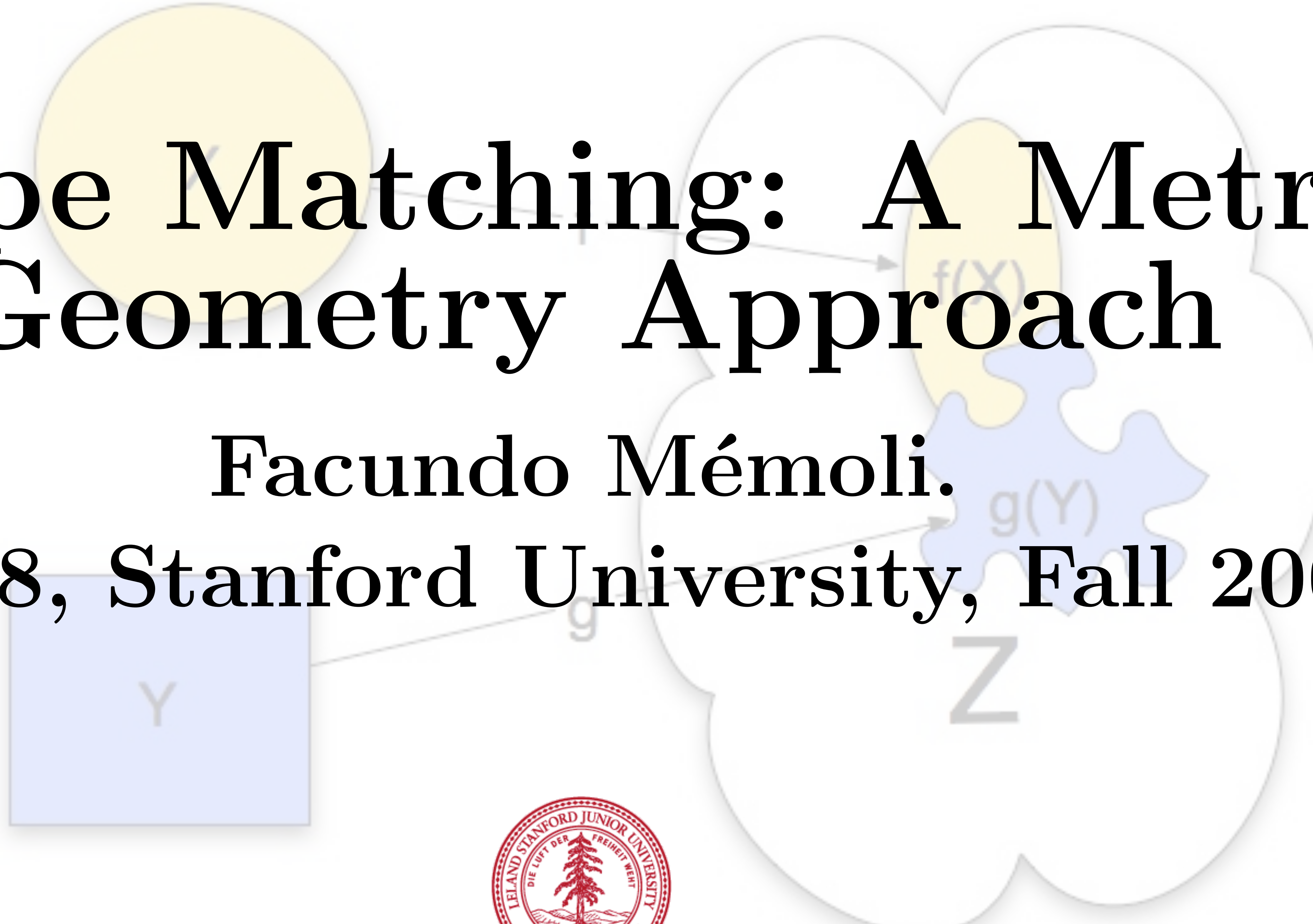
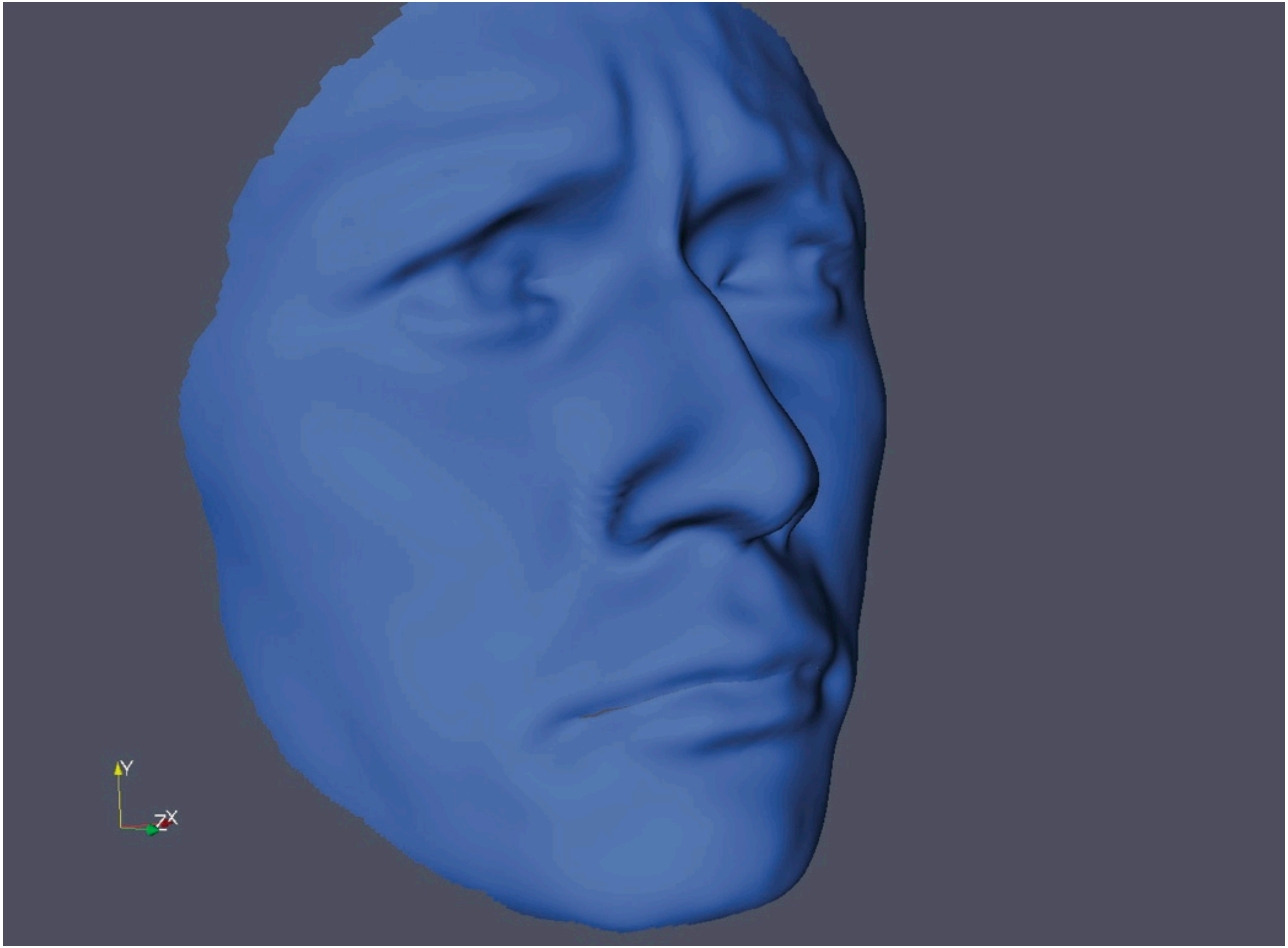
Shape Matching: A Metric Geometry Approach

