Shape Matching: A Metric Geometry Approach

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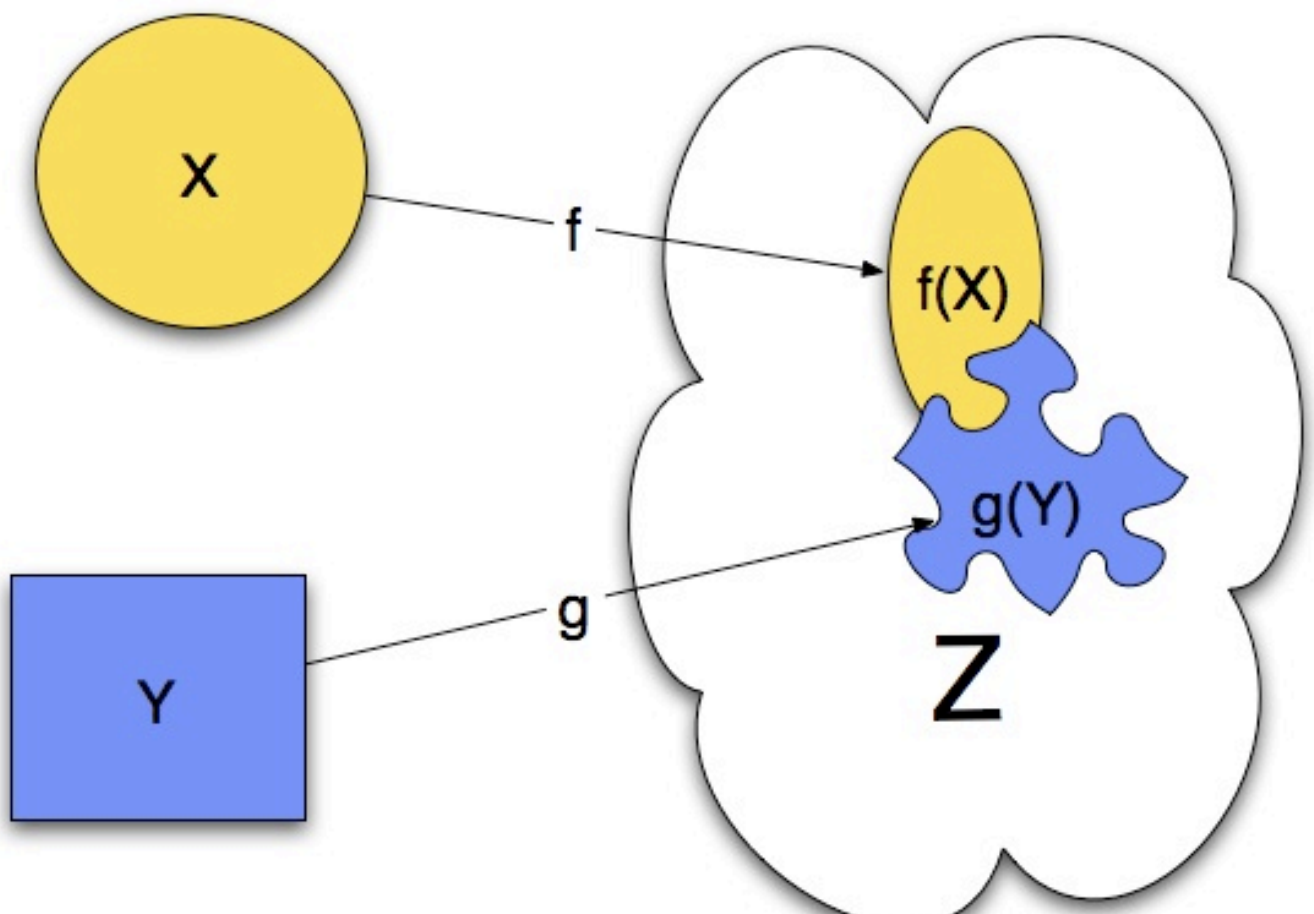
CS 468, Stanford University, Fall 2008.

GH: definition

$$d_{\mathcal{GH}}(X,Y) = \inf_{Z,f,g} d_{\mathcal{H}}^{Z}(f(X), g(Y))$$



Remember the expression:
For maps $\phi : X \to Y$, and $= \psi : Y \to X$ compute

$$\operatorname{dis}(\phi) = \max_{x,x'} |d_X(x,x') - d_Y(\phi(x), \phi(x'))|,$$

$$\operatorname{dis}(\psi) = \max_{y,y'} |d_Y(y,y') - d_X(\psi(y), \psi(y'))|$$

and

$$C(\phi, \psi) := \max_{x \in X, y \in Y} |d_X(x, \psi(y)) - d_Y(y, \phi(x))|.$$

Then

$$\boxed{d_{\mathcal{GH}}(X,Y) = \frac{1}{2} \min_{\phi,\psi} \max(\operatorname{dis}(\phi), \operatorname{dis}(\psi), C(\phi,\psi))}$$

where one minimizes over all choices of ϕ and ψ.

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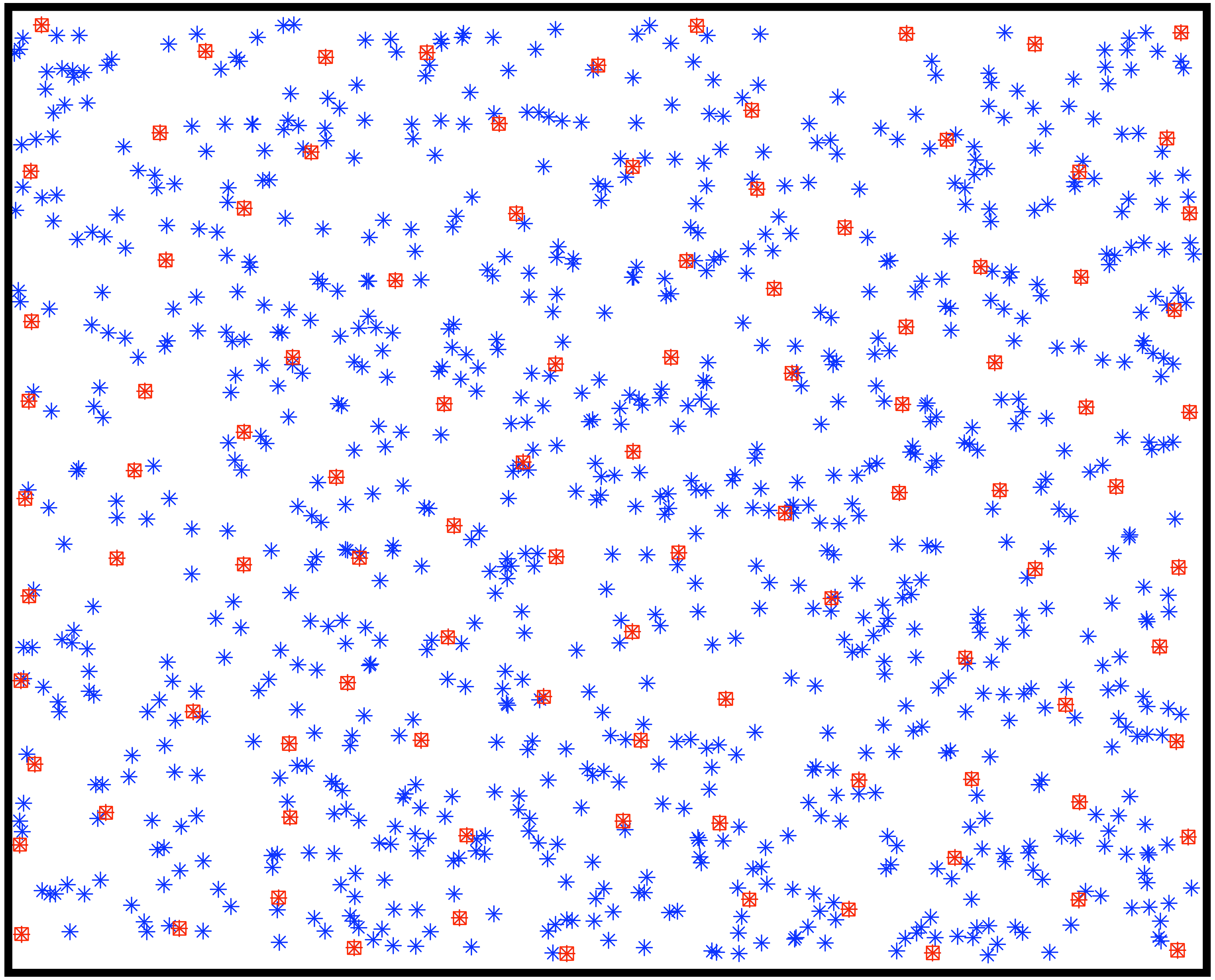
Approximation from [MS05]

- From the algorithmic point of view, we assume we know just
  - A dense point cloud $\mathbb{X}$ sampled from X
  - A dense point cloud $\mathbb{Y}$ sampled from Y
- Given a metric space (X, d_X), the discrete subset $\boxed{N_{X,n}^{(R,s)}}$ denotes a set of points $\{x_1, \ldots, x_n\} \subset X$ such that

  (1) $B_X(N_{X,n}^{(R,s)}, R) = X$,

  (2) $d_X(x_i, x_j) \geq s$ whenever $i \neq j$.

  In other words, the set constitutes a R-covering and the points in the set are not too close to each other.
- In practice, one constructs $N_{X,n}^{R,s}$ from $\mathbb{X}$!
- To fix ideas, say that $\#\mathbb{X} = 20,000$ and $n = 100$.



$\mathbb{X}$ is in blue, $N_{X,n}^{(R,s)}$ is in red.

So, the plan would be as follows:

- Given $\mathbb{X}$ and $\mathbb{Y}$ obtain $N_{\mathbb{X},n}^{(R,s)}$ and $N_{\mathbb{X},n}^{(R',s')}$.
- Write $N_{\mathbb{X},n}^{(R,s)} = \{x_i,\, i = 1, \ldots, n\}$ and $N_{\mathbb{Y},n}^{(R',s')} = \{y_i,\, i = 1, \ldots, n\}$.
- Compute

$$A := \min_{\bar{y}_1, \ldots, \bar{y}_n \in \mathbb{Y}} \max_{i,j} |d_X(x_i, x_j) - d_Y(\bar{y}_i, \bar{y}_j)|$$

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- and let $d_{\mathcal{F}}(\mathbb{X}, \mathbb{Y}) := \max(A, B)$.
- Can we relate $d_{\mathcal{F}}(\mathbb{X}, \mathbb{Y})$ to $d_{\mathcal{GH}}(\mathbb{X}, \mathbb{Y})$?

