z)$ for all *weighted* subsets of Z: that is, pairs (A, μ_A) where μ_A is a probability measure on Z s.t. $\operatorname{supp}[\mu_A] = A$.





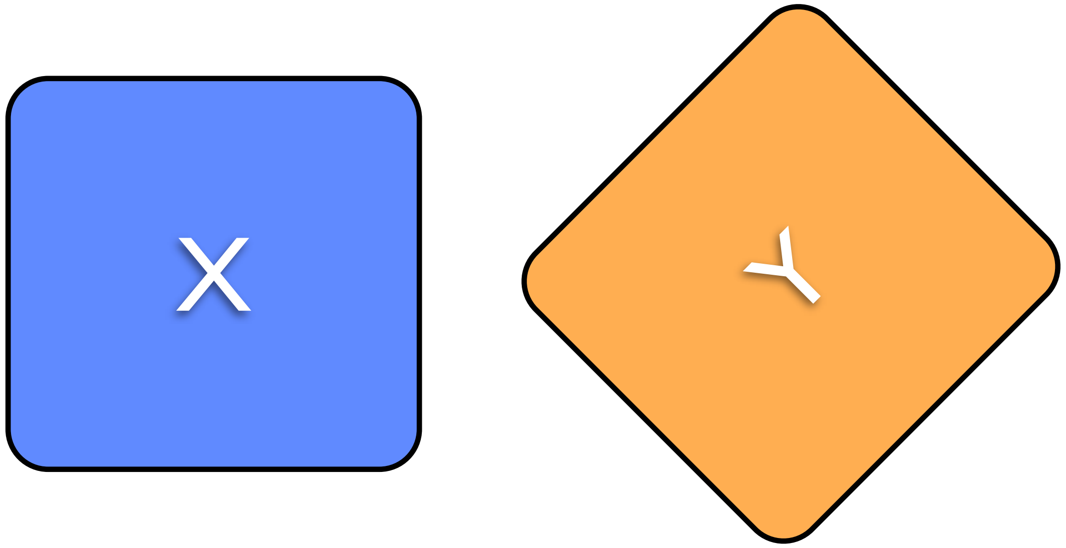
Ambient space isometries..(Extrinsic approach)

Fix a compact metric space (Z, d). Let $I(Z)$ denote the isometry group of Z (when $Z = \mathbb{R}^k$, $I(Z) = E(k)$, that is, all Euclidean isometries.)

- In this case, we considered either objects in $\mathcal{C}(Z)$ or in $\mathcal{C}_w(Z)$. Let $\mathcal{O}(Z)$ denote your choice.
- Let **dist** be the corresponding metric (Hausdorff or Wasserstein).
- Then, we constructed distances between objects in $\mathcal{O}(Z)$ that were blind to isometries:

$$\mathbf{dist}^{iso}(X, Y) := \inf_{T \in I(Z)} \mathbf{dist}(X, T(Y))$$

for all $X, Y \in \mathcal{O}(Z)$.



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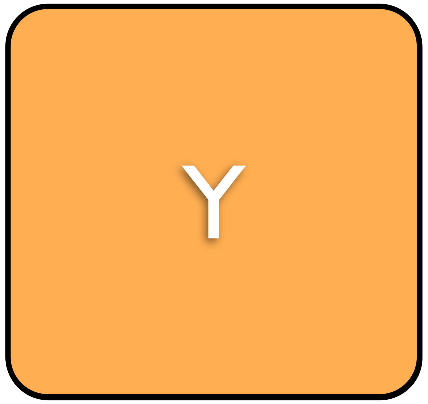

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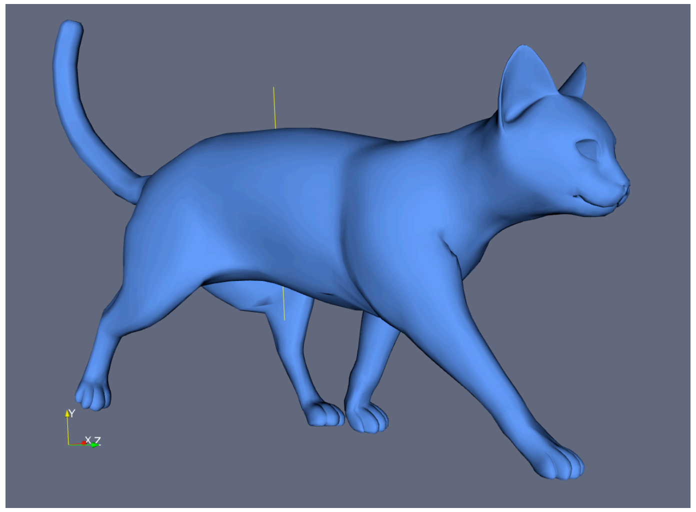
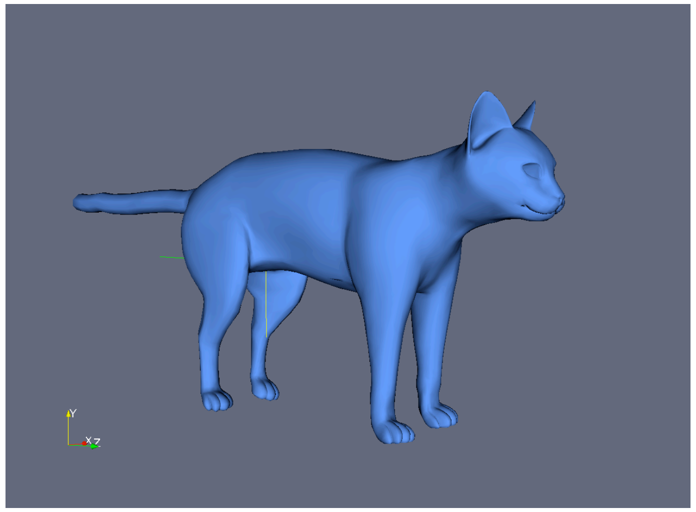
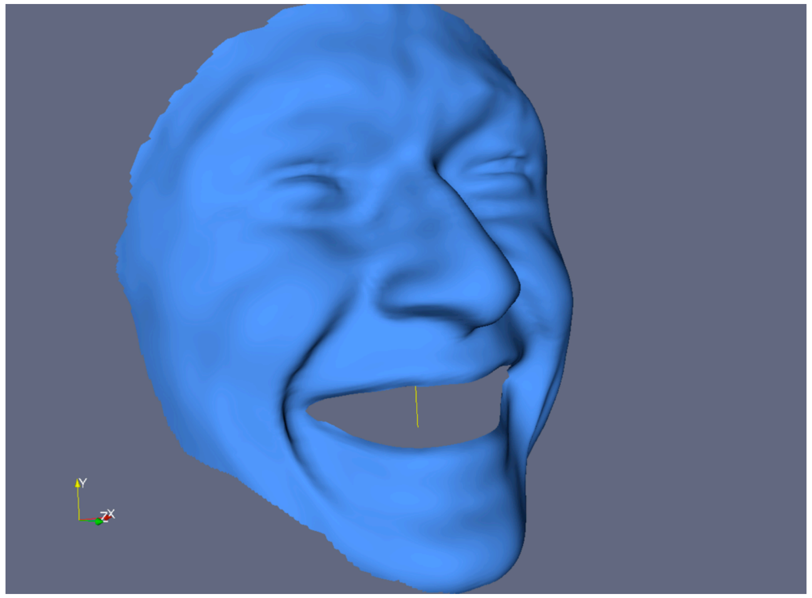
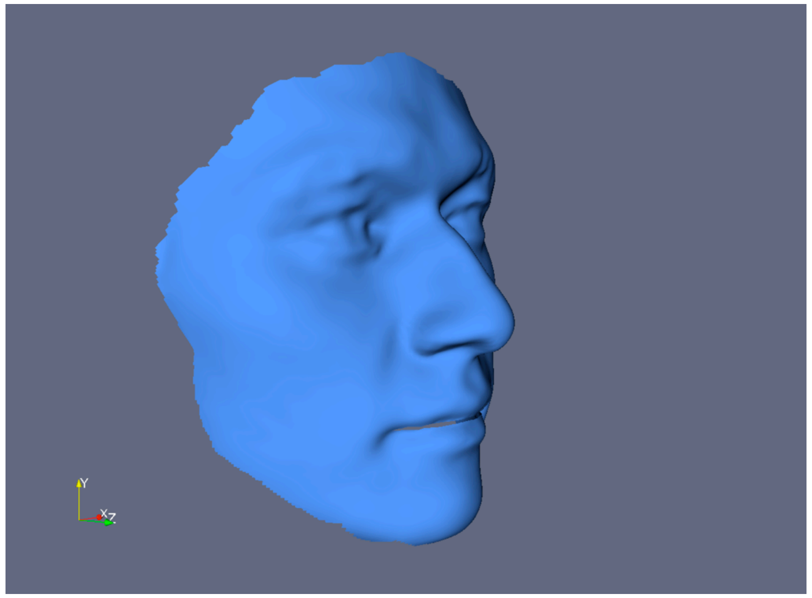
- In this case, we co… …Z). Let $\mathcal{O}(Z)$ denote your ch…
- Let **dist** be…
- Then, we… …d to isometries:

  for all X…

but this was not general enough...
remember BENDS

The intrinsic approach...