Facundo Mémoli.

CS 468, Stanford University, Fall 2008.





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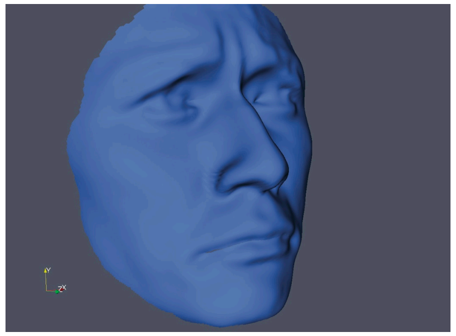

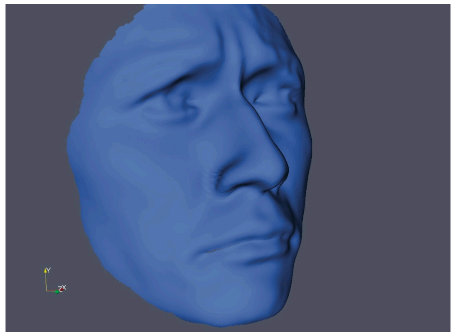

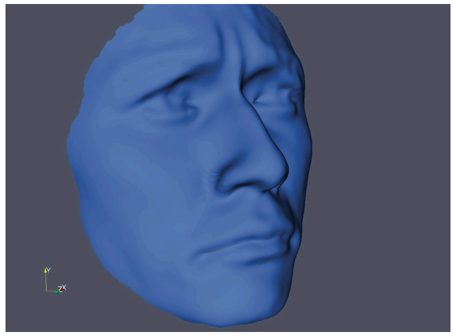

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Shape Distributions [Osada-et-al]