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Proposition. *Let* (X, d_X) *and* (Y, d_Y) *be compact metric spaces and let* $\eta = d_{\mathcal{GH}}(X, Y)$. *Also, let* $N_{X,n}^{(R,s)} = \{x_1, \ldots, x_n\}$ *be given. Then, given* $\alpha > 0$ *there exist points* $\mathbb{Y}^\alpha = \{y_1^\alpha, \ldots, y_n^\alpha\} \subset Y$ *such that*

1. $|d_X(x_i, x_j) - d_Y(y_i^\alpha, y_j^\alpha)| \leq 2(\eta + \alpha)$ *for all*

2. $\mathbb{Y}^\alpha$ *is a* $R + 2(\eta + \alpha)$ *covering of* Y*:*

$$B_Y\left(\{y_1^\alpha, \ldots, y_n^\alpha\}, R + 2(\eta + \alpha)\right) = Y$$

3. *Separation of* $\mathbb{Y}^\alpha$ *is* $\geq s - 2(\eta + \alpha)$*:*

$$d_Y(y_i^\alpha, y_j^\alpha) \geq s - 2(\eta + \alpha)$$

for $i \neq j$.

- This proposition tells us that if the metric spaces happen to be sufficiently close in a metric sense, then given a (s-separated) R-covering on one of them, one can find a (s'-separated) R'-covering in the other metric space such that the metric distorsion between those coverings (point clouds) is also small.

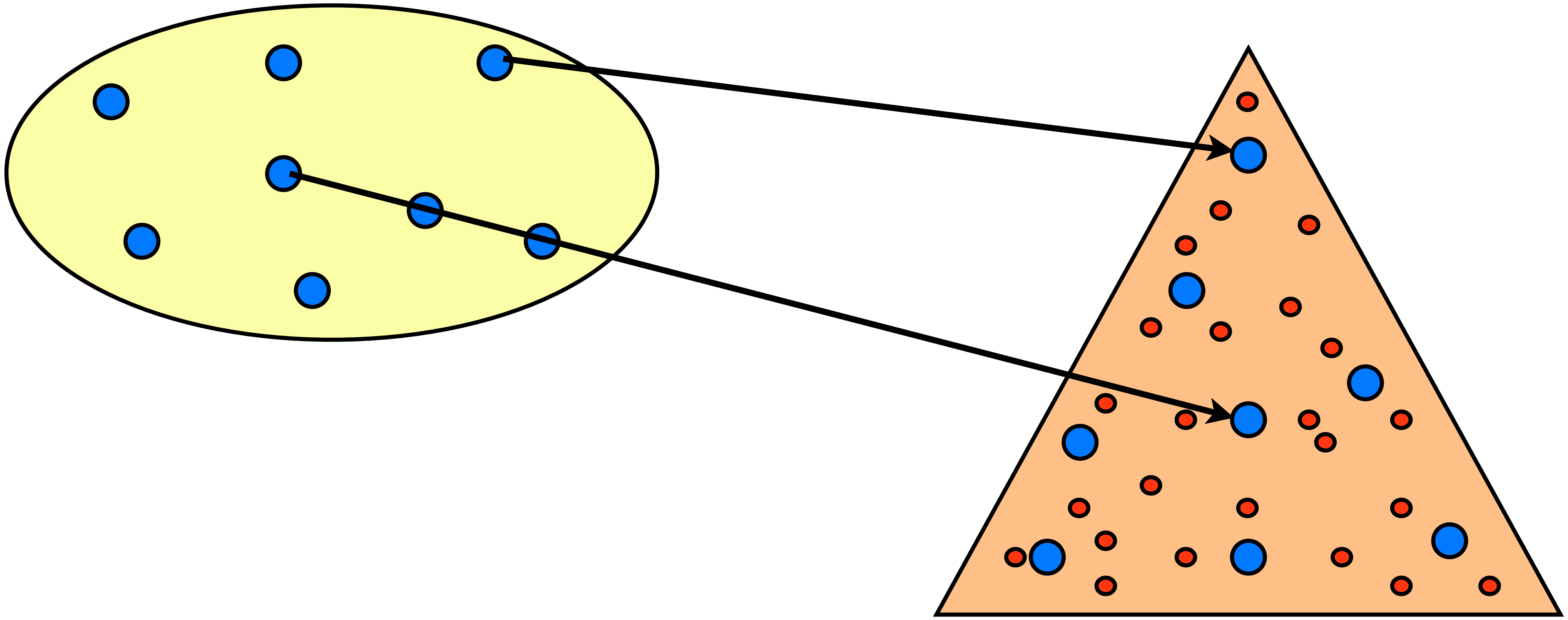
- Since by Tychonoff's Theorem the n-fold product space $Y \times \ldots \times Y$ is compact, if $s - 2 \quad \geq c > 0$ for some positive constant c, by passing to the limit along the subsequences of $\{y_1^\alpha, \ldots, y_n^\alpha\}_{\{\alpha>0\}}$ (if needed) above one can assume the existence of a set of **different** points $\{\bar{y}_1, \ldots, \bar{y}_n\} \subset Y$ such that

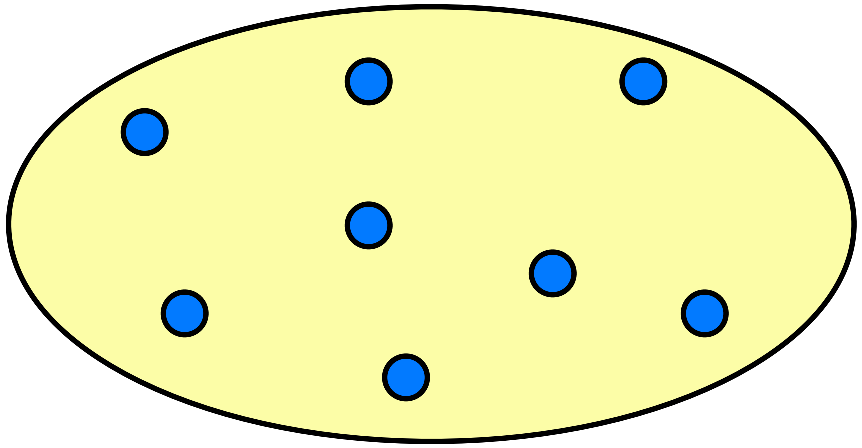
  1. $|d_X(x_i, x_j) - d_Y(\bar{y}_i, \bar{y}_j)| \leq 2$, all i, j
  2. $\min_{i \neq j} d_Y(\bar{y}_i, \bar{y}_j) \geq s - 2 \quad > 0$, and
  3. $B_Y(\{\bar{y}_1, \ldots, \bar{y}_n\}, R + 2 \quad) = Y$.

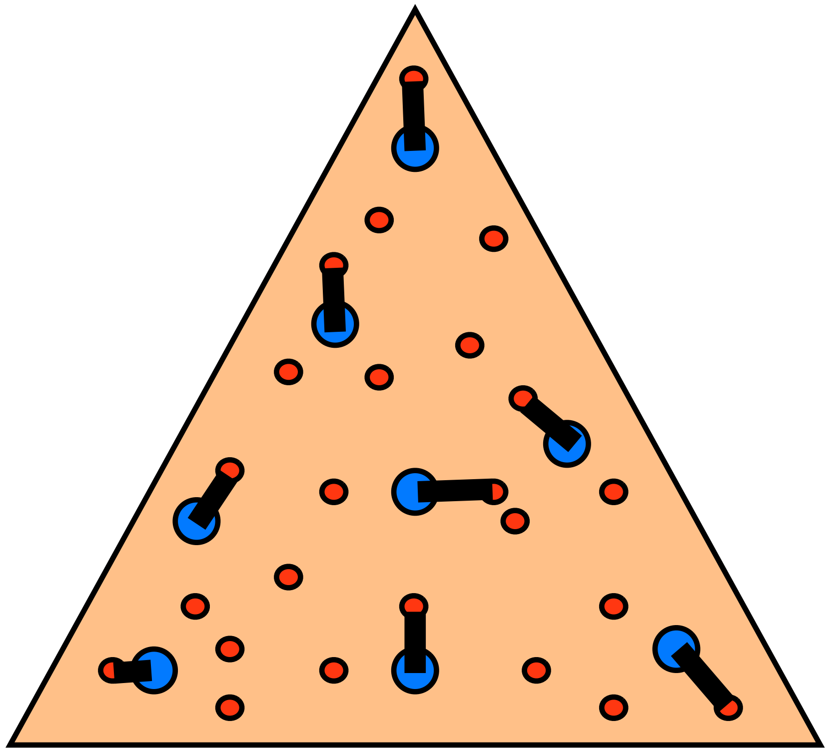
- But there's no reason to expect that $\bar{y}_i \in \mathbb{Y}$!! (we only have access to $\mathbb{Y}$..)

$\mathbb{Y}$ in red, and $\{\bar{y}_i\}$ in blue.

$N_{X,n}^{(R,s)}$







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$$B := \min_{\bar{x}_1,\ldots,\bar{x}_n \in \mathbb{X}} \max_{i,j} |d_Y(y_i, y_j) - d_X(\bar{x}_i, \bar{x}_j)|$$

- and let $d_{\mathcal{F}}(\mathbb{X}, \mathbb{Y}) := \max(A, B)$.
- Can we relate $d_{\mathcal{F}}(\mathbb{X}, \mathbb{Y})$ to $d_{\mathcal{GH}}(\mathbb{X}, \mathbb{Y})$?

Theorem 1 (**[MS05]**)**.** *Let X and Y be compact sub-manifolds of $\mathbb{R}^d$, and let $\eta = d_{\mathcal{GH}}(X, Y)$. Let $N_{X,n}^{(R,s)}$ be s.t. for some $c > 0$*

$$s > 2\eta + c.$$

Then, given $p \in (0, 1)$ there exist $m = m(c, p) \in$ s.t. if $\mathbb{Y}$ is formed by sampling i.i.d. uniformly m points from Y, then,

$$\mathbf{P}\left(A \leq 3\eta + R\right) \geq p$$

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Remark.