- Regard shapes as metric spaces in themselves: no reference to any ambient space.
- That is, objects/shapes now are metric spaces or mm-spaces (X, d_X) or (X, d_X, μ_X).
- Let $\mathcal{M}$ denote collection of all metric spaces and $\mathcal{M}_w$ the collections of all mm-spaces.
- Endow $\mathcal{M}$ and/or $\mathcal{M}_w$ with a metric.
- We saw the following constructions:
  - On $\mathcal{M}$ we put the Gromov-Hausdorff distance $d_{\mathcal{GH}}(,)$.
  - On $\mathcal{M}_w$ we put two metrics, $\mathfrak{S}_p$ and $\mathfrak{D}_p$. These metrics could be called Gromov-Wasserstein metrics.

$\mathcal{C}(Z)$	$\mathcal{C}_w(Z)$
$\mathcal{M}$	$\mathcal{M}_w$
$d_{\mathcal{H}}$	$d_{\mathcal{W},p}$
$d_{\mathcal{GH}}$	$\mathfrak{D}_p \quad \mathfrak{S}_p$

$\mathcal{C}(Z)$	$\mathcal{C}_w(Z)$
$\mathcal{M}$	$\mathcal{M}_w$
$d_{\mathcal{H}}$	$d_{\mathcal{W},p}$
$d_{\mathcal{GH}}$	$\mathfrak{D}_p$ $\mathfrak{S}_p$

Extrinsic approach

$\mathcal{C}(Z)$	$\mathcal{C}_w(Z)$
$\mathcal{M}$	$\mathcal{M}_w$
$d_{\mathcal{H}}$	$d_{\mathcal{W},p}$
$d_{\mathcal{GH}}$	$\mathfrak{D}_p$ $\mathfrak{S}_p$

Extrinsic approach

Intrinsic approach

What is the relationship between the Intrinsic and Extrinsic Approaches?

- Answer known for $Z = \mathbb{R}^k$, [**M08-euclidean**].
- In class we saw the case of $d_{\mathcal{GH}}$ vs. $d_{\mathcal{H}}^{iso}$: for all $X, Y \in \mathbb{R}^k$,

  $$d_{\mathcal{GH}}((X, \|\cdot\|), (Y, \|\cdot\|)) \leqslant \inf_{T \in E(k)} d_{\mathcal{H}}(X, T(Y)) \leqslant C_k \left(d_{\mathcal{GH}}((X, \|\cdot\|), (Y, \|\cdot\|))\right)^{1/2} M^{1/2}$$

  where $M = \max(\mathbf{diam}\,(X)\,, \mathbf{diam}\,(Y))$.
- There is a similar claim valid for $\mathfrak{S}_p$ vs. $d_{\mathcal{W},p}^{iso}$.

Main points

- Define notion of distance on shapes. Get sampling consistency + stability for free.
- GH distance leads to hard combinatorial optimization problems.
- Relaxations of these do not appear to be correct.
- Gromov-Wasserstein distances are better. Both $\mathfrak{D}_p$ and $\mathfrak{S}_p$ yield quadratic optimization probs. with linear constraints.
- Sturm's $\mathfrak{S}_p$ requires large nbr. of constraints, then we argue for $\mathfrak{D}_p$.
- Many lower bounds for $\mathfrak{D}_p$ are possible. These employ invariants previously used in the literature:
  - Shape Distributions
  - Eccentricities (Hamza-Krim)
  - Shape contexts

Main Technical concepts

- Metric spaces, mm-spaces, isometries, approximate isometries, probability measures.
- Correspondences, measure couplings, metric couplings.
- Hausdorff distance. Wasserstein distance. Mass transportation.
- Gromov-Hausdorff distance. Gromov-Wasserstein distances.
- Invariants of mm-spaces

correspondences and the Hausdorff distance

Definition [Correspondences]

For sets A and B, a subset $R \subset A \times B$ is a *correspondence* (between A and B) if and and only if

- $\forall\ a \in A$, there exists $b \in B$ s.t. $(a, b) \in R$
- $\forall\ b \in B$, there exists $a \in A$ s.t. $(a, b) \in R$

Let $\mathcal{R}(A, B)$ denote the set of all possible correspondences between sets A and B. Note that in the case $n_A = n_B$, correspondences are larger than bijections.

correspondences

Note that when A and B are finite, $R \in \mathcal{R}(A,B)$ can be represented by a matrix $((r_{a,b})) \in \{0,1\}^{n_A \times n_B}$ s.t.

$$\sum_{a \in A} r_{ab} \geq 1 \;\; \forall b \in B$$

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Proposition
Let (X, d) be a compact metric space and $A, B \subset X$ be compact. Then

$$d_{\mathcal{H}}(A, B) = \inf_{R \in \mathcal{R}(A,B)} \|d\|_{L^\infty(R)}$$

correspondences and measure couplings

Let (A, μ_A) and (B, μ_B) be compact subsets of the compact metric space (X, d) and μ_A and μ_B be **probability measures** supported in A and B respectively.

Definition [Measure coupling] Is a probability measure μ on $A \times B$ s.t. (in the finite case this means $((\mu_{a,b})) \in [0,1]^{n_A \times n_B}$)

- $\sum_{a \in A} \mu_{ab} = \mu_B(b) \;\; \forall b \in B$
- $\sum_{b \in B} \mu_{ab} = \mu_A(a) \;\; \forall a \in A$

Let $\mathcal{M}(\mu_A, \mu_B)$ be the set of all couplings of μ_A and μ_B.
Notice that in the finite case, $((\mu_{a,b}))$ must satisfy $n_A + n_B$ *linear* constraints.

correspondences and measure couplings

Proposition [($\mu \leftrightarrow R$)]

- Given (A, μ_A) and (B, μ_B), and $\mu \in \mathcal{M}(\mu_A, \mu_B)$, then

$$R(\mu) := \mathrm{supp}(\mu) \in \mathcal{R}(A, B).$$

- König's Lemma. [gives conditions for $R \to \mu$]

Wasserstein distance

$$d_{\mathcal{H}}(A,B) = \inf_{R \in \mathcal{R}(A,B)} \|d\|_{L^\infty(R)}$$

$$\Downarrow \; (R \leftrightarrow \mu)$$

$$d_{\mathcal{W},\infty}(A,B) = \inf_{\mu \in \mathcal{M}(\mu_A,\mu_B)} \|d\|_{L^\infty(R(\mu))}$$

$$\Downarrow \; (L^\infty \leftrightarrow L^p)$$

$$d_{\mathcal{W},p}(A,B) = \inf_{\mu \in \mathcal{M}(\mu_A,\mu_B)} \|d\|_{L^p(A \times B,\mu)}$$

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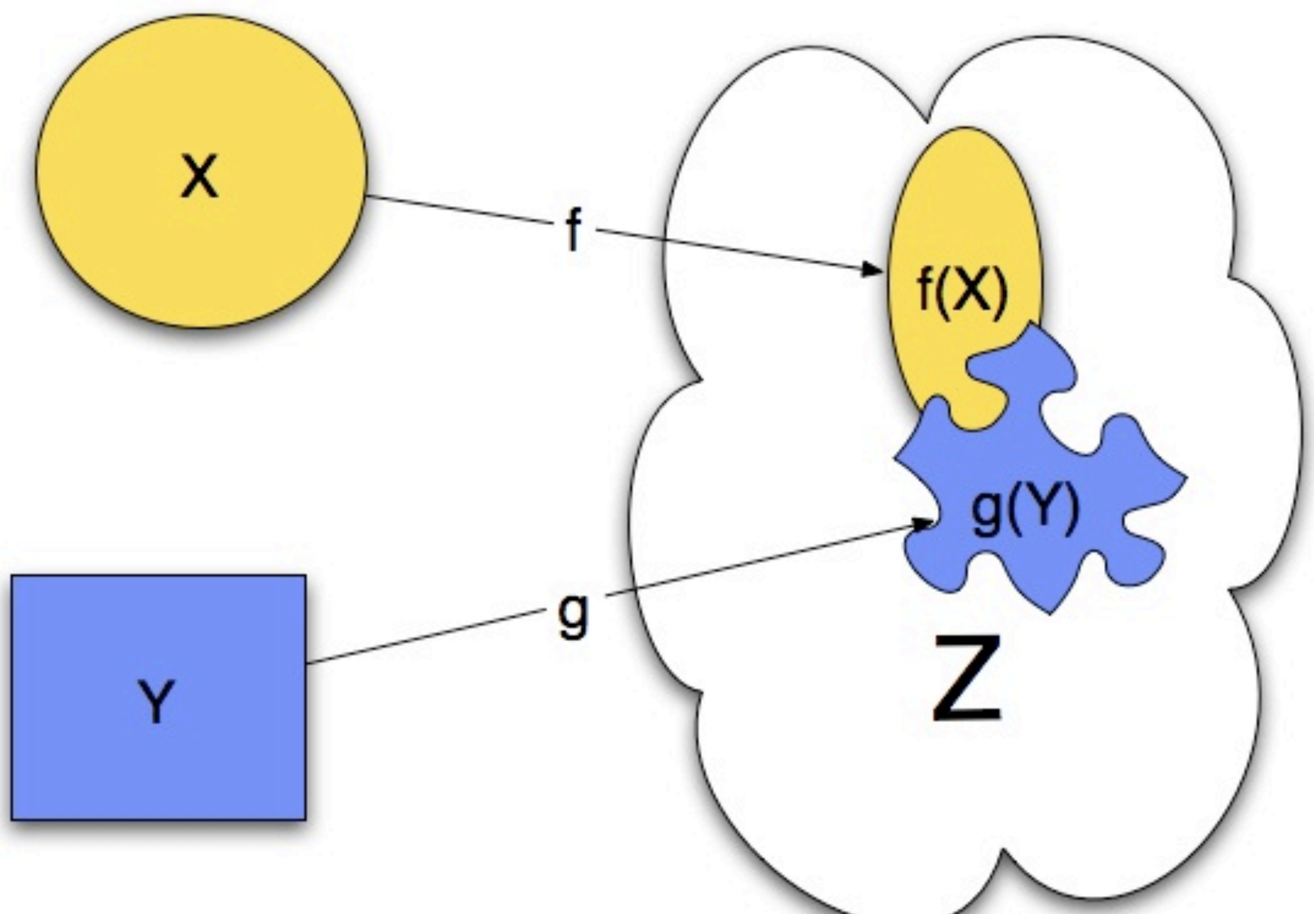
GH distance

GH: definition

$$d_{\mathcal{GH}}(X,Y) = \inf_{Z,f,g} d_{\mathcal{H}}^{Z}(f(X), g(Y))$$



correspondences and GH distance

The GH distance between (X, d_X) and (Y, d_Y) admits the following expression:

$$\boxed{d_{\mathcal{GH}}^{(1)}(X,Y) = \inf_{d \in \mathcal{D}(d_X, d_Y)} \inf_{R \in \mathcal{R}(X,Y)} \|d\|_{L^\infty(R)}}$$

where $\mathcal{D}(d_X, d_Y)$ is a metric on $X \sqcup Y$ that reduces to d_X and d_Y on $X \times X$ and $Y \times Y$, respectively.

$$\begin{array}{cc} & \begin{array}{cc} X & \quad Y \end{array} \\ \begin{array}{c} X \\ Y \end{array} & \begin{pmatrix} d_X & \mathbf{D} \\ \mathbf{D}^T & d_Y \end{pmatrix} = d \end{array}$$

In other words: you need to **glue** X and Y in an optimal way. Note that $\mathbf{D}$ consists of $n_X \times n_Y$ positive reals that must satisfy $\sim n_X \cdot C_2^{n_Y} + n_Y \cdot C_2^{n_X}$ linear constraints.

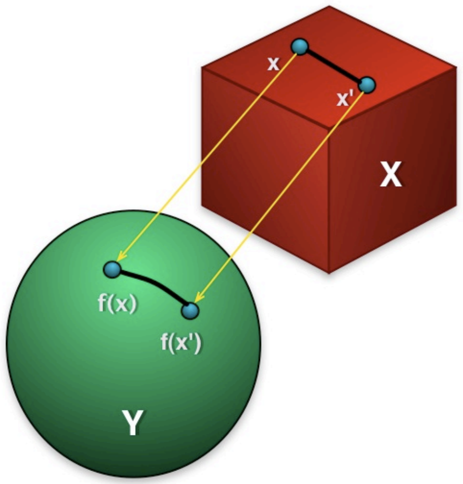
Another expression for the GH distance

For compact spaces (X, d_X) and (Y, d_Y) let

$$d_{\mathcal{GH}}^{(2)}(X,Y) = \frac{1}{2}\inf_R \max_{(x,y),(x',y')\in R} |d_X(x,x') - d_Y(y,y')|$$

We write, compactly,

$$\boxed{d_{\mathcal{GH}}^{(2)}(X,Y) = \frac{1}{2}\inf_R \|d_X - d_Y\|_{L^\infty(R\times R)}}$$



Equivalence thm:

Theorem [Kalton-Ostrovskii]
For all X, Y compact,

$$\begin{tikzcd} d_{\mathcal{GH}}^{(1)} \arrow[r, equal] \arrow[d, equal] & d_{\mathcal{GH}}^{(2)} \arrow[d, equal] \\ \inf_{d,R} \|d\|_{L^\infty(R)} & \frac{1}{2} \inf_R \|d_X - d_Y\|_{L^\infty(R\times R)} \end{tikzcd}$$

Relaxing the notion of correspondence from GH to GW

$$
\begin{tikzcd}
d^{(1)}_{\mathcal{GH}} \arrow[rr, equal] \arrow[dd] & & d^{(2)}_{\mathcal{GH}} \arrow[dd] \\
& d_{\mathcal{H}} \arrow[ul] \arrow[ur] \arrow[dd, Rightarrow] & \\
d^{(1)}_{\mathcal{GW},p} \arrow[rr, equal, "?"] & & d^{(2)}_{\mathcal{GW},p} \\
& d_{\mathcal{W},p} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Shapes as mm-spaces, [M07]

Remember:

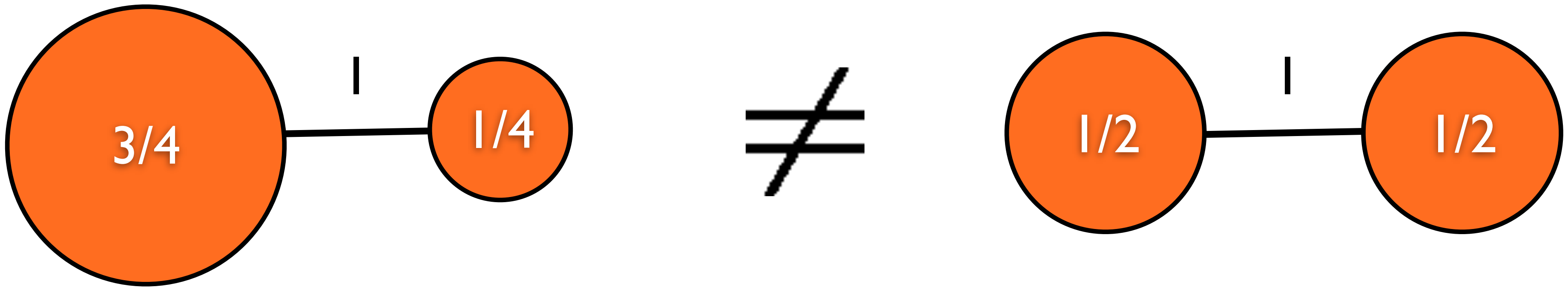
$$(X, d_X, \mu_X)$$

1. Specify representation of shapes.
2. Identify invariances that you want to mod out.
3. Describe notion of isomorphism between shapes (this is going to be the zero of your metric)
4. Come up with a *metric* between shapes (in the representation of 1.)

- Now we are talking of triples (X, d_X, μ_X) where X is a set, d_X a metric on X and μ_X a probability measure on X.
- These objects are called *measure metric spaces*, or mm-spaces for short.
- two mm-spaces X and Y are deemed *equal* or *isomorphic* whenever there exists an isometry $\Phi : X \to Y$ s.t. $\mu_Y(B) = \mu_X(\Phi^{-1}(B)$ for all (measurable) sets $B \subset Y$.

Remember

Now, one works with **mm-spaces**: triples (X, d, ν) where (X, d) is a compact metric space and ν is a Borel probability measure. Two mm-spaces are *isomorphic* iff there exists isometry $\Phi : X \to Y$ s.t. $\mu_X(\Phi^{-1}(B)) = \mu_Y(B)$ for all measurable $B \subset Y$.