- *This essentially gives control over $d_{\mathcal{F}}(\mathbb{X}, \mathbb{Y})$. We obtain that for some constants A, B, α, β, (α, β can be controlled to zero)*

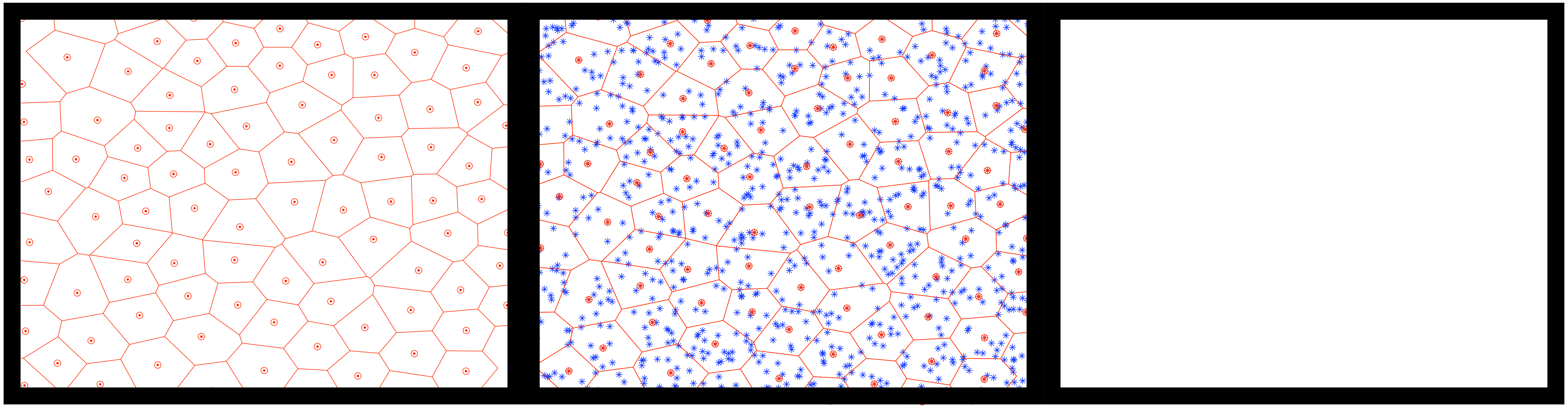
  $$A(d_{\mathcal{GH}}(X, Y) - \alpha) \leq d_{\mathcal{F}}(\mathbb{X}, \mathbb{Y}) \leq B(d_{\mathcal{GH}}(X, Y) - \beta)$$

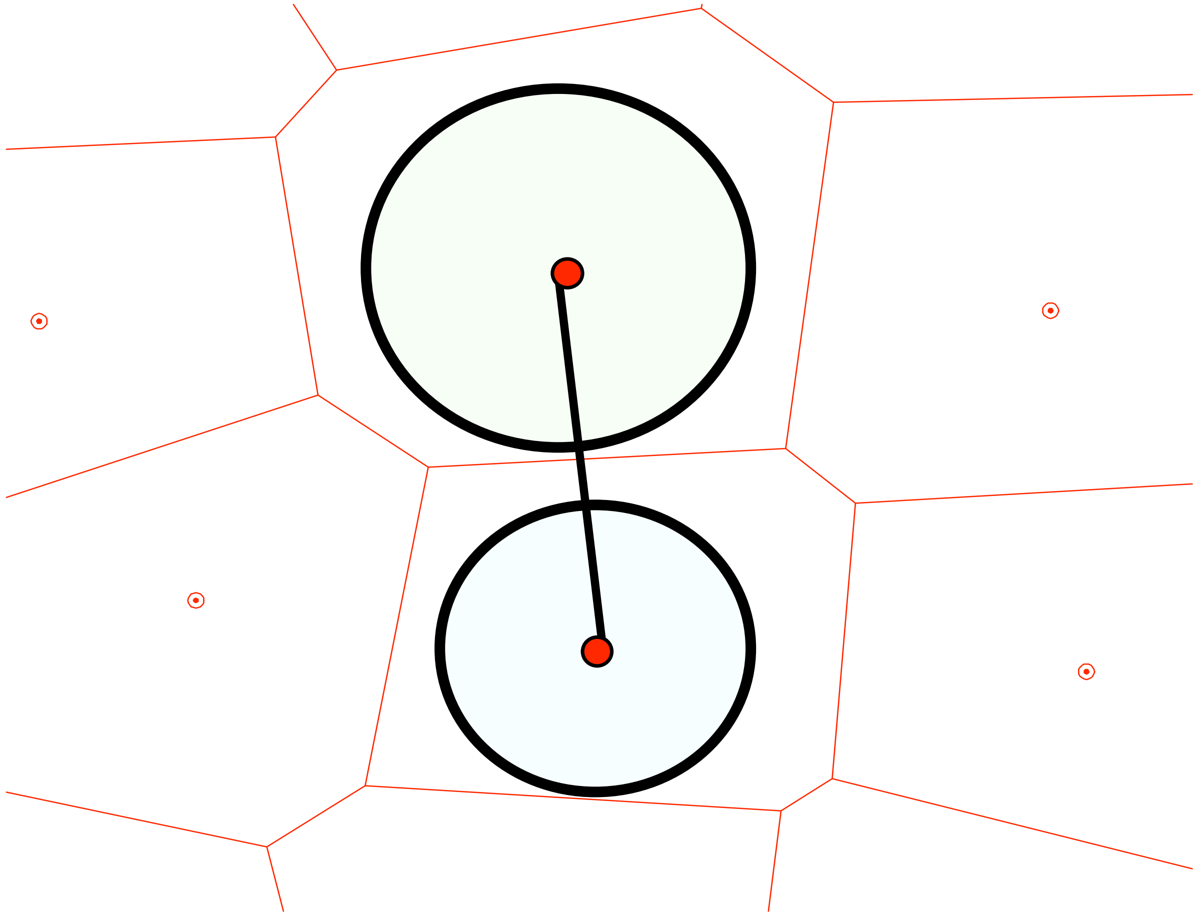
  *with **controllable probability**.*

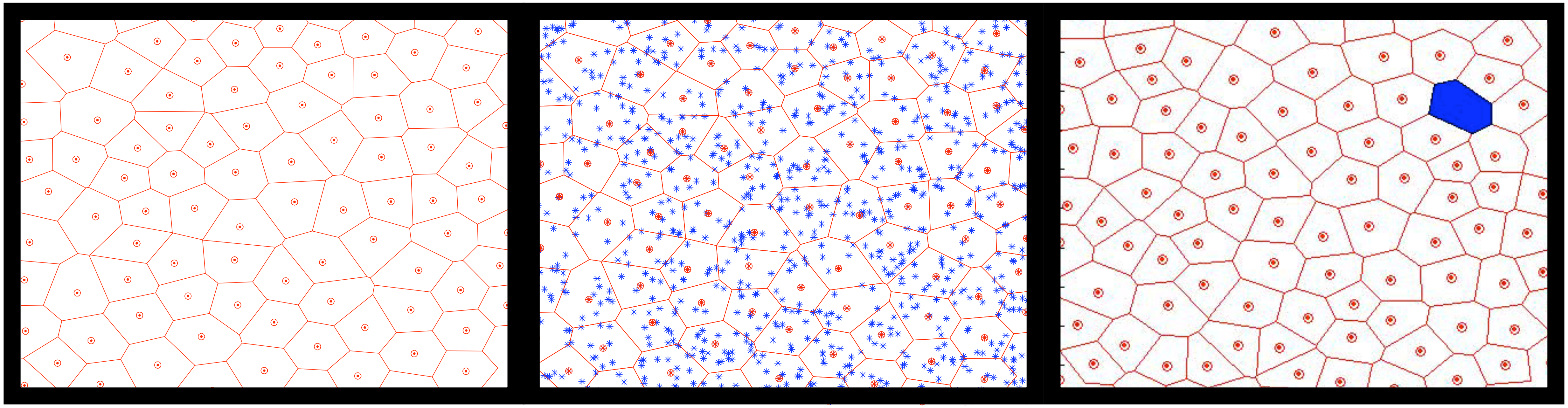
- *The proof is based on using the **Coupon collector problem**: each Voronoi cell defined by $\{\bar{y}_i\}$ wants to collect a coupon (a point from $\mathbb{Y}$). How many times do I have to go to the store until I get all coupons?*

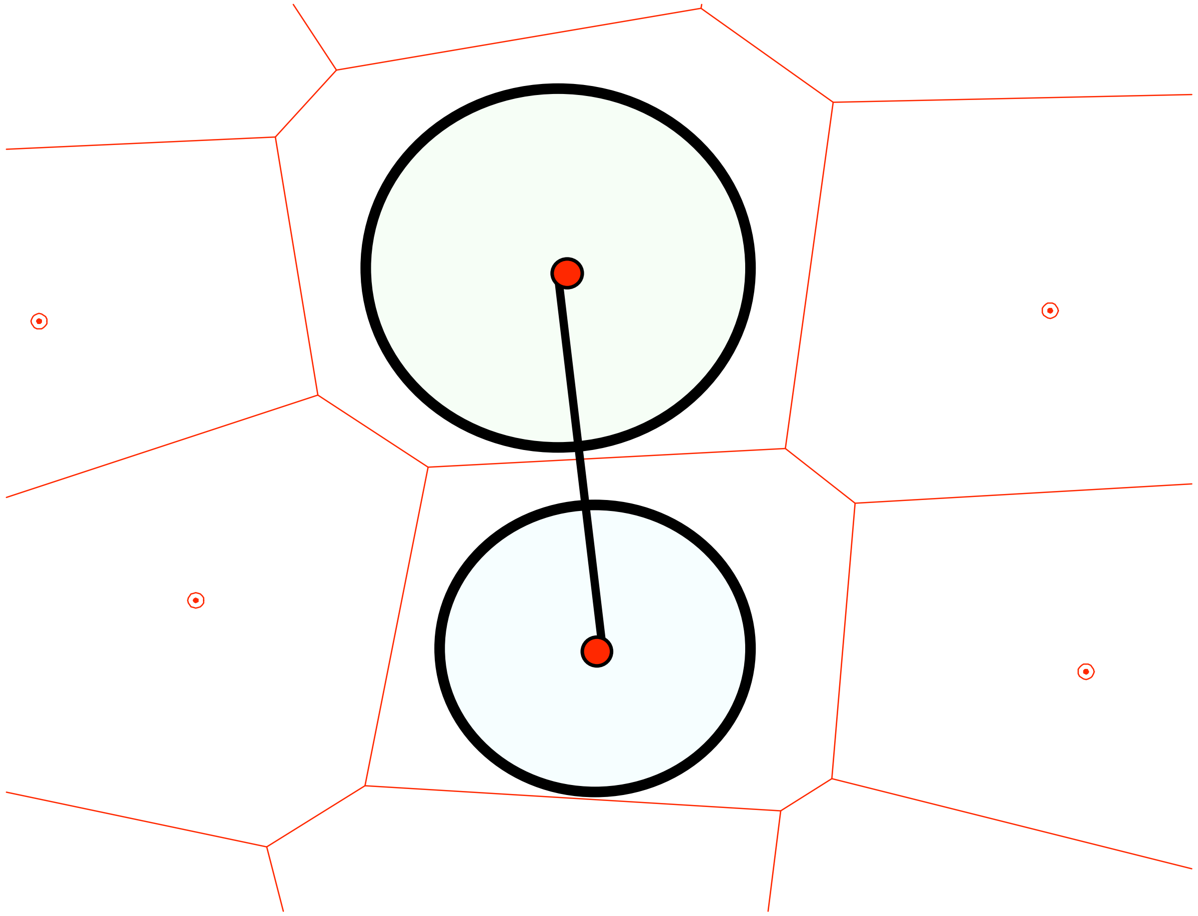
- *This particular instance of the CCP has to deal with* unequal probabilities *for each coupon: the areas of the Voronoi cells are different.*

- *We impose that $c > 0$ so that we have non-zero probability of getting all coupons.*









What do I need to compute?