The plan

$$\begin{tikzcd}
d^{(1)}_{\mathcal{GH}} \arrow[rr, equal] \arrow[dd] & & d^{(2)}_{\mathcal{GH}} \arrow[dd] \\
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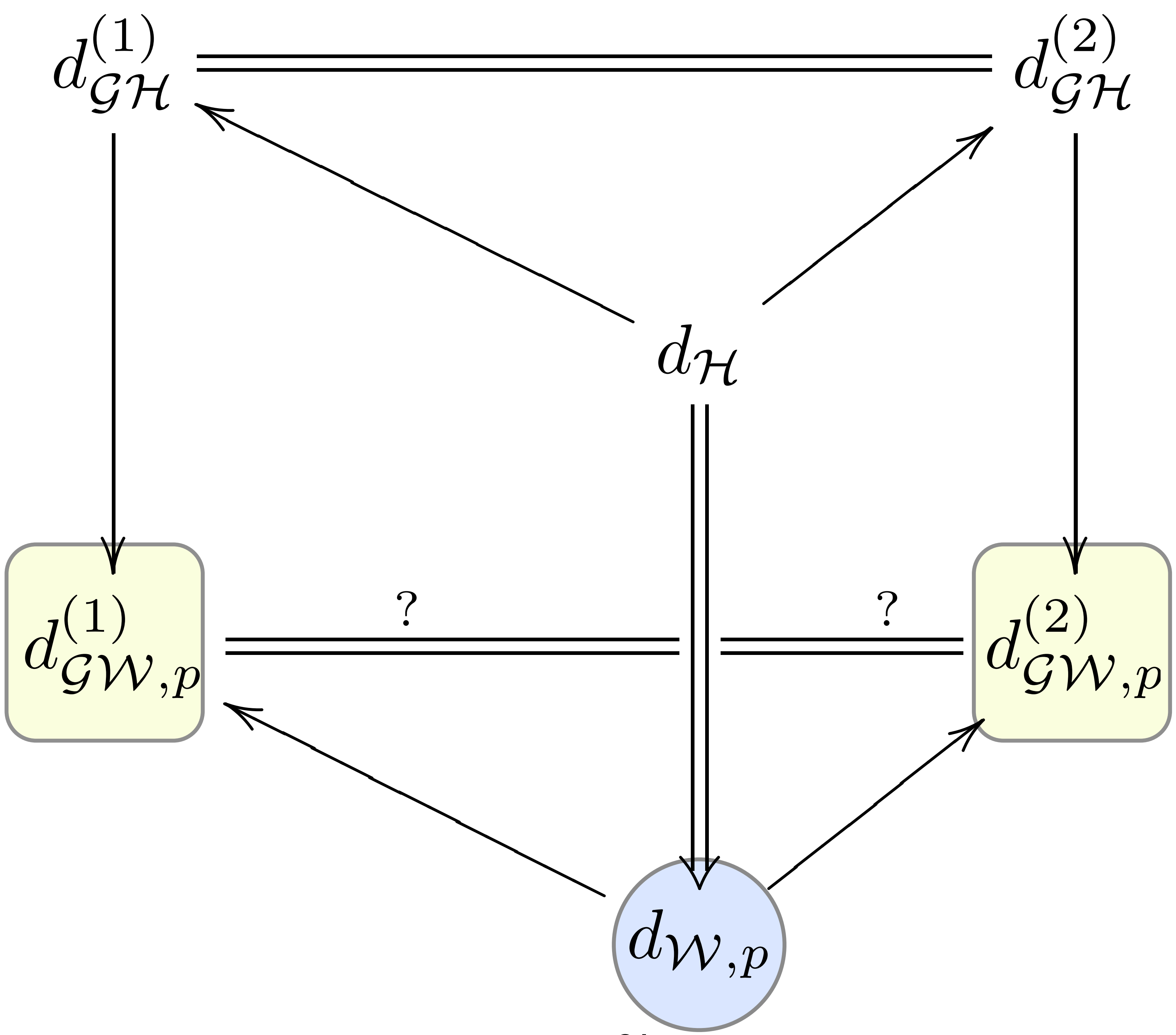
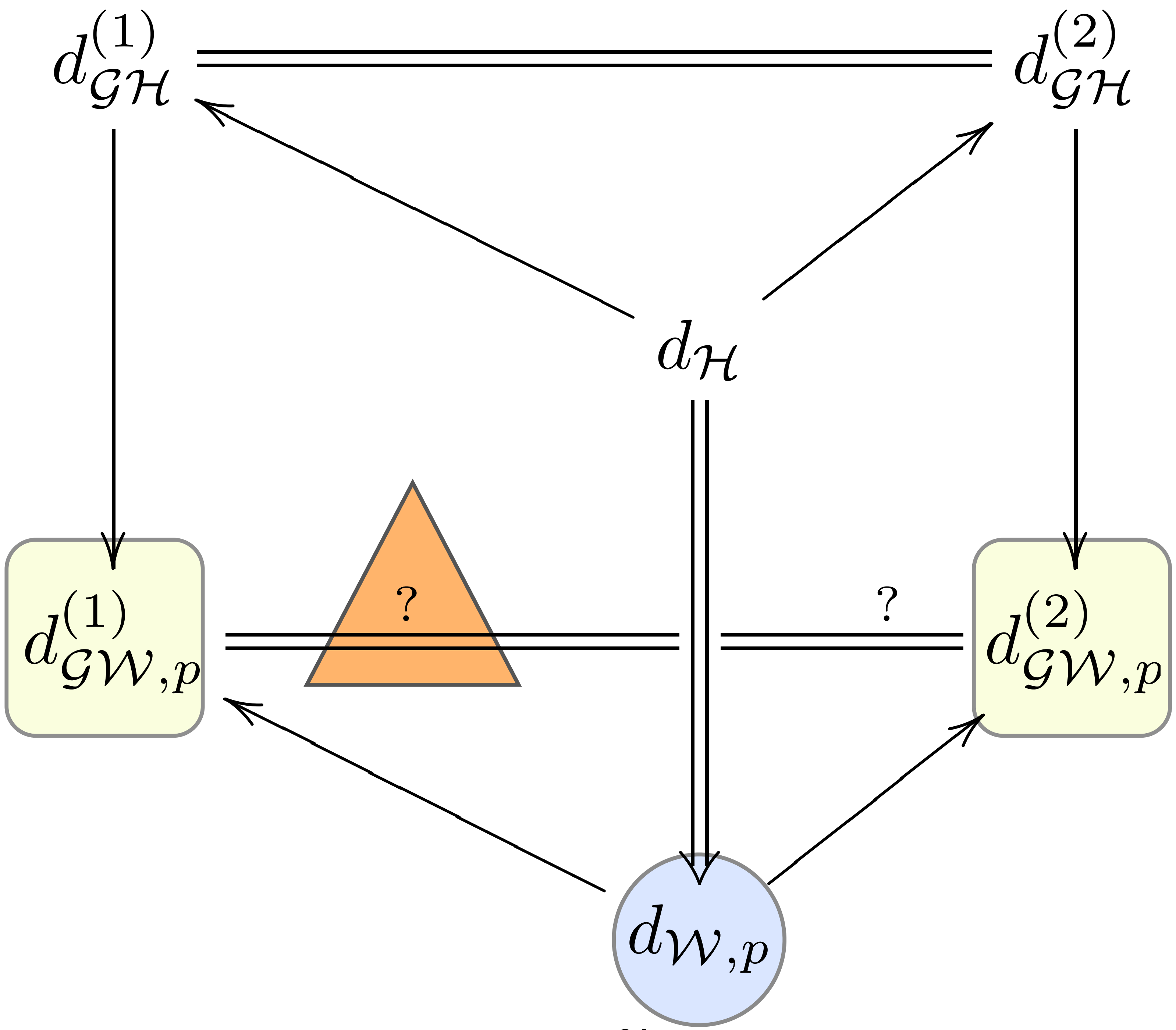
\end{tikzcd}$$

The plan



$d_{\mathcal{GH}}^{(1)}$
$d_{\mathcal{GH}}^{(2)}$
$d_{\mathcal{H}}$
$d_{\mathcal{GW},p}^{(1)}$
?
?
$d_{\mathcal{GW},p}^{(2)}$
$d_{\mathcal{W},p}$

The plan



$d^{(1)}_{\mathcal{GH}}$
$d^{(2)}_{\mathcal{GH}}$
$d_{\mathcal{H}}$
$d^{(1)}_{\mathcal{GW},p}$
?
?
$d^{(2)}_{\mathcal{GW},p}$
$d_{\mathcal{W},p}$

$$
\begin{tikzcd}
d_{\mathcal{GH}}^{(1)} \arrow[r, equal] \arrow[d, equal] & d_{\mathcal{GH}}^{(2)} \arrow[d, equal] \\
\inf_{d,R} \|d\|_{L^\infty(R)} & \frac{1}{2}\inf_R \|\Gamma_{X,Y}\|_{L^\infty(R\times R)}
\end{tikzcd}
$$

$$
\begin{tikzcd}
d_{\mathcal{GH}}^{(1)} \arrow[r, equal] \arrow[d, equal] & d_{\mathcal{GH}}^{(2)} \arrow[d, equal] \\
\inf_{d,R} \|d\|_{L^\infty(R)} \arrow[d] & \frac{1}{2} \inf_R \|\Gamma_{X,Y}\|_{L^\infty(R\times R)} \arrow[d] \\
\inf_{d,\mu} \|d\|_{L^p(\mu)} \arrow[d, equal] & \frac{1}{2} \inf_{\mu} \|\Gamma_{X,Y}\|_{L^p(\mu\otimes\mu)} \arrow[d, equal] \\
d_{\mathcal{GW},p}^{(1)} & d_{\mathcal{GW},p}^{(2)}
\end{tikzcd}
$$

$$
\begin{tikzcd}
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\boxed{\inf_{d,\mu} \|d\|_{L^p(\mu)}} \arrow[d, equal] & \frac{1}{2}\inf_{\mu} \|\Gamma_{X,Y}\|_{L^p(\mu\otimes\mu)} \arrow[d, equal] \\
d^{(1)}_{\mathcal{GW},p} & d^{(2)}_{\mathcal{GW},p}
\end{tikzcd}
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$$
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d^{(1)}_{\mathcal{GH}} \arrow[r, equal] \arrow[d, equal] & d^{(2)}_{\mathcal{GH}} \arrow[d, equal] \\
\inf_{d,R} \|d\|_{L^\infty(R)} \arrow[d] & \frac{1}{2} \inf_R \|\Gamma_{X,Y}\|_{L^\infty(R\times R)} \arrow[d] \\
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\boxed{\mathfrak{S}_p} & d^{(2)}_{\mathcal{GW},p}
\end{tikzcd}
$$

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d^{(1)}_{\mathcal{GH}} \arrow[r, equal] \arrow[d, equal] & d^{(2)}_{\mathcal{GH}} \arrow[d, equal] \\
\inf_{d,R} \|d\|_{L^\infty(R)} \arrow[d] & \frac{1}{2} \inf_{R} \|\Gamma_{X,Y}\|_{L^\infty(R\times R)} \arrow[d] \\
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\boxed{\mathfrak{S}_p} & & \boxed{\mathfrak{D}_p}
\end{tikzcd}$$

Can $\mathfrak{S}_p$ be equal to $\mathfrak{D}_p$?

- Using the same proof as in the Kalton-Ostrovskii Thm., one can prove that
$$\mathfrak{S}_\infty = \mathfrak{D}_\infty.$$

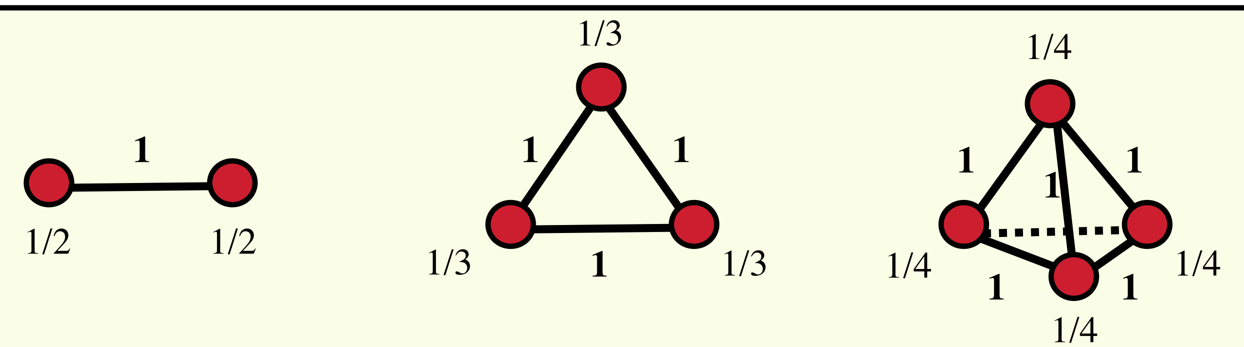
- Also, it is easy to see that for all $p \geq 1$
$$\mathfrak{S}_p \geq \mathfrak{D}_p.$$

- But the equality does not hold in general. One counterexample is as follows: take $X = (\Delta_{n-1}, ((d_{ij} = 1)), (\nu_i = 1/n))$ and $Y = (\{q\}, ((0)), (1))$. Then, for $p \in [1, \infty)$
$$\mathfrak{S}_1(X,Y) = \frac{1}{2} > \frac{1}{2}\left(\frac{n-1}{n}\right)^{1/p} = \mathfrak{D}_1(X,Y)$$

- Furthermore, these two (tentative) distances are **not Lipschitz equivalent!!** This forces us to analyze them separately. The delicate step is proving that $\mathrm{dist}(X,Y) = 0$ implies $X \simeq Y$.

- K. T. Sturm has analyzed $\mathfrak{S}_p$. Analysis of $\mathfrak{D}_p$ is in [**M07**].

$$\{\Delta_n\}_{n=1}^{\infty}$$



Properties of $\mathfrak{D}_p$

Theorem 1 ([M07]). *1. Let X, Y and Z mm-spaces then*

$$\mathfrak{D}_p(X,Y) \leqslant \mathfrak{D}_p(X,Z) + \mathfrak{D}_p(Y,Z).$$

2. If $\mathfrak{D}_p(X,Y) = 0$ then X and Y are isomorphic.

3. Let $\mathbb{X}_n = \{x_1, \ldots, x_n\} \subset X$ be a subset of the mm-space (X, d, ν). Endow $\mathbb{X}_n$ with the metric d and a prob. measure ν_n, then

$$\mathfrak{D}_p(X, \mathbb{X}_n) \leqslant d_{\mathcal{W},p}(\nu, \nu_n).$$

4. $p \geqslant q \geqslant 1$, then $\mathfrak{D}_p \geqslant \mathfrak{D}_q$.

5. $\mathfrak{D}_\infty \geqslant d_{\mathcal{GH}}$.

The parameter p is not superfluous

For $p \in [1,\infty]$ let $\mathbf{diam}_p(X) = \|d_X\|_{L^p(\mu_X \otimes \mu_X)}$.

The simplest lower bound for $\mathfrak{D}_p$ one has is based on the triangle inequality plus the observation that

$$\mathfrak{D}_p((X, d_X, \mu_X), (\{q\}, 0, 1)) = \mathbf{diam}_p(X)$$

Then,

$$\mathfrak{D}_p(X,Y) \geqslant \frac{1}{2}\big|\mathbf{diam}_p(X) - \mathbf{diam}_p(Y)\big|$$

For example, when $X = S^n$ (spheres with uniform measure and usual intrinsic metric):

- $p = \infty$ gives $\mathbf{diam}_\infty(S^n) = \pi$ for all $n \in \mathbb{N}$
- $p = 1$ gives $\mathbf{diam}_1(S^n) = \pi/2$ for all $n \in \mathbb{N}$
- $p = 2$ gives $\mathbf{diam}_2(S^1) = \pi/\sqrt{3}$ and $\mathbf{diam}_2(S^2) = \sqrt{\pi^2/2 - 2}$

Lower bounds for $\mathfrak{D}_p$ in terms of invariants

. Recall the invariants we defined before. Fix $X \in \mathcal{G}_w$ and $p \geqslant 1$.

- **distribution of distances**: $F_X : [0,\infty) \to [0,1]$,

$$t \mapsto (\mu_X \otimes \mu_X)(\{(x,x') |\, d_X(x,x') \leqslant t\}).$$

- **local distribution of distances**: $C_X : X \times [0,\infty) \to [0,1]$,

$$t \mapsto \mu_X(\{x' |\, d_X(x,x') \leqslant t\}).$$

- **eccentricities**: $s_{X,p} : X \to \mathbb{R}^+$,

$$x \mapsto \|d_X(x,\cdot)\|_{L^p(\mu_X)}.$$

- There are explicit lower bounds for $\mathfrak{D}_p$ in terms of these invariants, [**M07**].
- These lower bounds are important in practice: yield **LOP**s, easy optimization problems.
  Solution can be used as initial condition for solving $\mathfrak{D}_p$.

Shape Distributions [Osada-et-al]

$$\begin{pmatrix} 0 & d_{12} & d_{13} & d_{14} & \cdots \\ d_{12} & 0 & d_{23} & d_{24} & \cdots \\ d_{13} & d_{23} & 0 & d_{34} & \cdots \\ d_{14} & d_{24} & d_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Shape Distributions [Osada-et-al]

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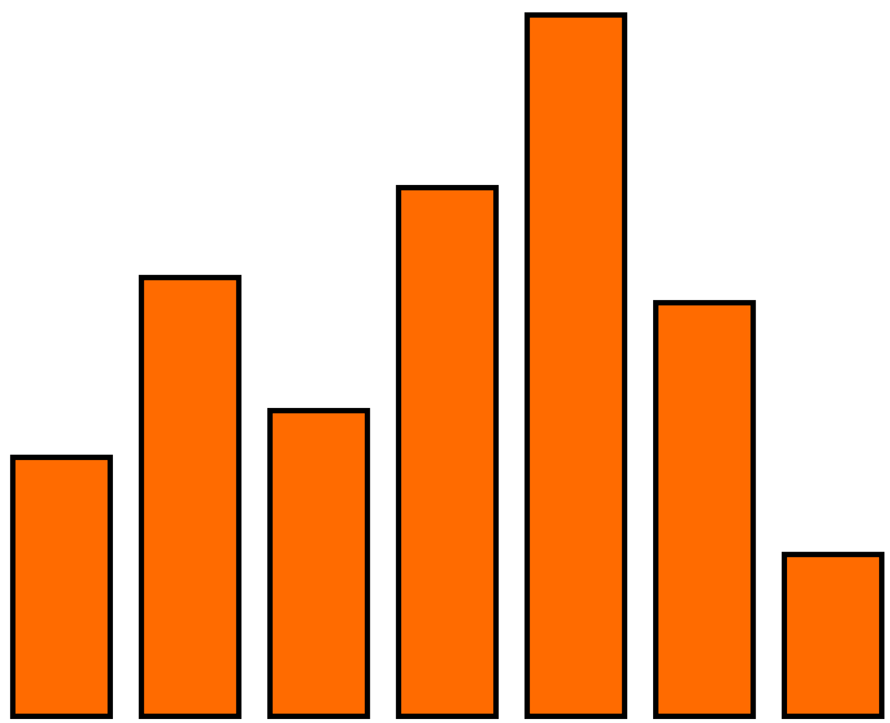

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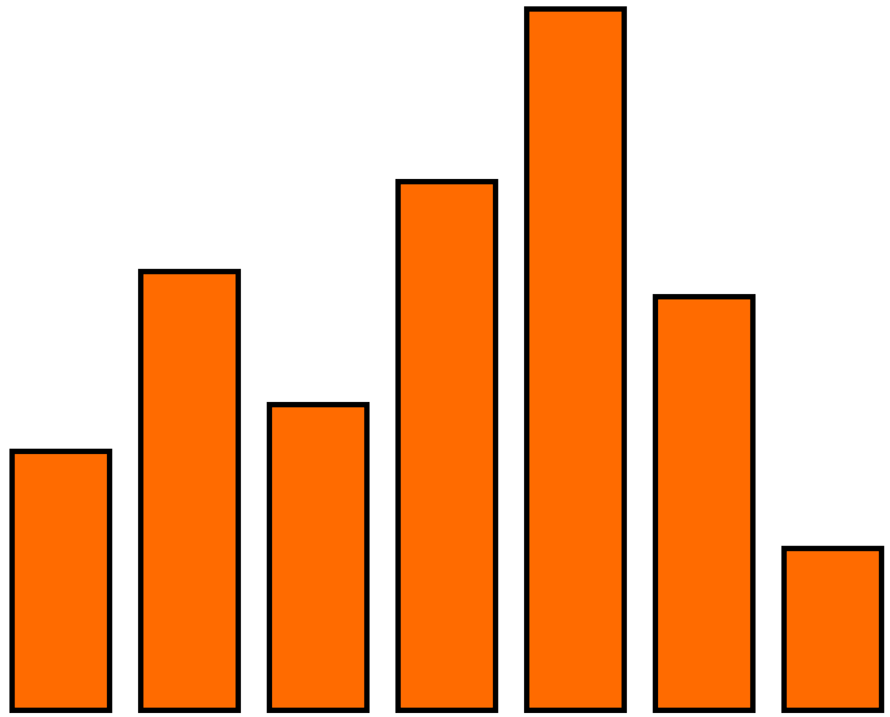