You are given $\mathbb{X}$ and $\mathbb{Y}$.

- Construct $N_{X,n}^{(R,s)}$ and $N_{Y,n}^{(R',s')}$.
- Compute A and B: for A, you fix $N_{X,n}^{(R,s)}$ and find the n points in $\mathbb{Y}$ that match the distance matrix of $N_{X,n}^{(R,s)}$ as closely as possible:

$$A = \min_{\{\bar{y}_i\}} \max_{i,j} |d_X(x_i, x_j) - d_Y(\bar{y}_i, \bar{y}_j)|$$

- Do the same for B. These are combinatorial problems. There are no exact algorithms to my knowledge. Have heuristic.
- **Problem:** can anything be proved for these heuristics?

- Computations are combinatorial in nature
- Assumed shapes are smooth surfaces so that I could talk about uniform probability distribution.
- The proofs do not depend on this fact, so arguments and guarantees carry over to measure metric spaces.
- Again, ugly combinatorial problems..

Approximation from [BBK06]

- Remember $d_{\mathcal{GH}}(X,Y) = \min_{\phi,\psi} \max(\mathrm{dis}(\phi), \mathrm{dis}(\psi), C(\phi,\psi))$.
- Concentrate on minimizing $\mathrm{dis}(\phi)$ alone:

$$\min_{\phi} \max_{i,j} |d_X(x_i, x_j) - d_Y(\phi(x_i), \phi(x_j))|.$$

- In review of **[MS05]** we wrote this as

$$\min_{y_1,\ldots,y_m \in \mathbb{Y}} \max_{i,j} |d_X(x_i, x_j) - d_Y(y_i, y_j)|.$$

- The reason the problem is combinatorial is that y_i are constrained to lie on $\mathbb{Y}$.
- Think for one second of 'moving' the y_i continuously. In other words, I'd like to think that I am allowed to change points infinitesimally: $y_i \to y_i + \delta_i$.
- This requires being able to compute $d_Y(y_i + \delta_i, y_j + \delta_j)$.

- This requires having a smooth underlying structure: smooth surfaces.
- need to create a local parameterization using for instance the meshes.
- then, one can use standard interpolation algorithms for computing $d_Y(y_i + \delta_i, y_j + \delta_j)$.
- Then, in order to find approximation to $\min_\phi \mathrm{dis}(\phi)$ one minimizes the functional $F(\mathbf{y}_i, \ldots, \mathbf{y}_j) := \max_{ij} |d_X(x_i, x_j) - d_Y(\mathbf{y}_i, \mathbf{y}_j)|$.
- Proposed method uses gradient descent. Need to compute numerical derivatives of $d_Y(y_i + \delta_i, y_j + \delta_j)$.
- Now, if you want to use these ideas for estimating the GH distance, you need to solve three **coupled problems**:

$$d_{\mathcal{GH}}(\mathbb{X}, \mathbb{Y}) = \frac{1}{2} \min_{\phi,\psi} \max(\mathrm{dis}(\phi), \mathrm{dis}(\psi), C(\phi, \psi)).$$

- the resulting problem is highly nonlinear, the number of variables is $\#\mathbb{X} + \#\mathbb{Y} + 1$ and the number of *nonlinear* constraints is $\sim (\#X + \#Y)^2$.

- Authors cope with the problem of having so many nonlinear constraints by using the *penalty barrier*.
- The L^∞ aspect of the GH distance complicates the numerical computations, they apply an L^p relaxation
- all in all they end up solving a *unconstrained* nonlinear problem.
- What is the relationship between the result and the GH distance? If this is hard to answer then..
- What are the properties of the thing they minimize? does that measure something like a metric on the class of shapes that you set out to study?

These questions are difficult to answer.
Better alternative: write down a notion of metric between metric spaces that translates directly into what you should compute in practice!

Critique

- Was not able to show connections with (sufficiently many) pre-existing approaches
- Computationally hard: currently only two attempts have been made:
  - [MS04,MS05] and [BBK06] only for surfaces.
  - [MS05] gives probabilistic guarantees for estimator based on sampling parameters.
  - Full generality leads to a hard **combinatorial optimization problem**: QAP.

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Desiderata

- Obtain an L^p version of the GH distance that:
  - retains theoretical underpinnings
  - its implementation leads to easier (continuous, quadratic, with linear constraints) optimization problems
  - can be related to pre-existing approaches (shape contexts, shape distributions, Hamza-Krim,..) via lower/upper bounds.

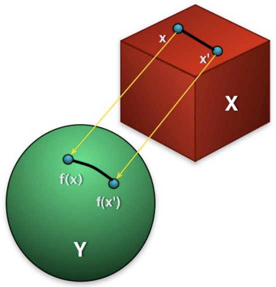
remember: Naive relaxation

Another expression for the GH distance

A theorem, [BuBuIv]

For compact metric spaces (X, d_X) and (Y, d_Y),

$$d_{\mathcal{GH}}(X,Y) = \frac{1}{2}\inf_{R} \max_{(x,y),(x',y')\in R} |d_X(x,x') - d_Y(y,y')|$$



First attempt: naive relaxation

Remember that

$$d_{\mathcal{GH}}(X,Y) = \frac{1}{2} \inf_R \max_{(x,y),(x',y') \in R} |d_X(x,x') - d_Y(y,y')|$$

where $R \in \mathcal{R}(X,Y)$. Using the matricial representation of R one can write

$$d_{\mathcal{GH}}(X,Y) = \frac{1}{2} \inf_R \max_{x,x',y,y'} |d_X(x,x') - d_Y(y,y')| \, r_{x,y} \, r_{x',y'}$$

where $R = ((r_{x,y})) \in \{0,1\}^{n_X \times n_B}$ s.t.

$$\sum_{x \in X} r_{xy} \geq 1 \;\; \forall y \in Y$$

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First attempt: naive relaxation (continued)

- The idea would be to use L^p norm instead of L^∞ ($\max\max$)
- relax $r_{x,y}$ to be in $[0,1]$ (!)

Then, the idea would be to compute (for some $p \geq 1$):

$$\widehat{d}_{\mathcal{GH}}(X,Y) = \frac{1}{2}\inf_R \left(\sum_{x,x',y,y'} |d_X(x,x') - d_Y(y,y')|^{\mathbf{p}}\, r_{x,y}\, r_{x',y'} \right)^{1/\mathbf{p}}$$

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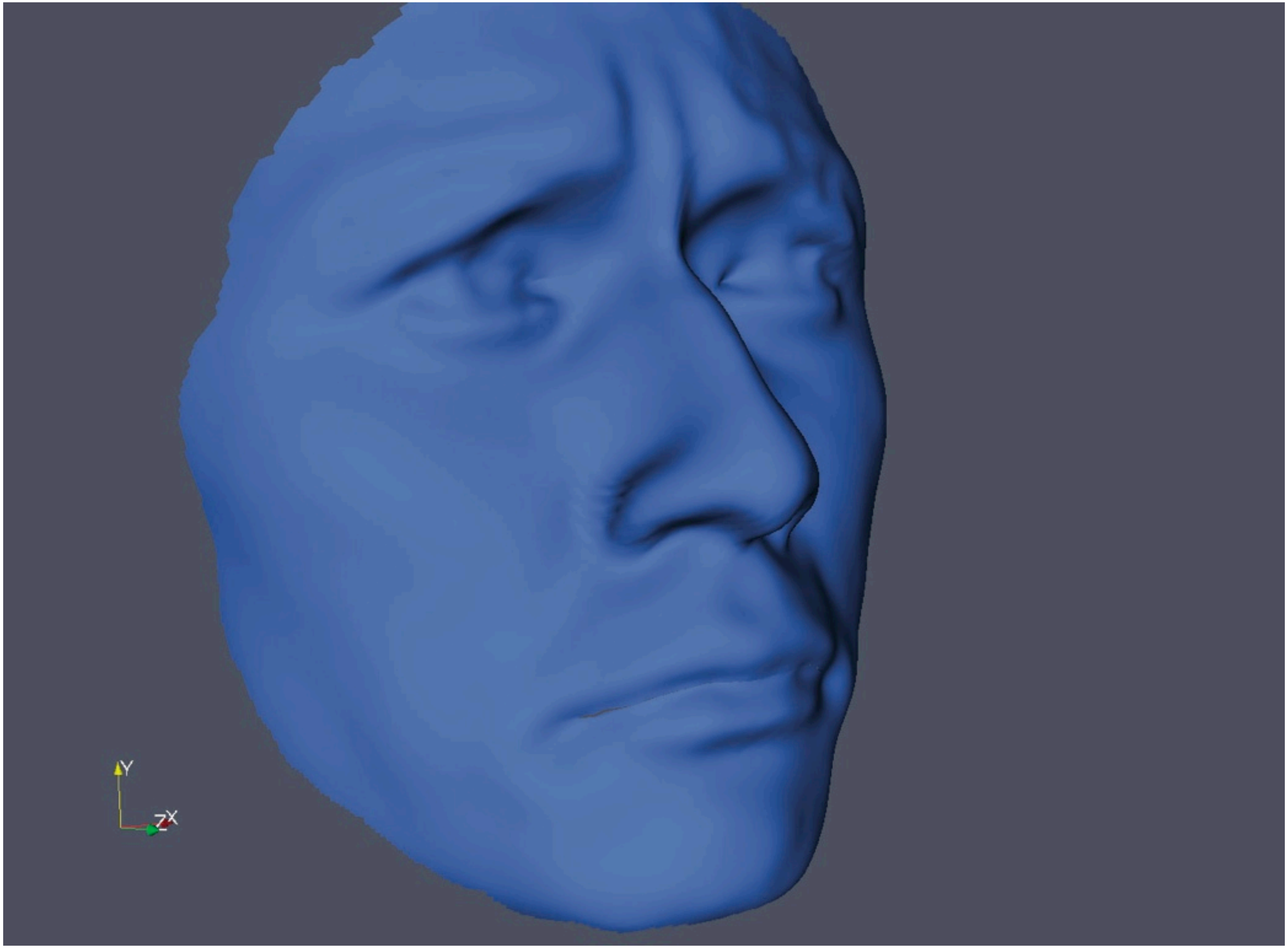
- The resulting problem is a continuous variable QOP with linear constraints, but..
- there is no limit problem.. this discretization cannot be connected to the GH distance..

we need to identify the **correct** relaxation of the GH distance. More precisely, the correct notion of *relaxed correspondence*.

Quick review of other methods for shape matching

- Shape distributions
- Shape contexts
- Hamza-Krim

Assignment: write a 1/2 page summary of each of those approaches. Papers are posted on course webpage.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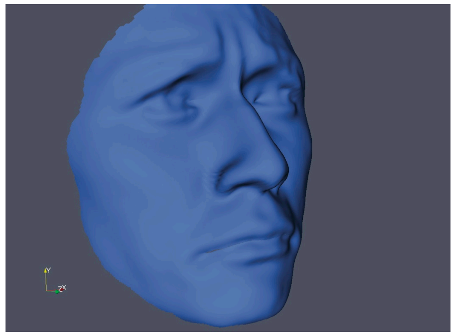

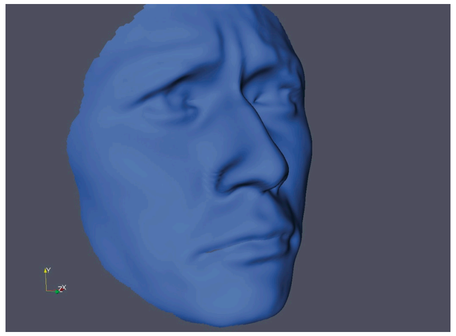

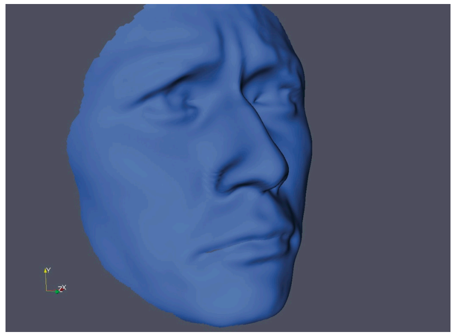

$$\begin{pmatrix} 0 & d_{12} & d_{13} & d_{14} & \dots \\ d_{12} & 0 & d_{23} & d_{24} & \dots \\ d_{13} & d_{23} & 0 & d_{34} & \dots \\ d_{14} & d_{24} & d_{34} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Shape Distributions [Osada-et-al]

$$\begin{pmatrix} 0 & d_{12} & d_{13} & d_{14} & \cdots \\ d_{12} & 0 & d_{23} & d_{24} & \cdots \\ d_{13} & d_{23} & 0 & d_{34} & \cdots \\ d_{14} & d_{24} & d_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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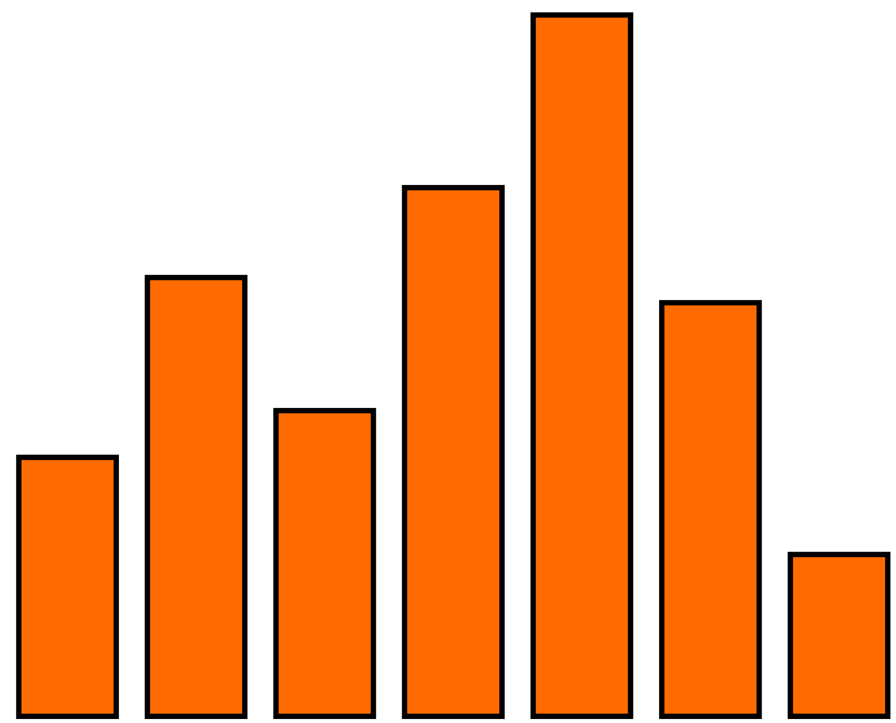
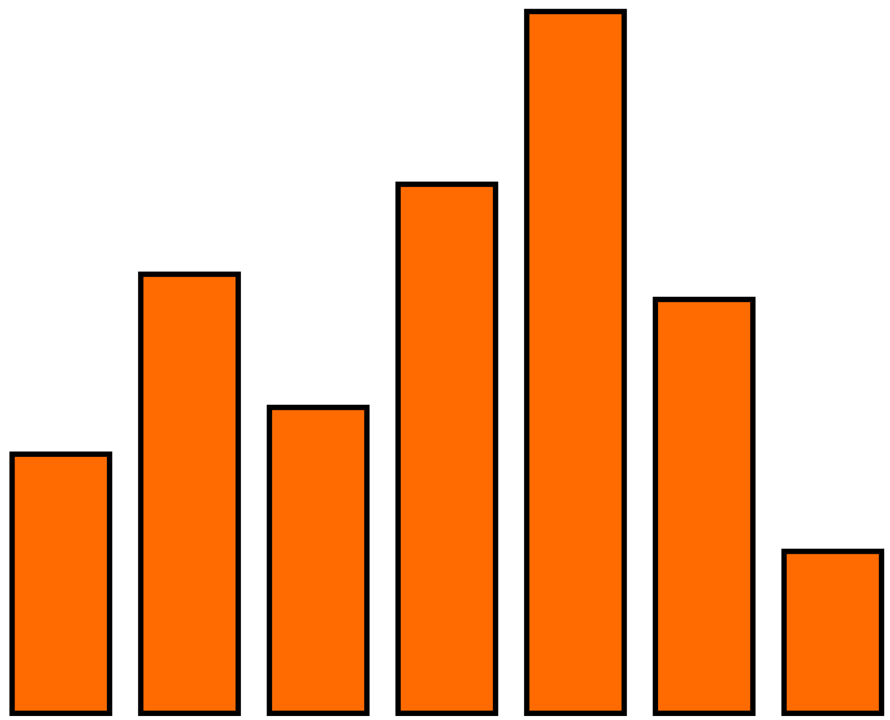


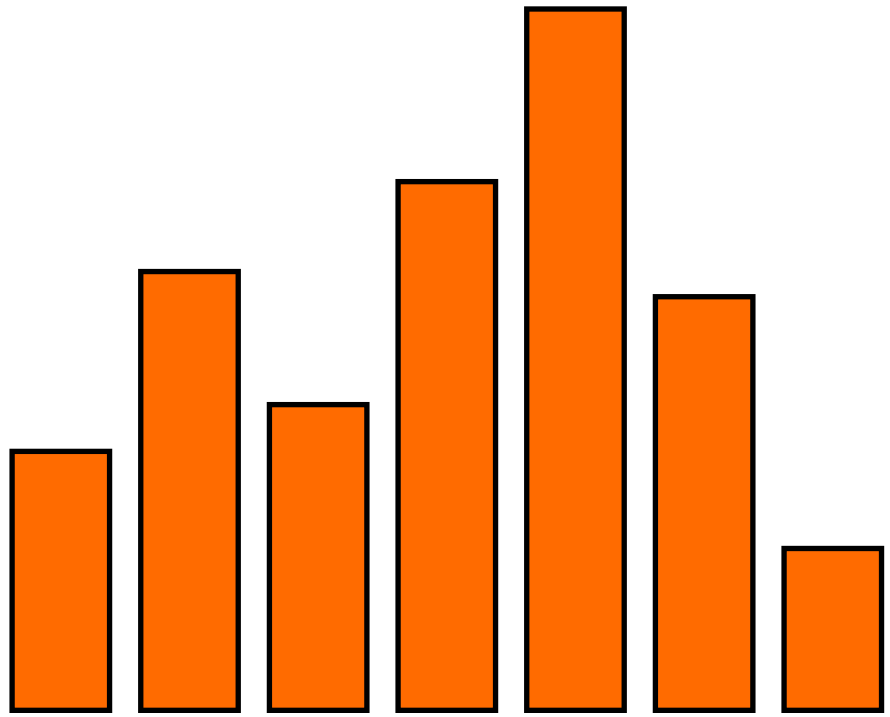
Shape Distributions [Osada-et-al]

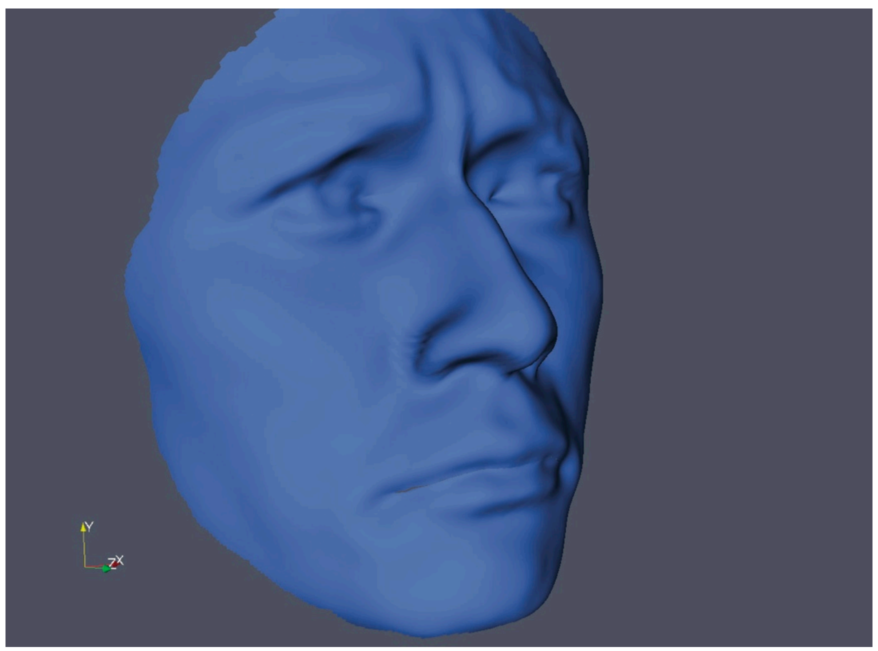
$$\begin{pmatrix} 0 & d_{12} & d_{13} & d_{14} & \cdots \\ d_{12} & 0 & d_{23} & d_{24} & \cdots \\ d_{13} & d_{23} & 0 & d_{34} & \cdots \\ d_{14} & d_{24} & d_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