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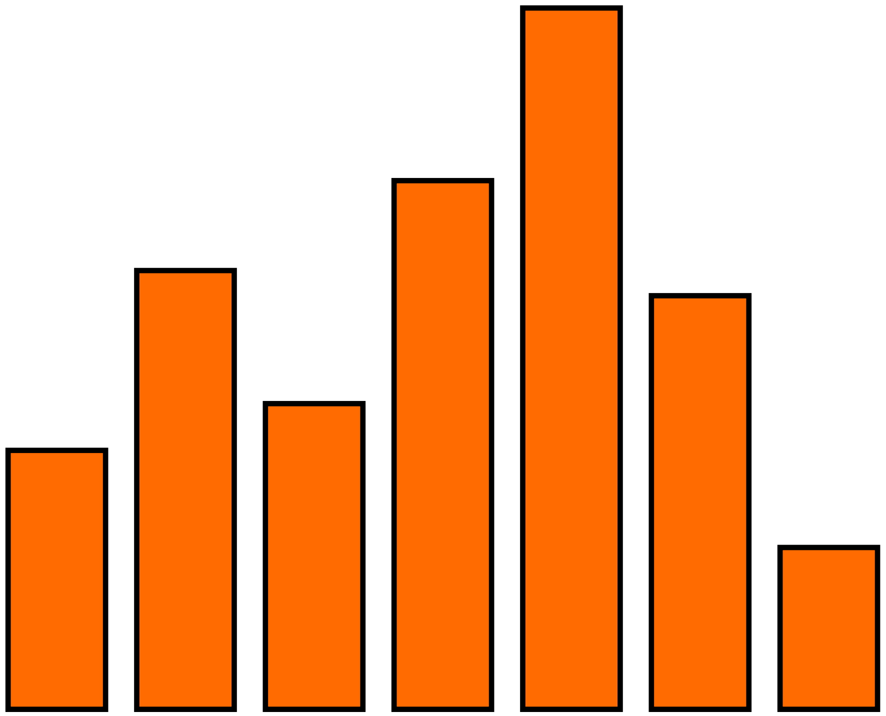
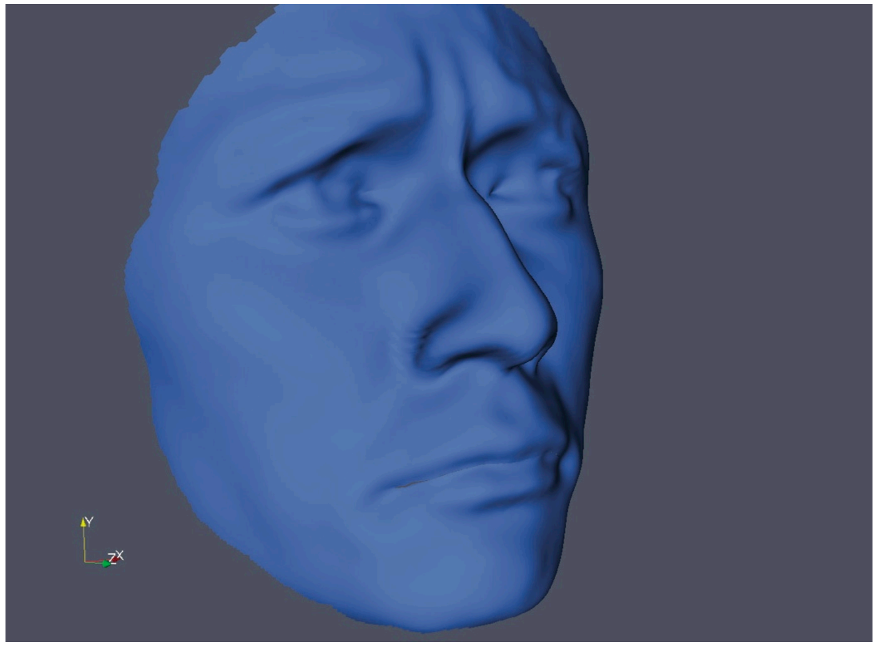
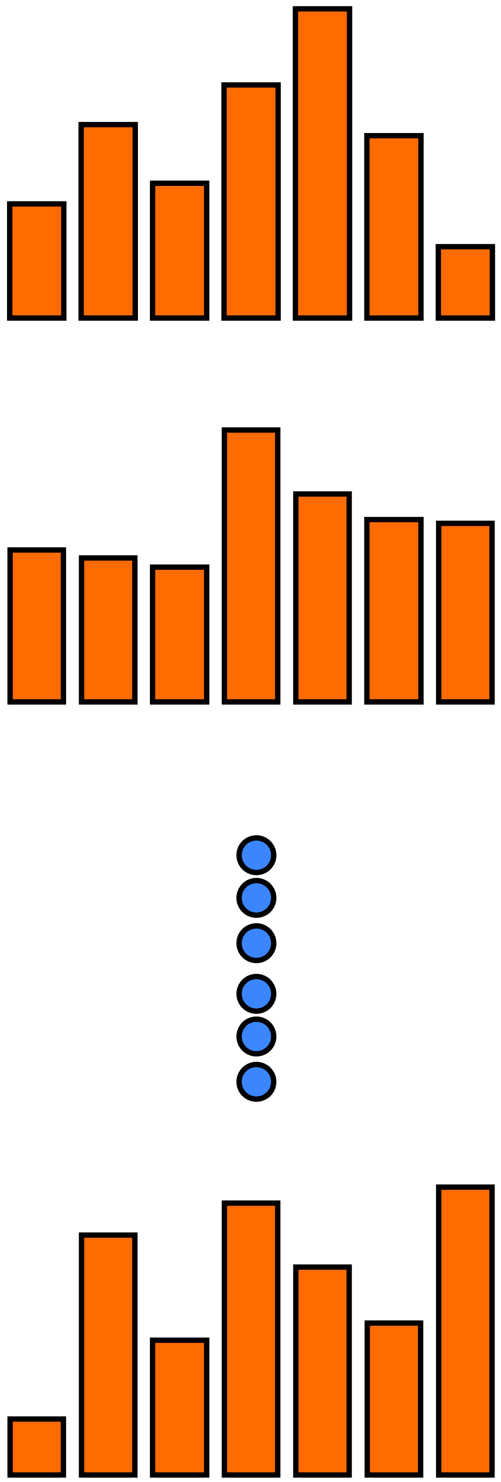




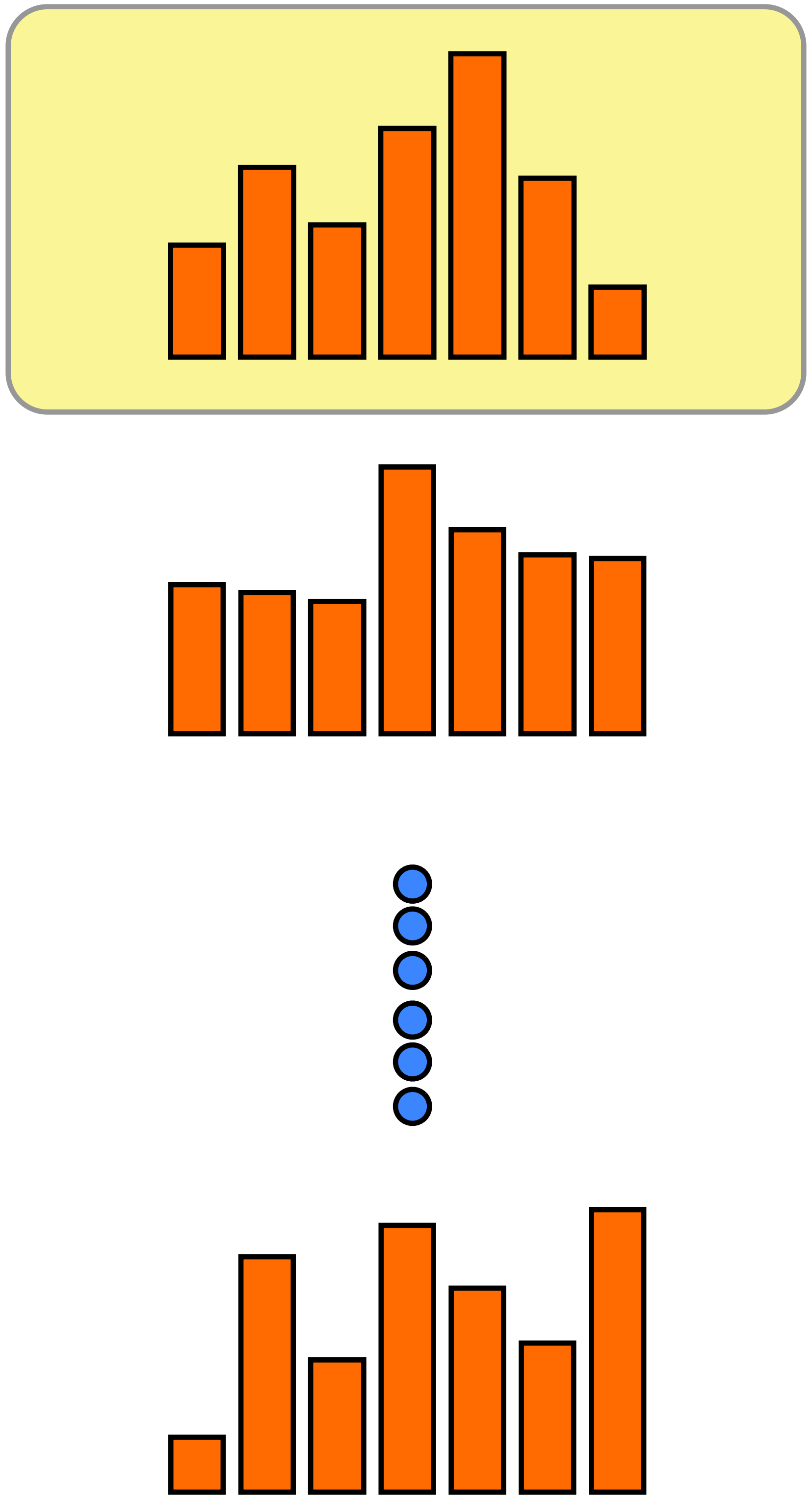
Shape Contexts

$$\begin{pmatrix} 0 & d_{12} & d_{13} & d_{14} & \cdots \\ d_{12} & 0 & d_{23} & d_{24} & \cdots \\ d_{13} & d_{23} & 0 & d_{34} & \cdots \\ d_{14} & d_{24} & d_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$



Shape Contexts

$$\begin{pmatrix} \boxed{0 & d_{12} & d_{13} & d_{14} & \dots} \\ d_{12} & 0 & d_{23} & d_{24} & \dots \\ d_{13} & d_{23} & 0 & d_{34} & \dots \\ d_{14} & d_{24} & d_{34} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$



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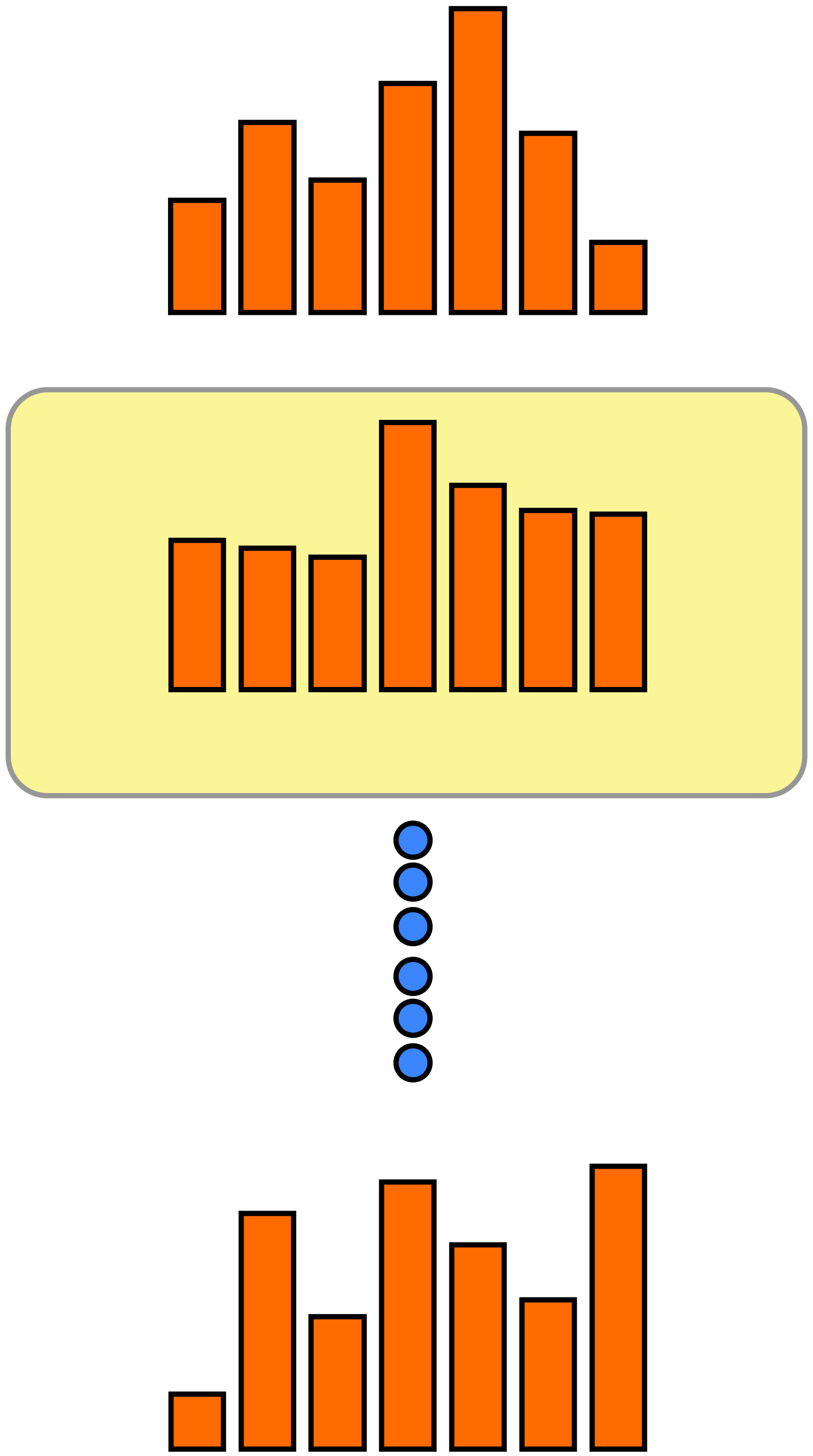

Hamza-Krim

$$\frac{\sum_j d_{1,j}}{N}$$

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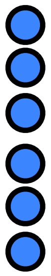
$$\frac{\sum_j d_{2,j}}{N}$$



$$\frac{\sum_j d_{N,j}}{N}$$

Hamza-Krim

$$\frac{\sum_j d_{1,j}}{N}$$

$$\begin{pmatrix} 0 & d_{12} & d_{13} & d_{14} & \dots \\ d_{12} & 0 & d_{23} & d_{24} & \dots \\ d_{13} & d_{23} & 0 & d_{34} & \dots \\ d_{14} & d_{24} & d_{34} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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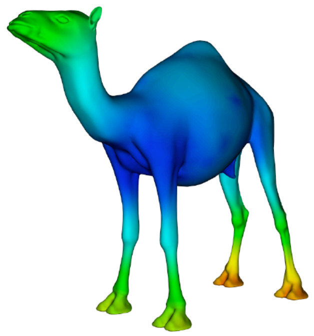

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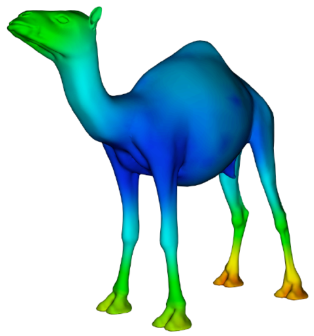

$$\frac{\sum_j d_{N,j}}{N}$$

The discrete case.

The option proposed and analyzed by K.L Sturm [St06], reads

$$\mathfrak{S}_p(X,Y) = \inf_{d \in \mathcal{D}(d_X,d_Y)} \inf_{\mu \in \mathcal{M}(\mu_X,\mu_Y)} \left(\sum_{x,y} d^p(x,y)\mu_{x,y} \right)^{1/p}$$

The second option reads **[M07]**

$$\mathfrak{D}_p(X,Y) = \inf_{\mu \in \mathcal{M}(\mu_X,\mu_Y)} \left(\sum_{x,y} \sum_{x',y'} |d_X(x,x') - d_Y(y,y')|^p \mu_{x,y}\mu_{x',y'} \right)^{1/p}$$

The **first** option,

$$\mathfrak{S}_p = \inf_{d \in \mathcal{D}(d_X, d_Y)} \inf_{\mu \in \mathcal{M}(\mu_X, \mu_Y)} \left(\sum_{x,y} d^p(x,y) \mu_{x,y} \right)^{1/p}$$

requires $\mathbf{2(n_X \times n_Y)}$ variables and $\mathbf{n_X + n_Y}$ plus $\sim \mathbf{n_Y \cdot C_2^{n_X} + n_X \cdot C_2^{n_Y}}$ linear constraints. When $p = 1$ it yields a *bilinear* optimization problem.

Our **second** option,

$$\mathfrak{D}_p(X,Y) = \inf_{\mu \in \mathcal{M}(\mu_X, \mu_Y)} \left(\sum_{x,y} \sum_{x',y'} |d_X(x,x') - d_Y(y,y')|^p \mu_{x,y} \mu_{x',y'} \right)^{1/p}$$

requires $\mathbf{n_X \times n_Y}$ variables and $\mathbf{n_X + n_Y}$ linear constraints. It is a *quadratic* (generally non-convex :-( ) optimization problem (with linear and bound constraints) for all p.

Future

- Study families of shapes.
- Statistic of families of shapes.
- Partial shape matching.
- Connections with Persistent topology invariants (Frosini+others... Yi will describe Frosini's work)
- Comparison/matching of animated geometries (Peter will talk about this)

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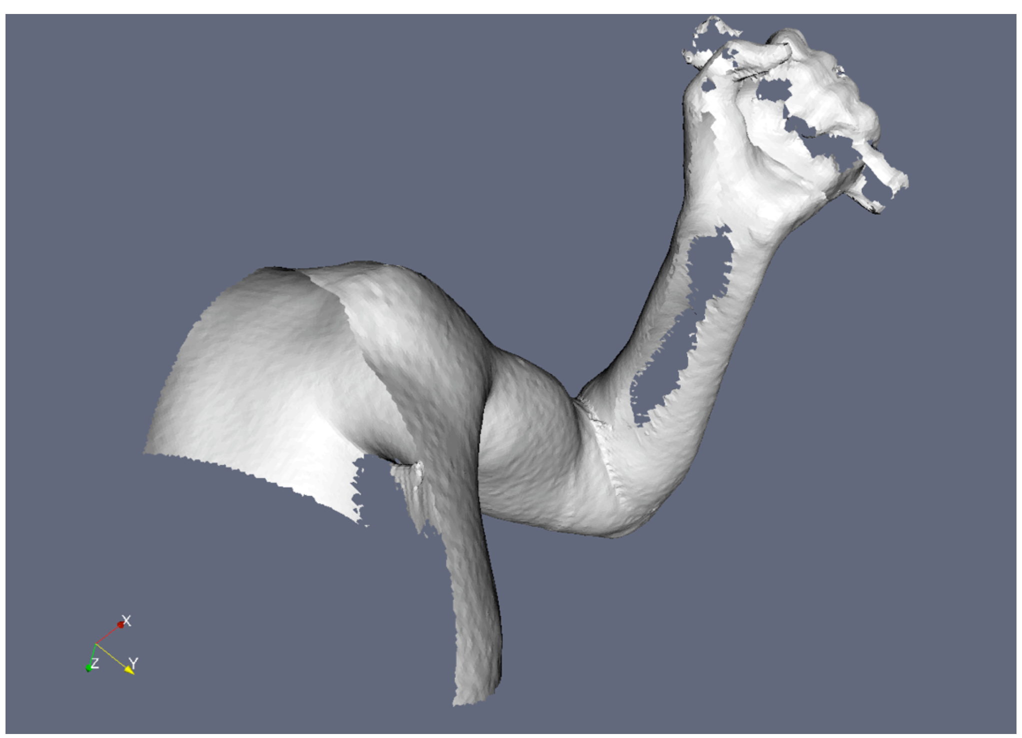