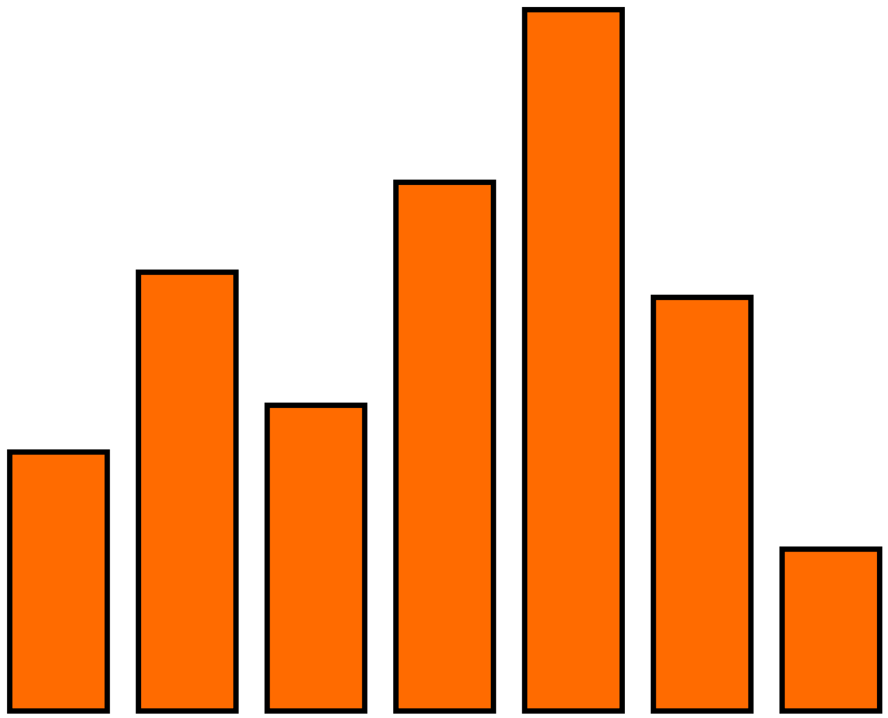


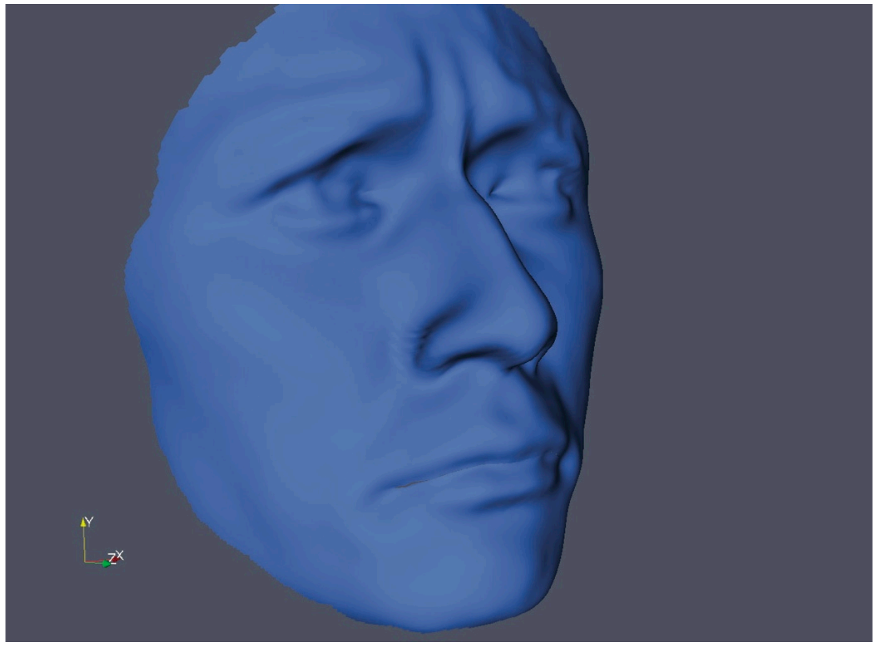
Shape Distributions [Osada-et-al]

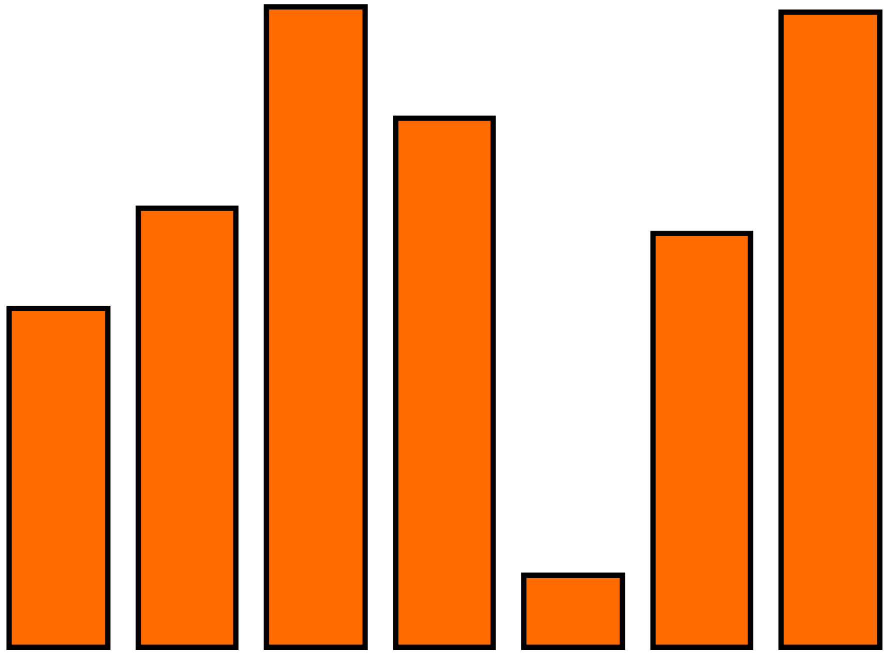
$$\begin{pmatrix} 0 & d_{12} & d_{13} & d_{14} & \cdots \\ d_{12} & 0 & d_{23} & d_{24} & \cdots \\ d_{13} & d_{23} & 0 & d_{34} & \cdots \\ d_{14} & d_{24} & d_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

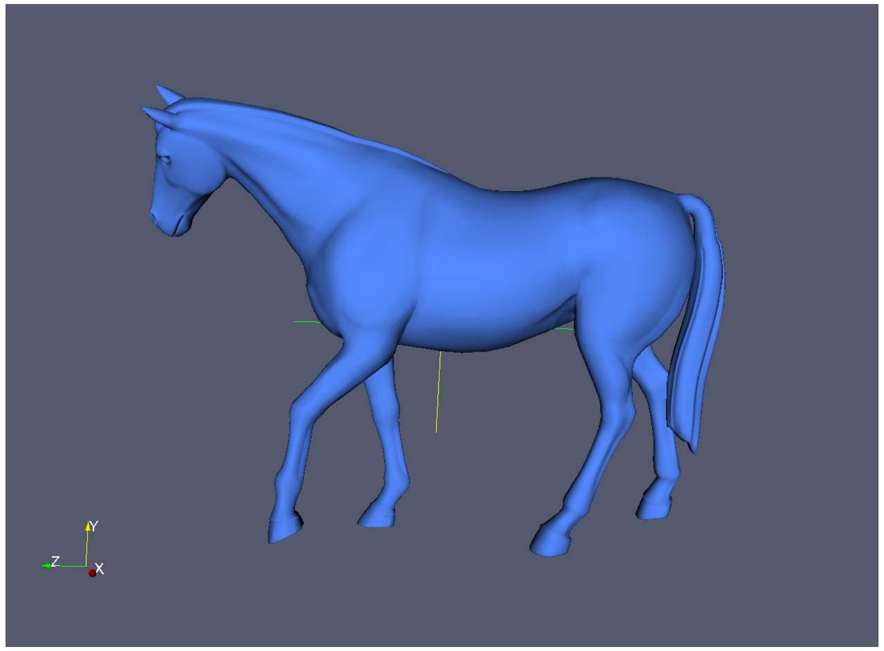


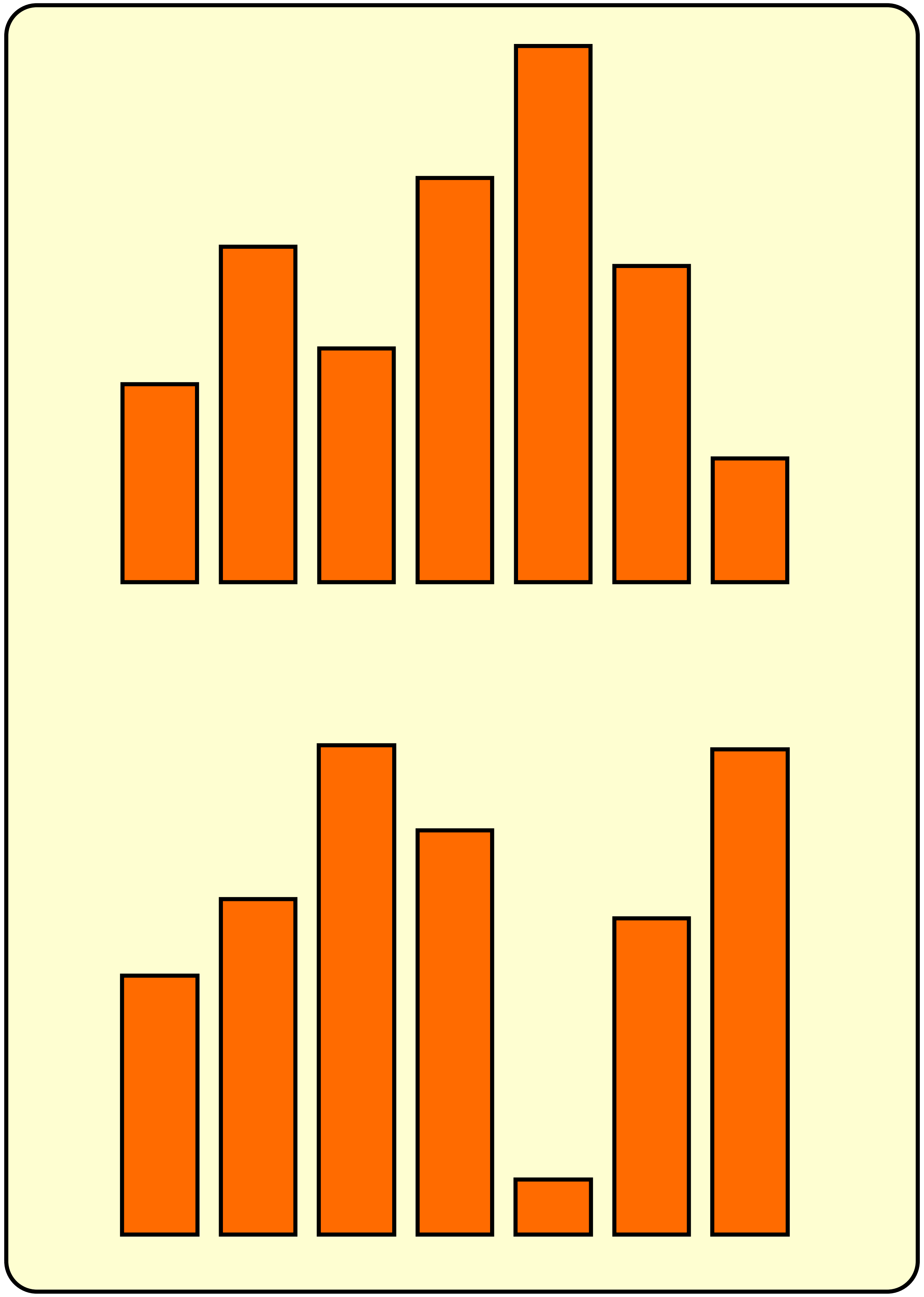


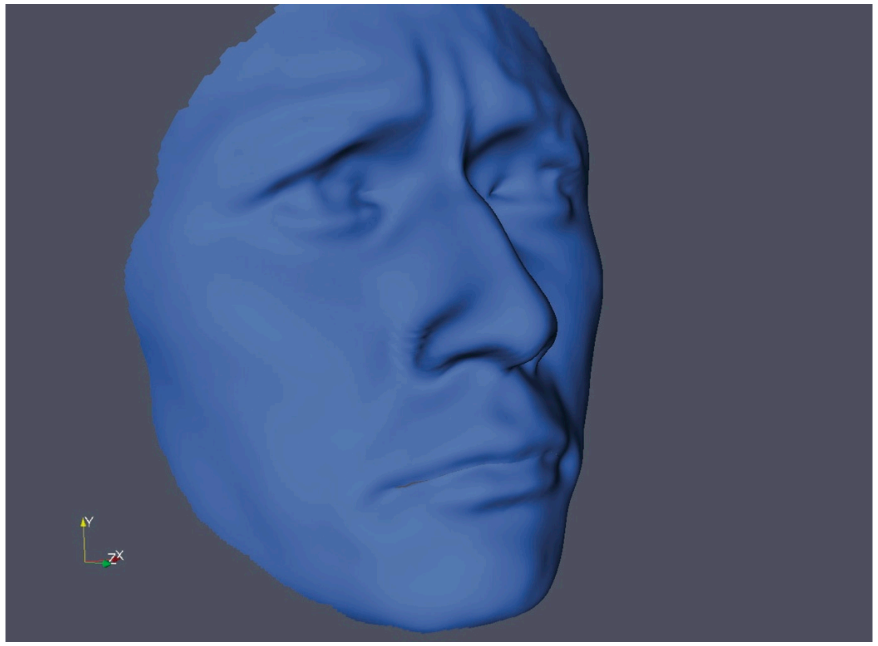


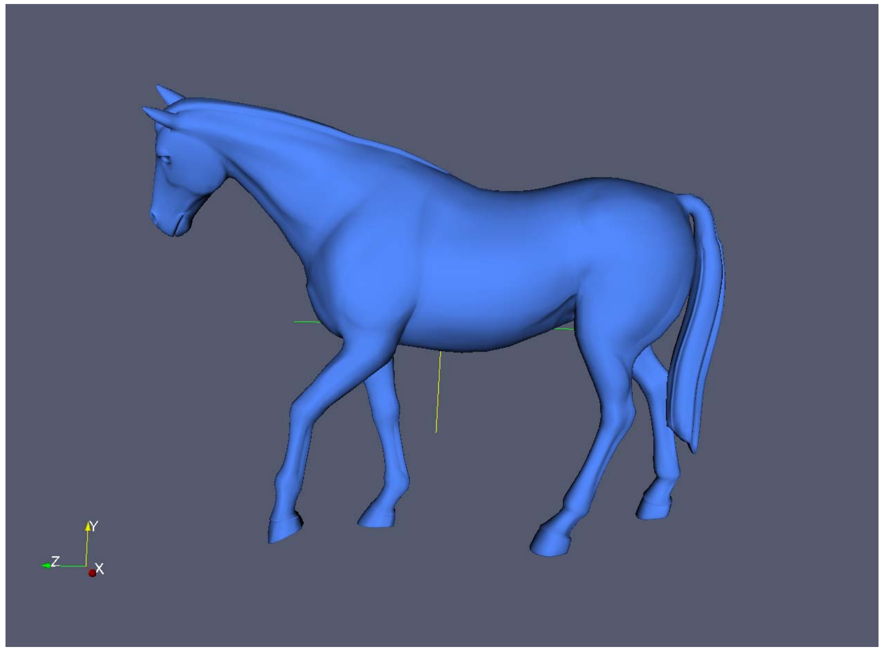






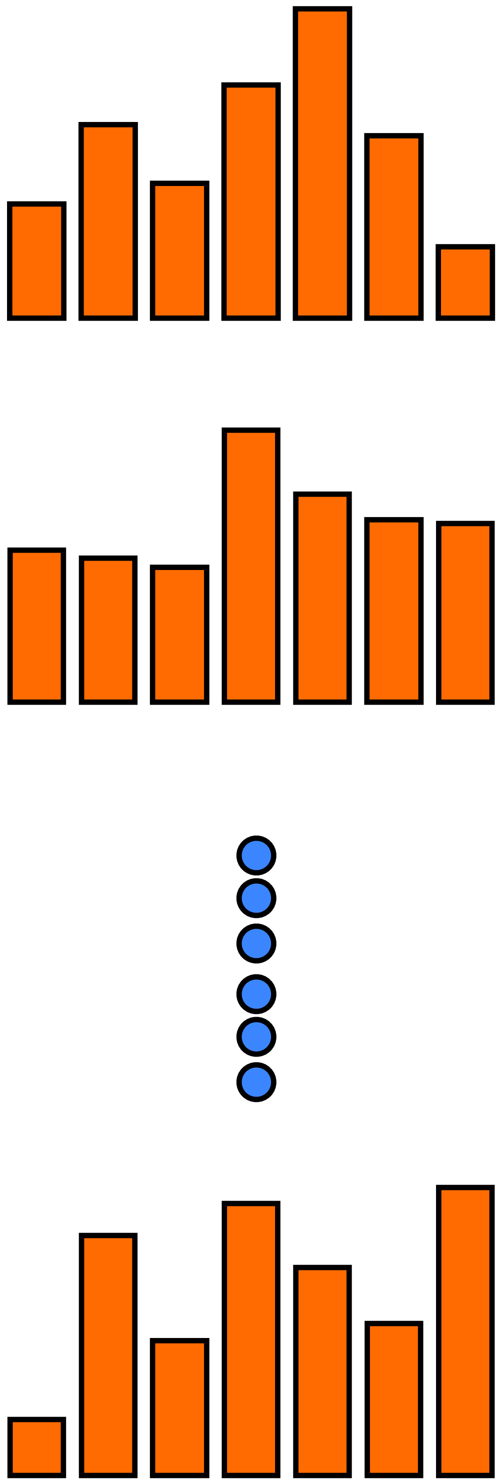






Shape Contexts

$$\begin{pmatrix} 0 & d_{12} & d_{13} & d_{14} & \cdots \\ d_{12} & 0 & d_{23} & d_{24} & \cdots \\ d_{13} & d_{23} & 0 & d_{34} & \cdots \\ d_{14} & d_{24} & d_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$



Shape Contexts

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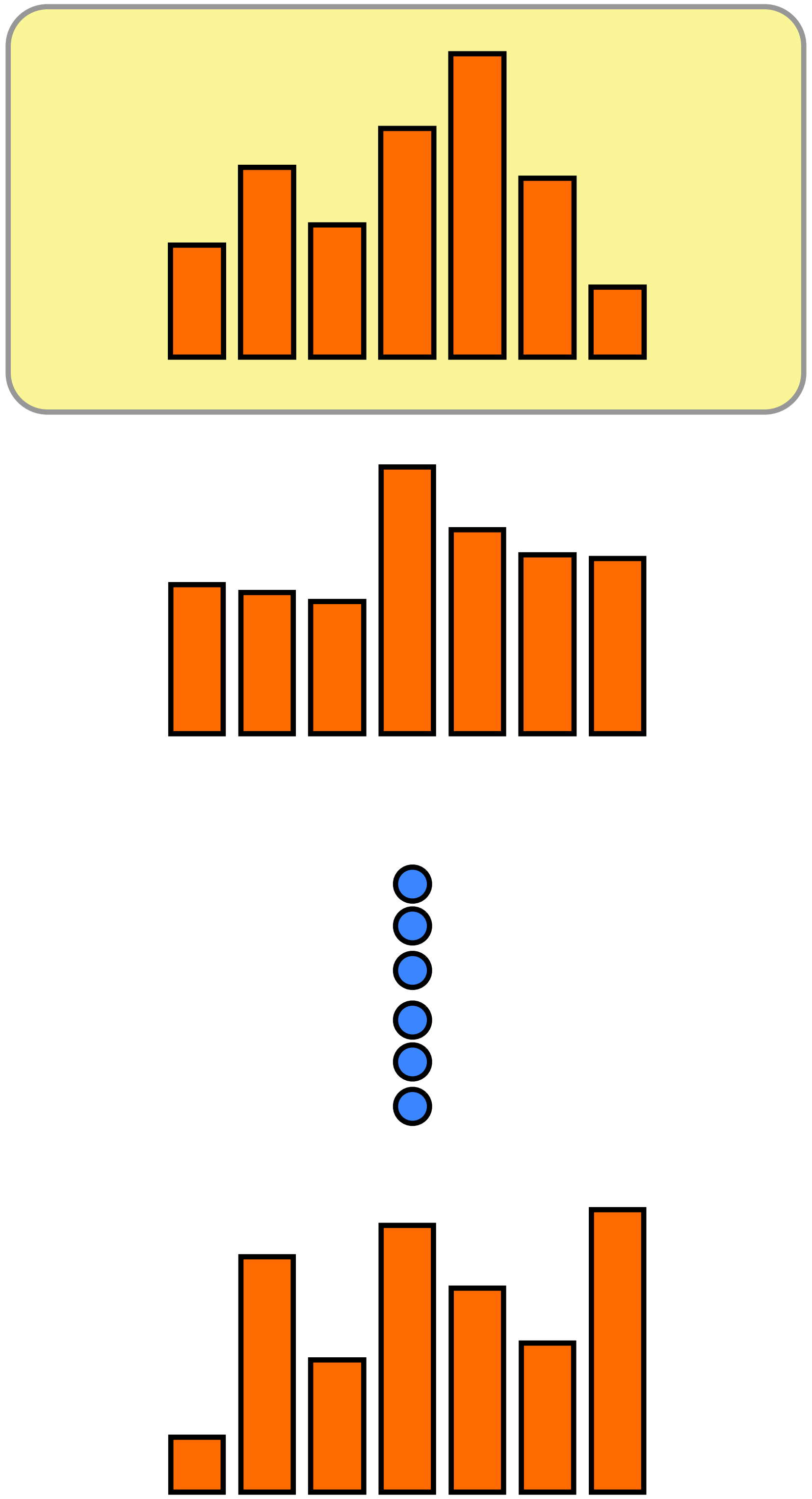