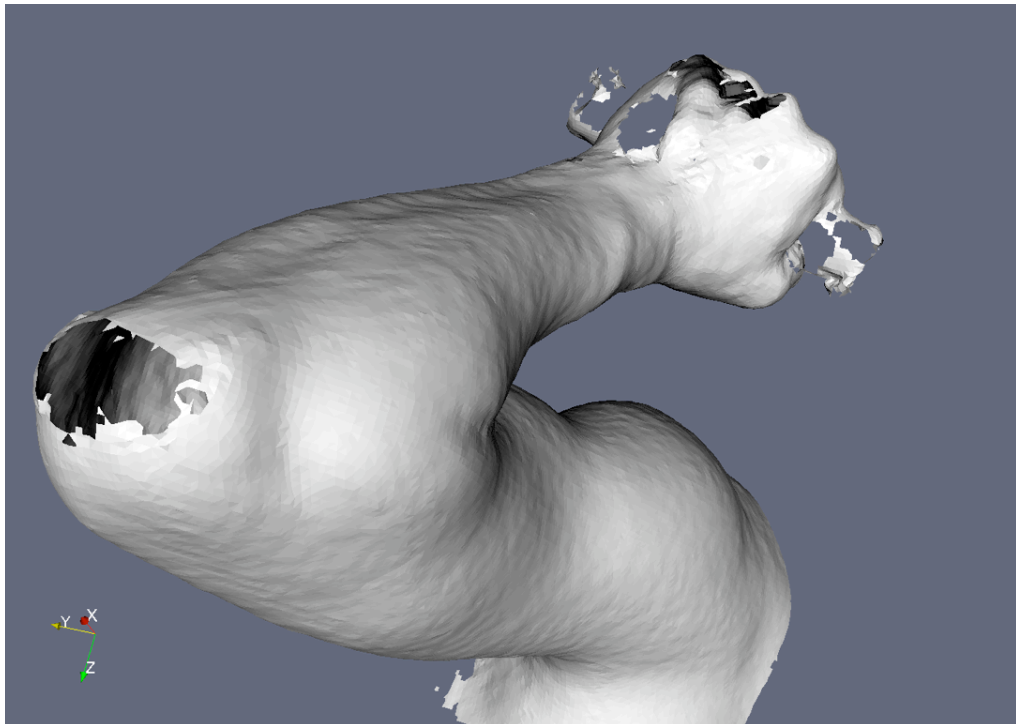

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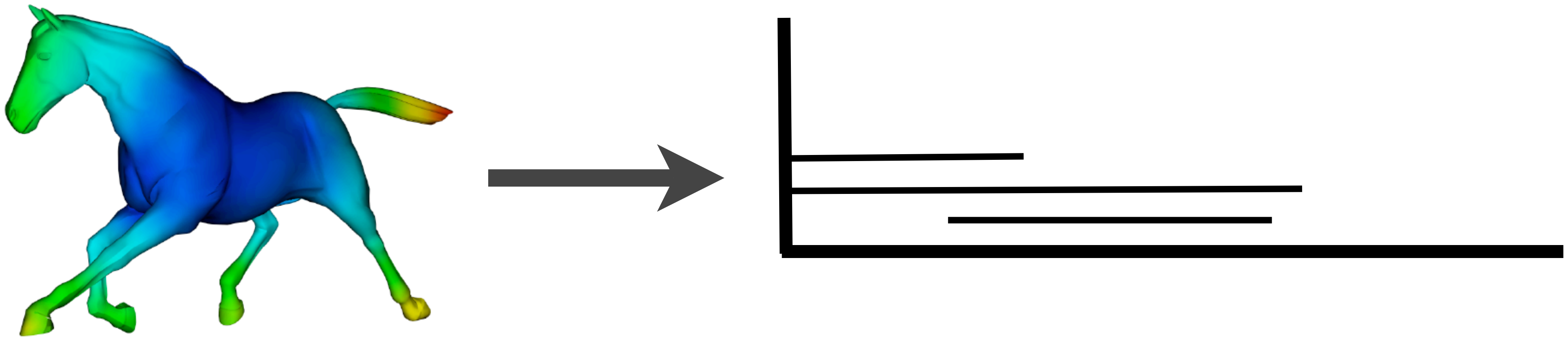

and more

- Explain/relate more methods using these ideas: what about eigenvalue based methods?
  - Shape signatures [**Reuter-et-al**]: associate to each shape the sorted list of eigenvalues of the Laplacian on the shape.
  - Leordaneau... from matching of pairs of points to matching of points.
- The GH distance and related Metric Geometry ideas are very powerful and can probably help uniformizing the treatment of many algorithmic procedure out there.

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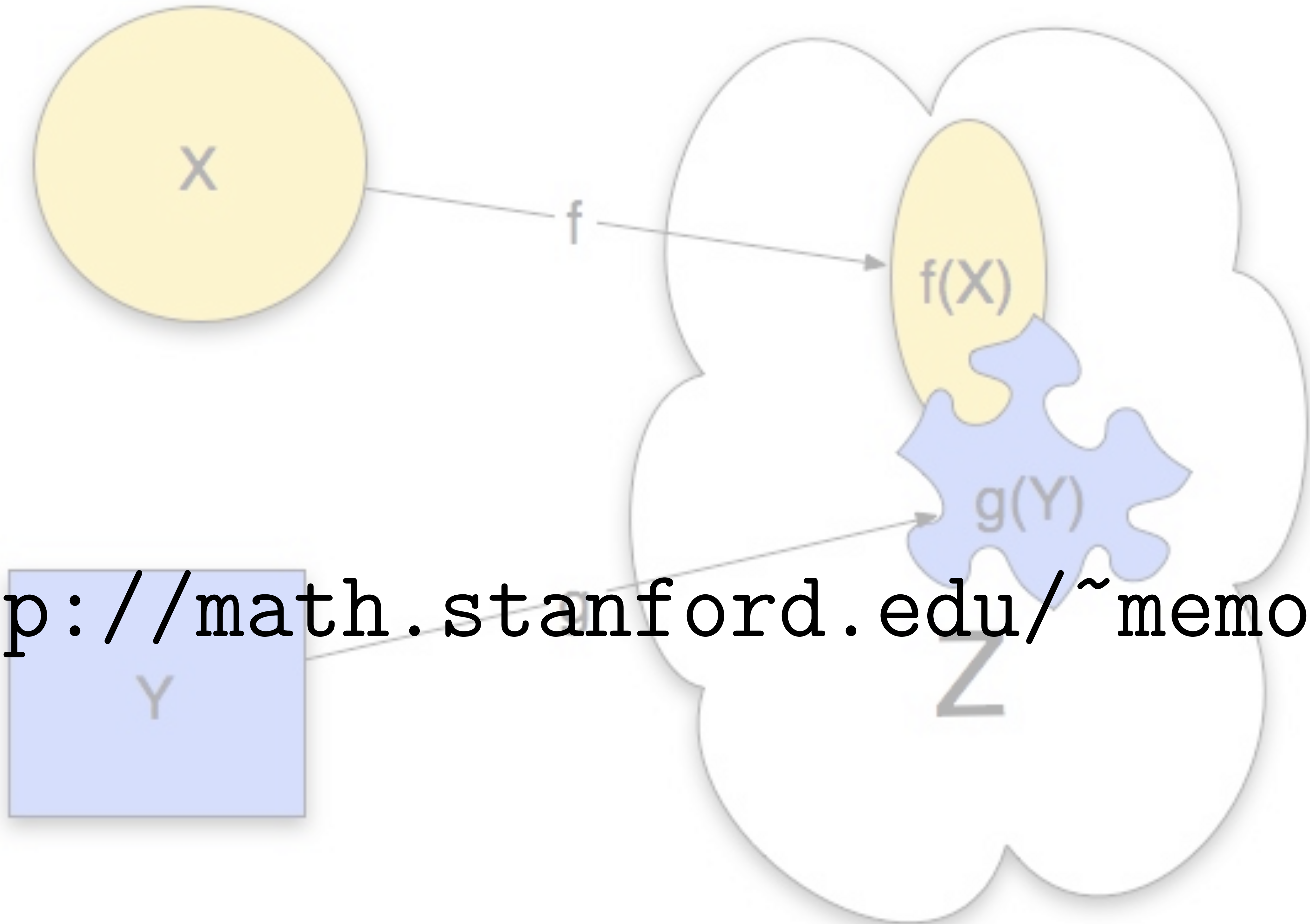

http://math.stanford.edu/~memoli