Shape Contexts

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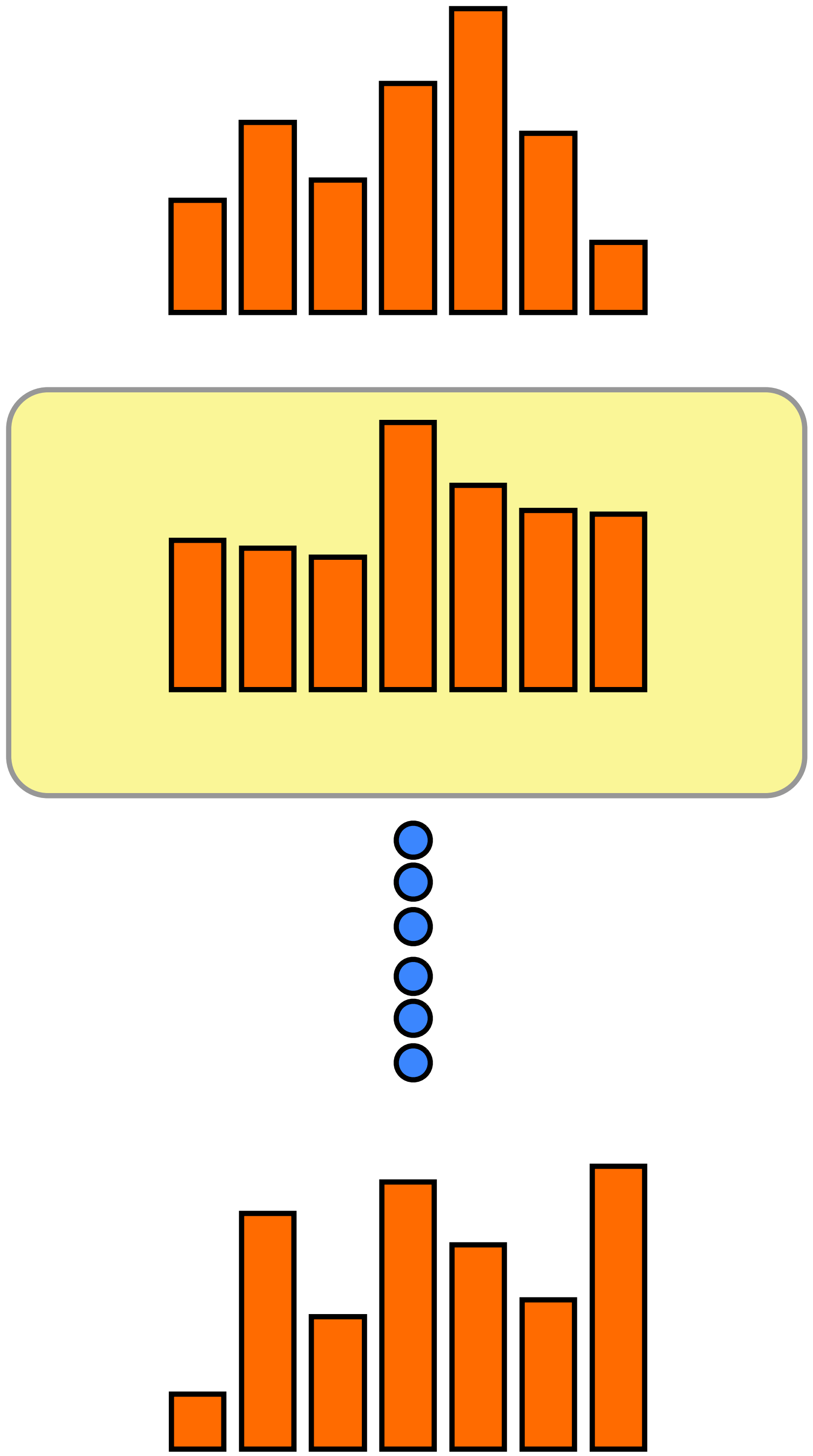


Connections with other approaches

- Shape Distributions [**Osada-et-al**]
- Shape contexts [**SC**]
- Hamza-Krim, Hilaga et al approach [**HK**]
- Rigid isometries invariant Hausdorff [**Goodrich**]
- Gromov-Hausdorff distance [**MS04**] [**MS05**]
- Elad-Kimmel idea [**EK**]
- Topology based methods

What lies ahead: mm-spaces

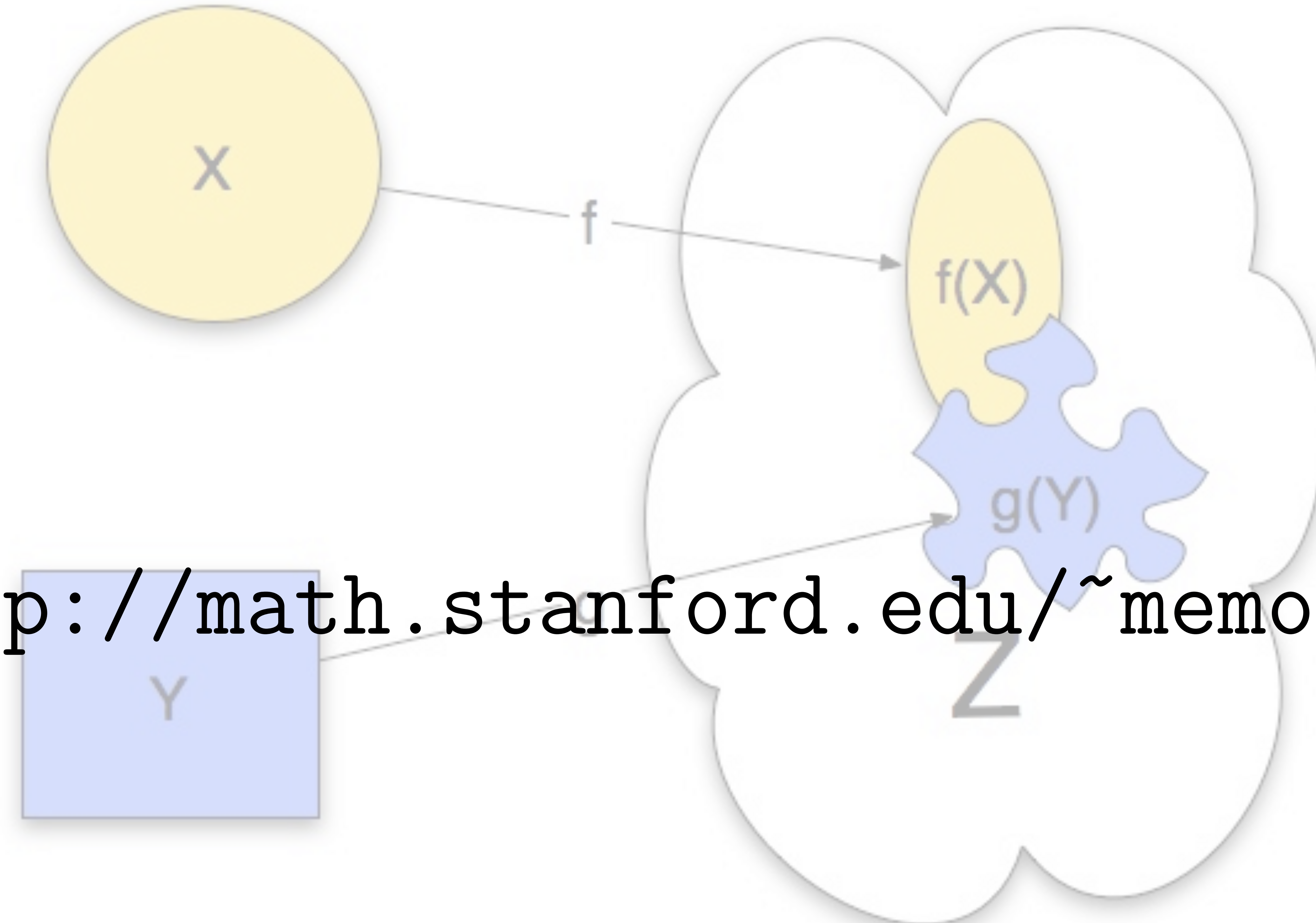
Shapes as mm-spaces, [M07]

Remember:

$$(X, d_X, \mu_X)$$

1. Specify representation of shapes.
2. Identify invariances that you want to mod out.
3. Describe notion of isomorphism between shapes (this is going to be the zero of your metric)
4. Come up with a *metric* between shapes (in the representation of 1.)

- Now we are talking of triples (X, d_X, μ_X) where X is a set, d_X a metric on X and μ_X a probability measure on X.
- These objects are called *measure metric spaces*, or mm-spaces for short.
- two mm-spaces X and Y are deemed *equal* or *isomorphic* whenever there exists an isometry $\Phi : X \to Y$ s.t. $\mu_Y(B) = \mu_X(\Phi^{-1}(B)$ for all (measurable) sets $B \subset Y$.



http://math.stanford.edu/~memoli