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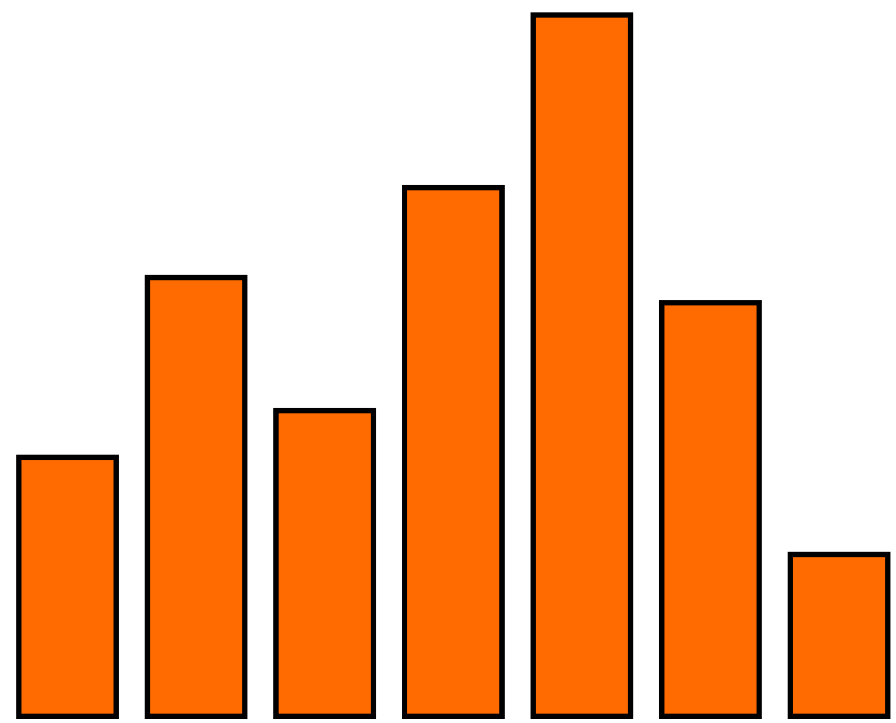

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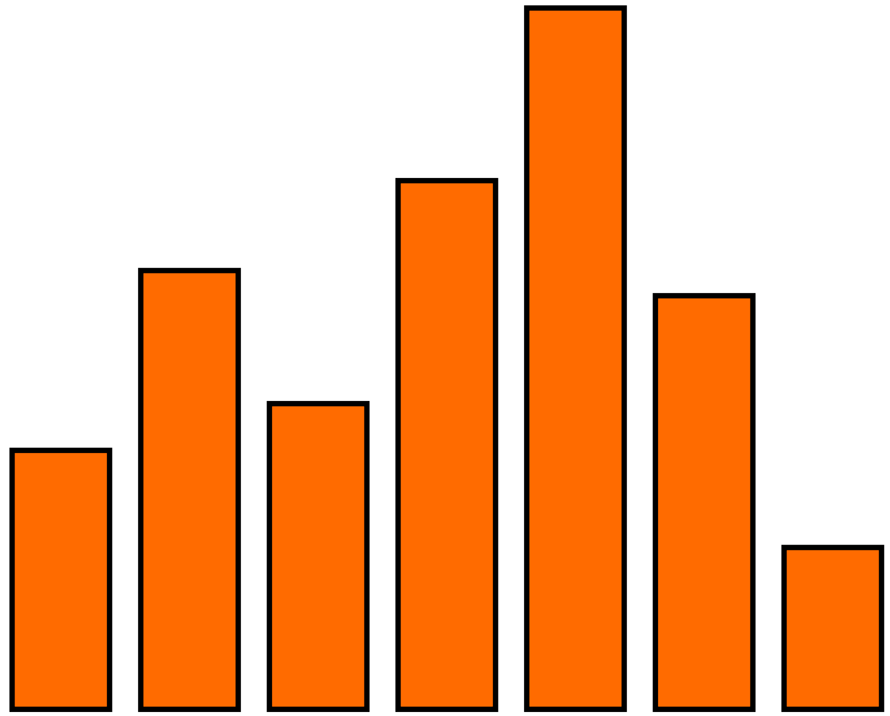

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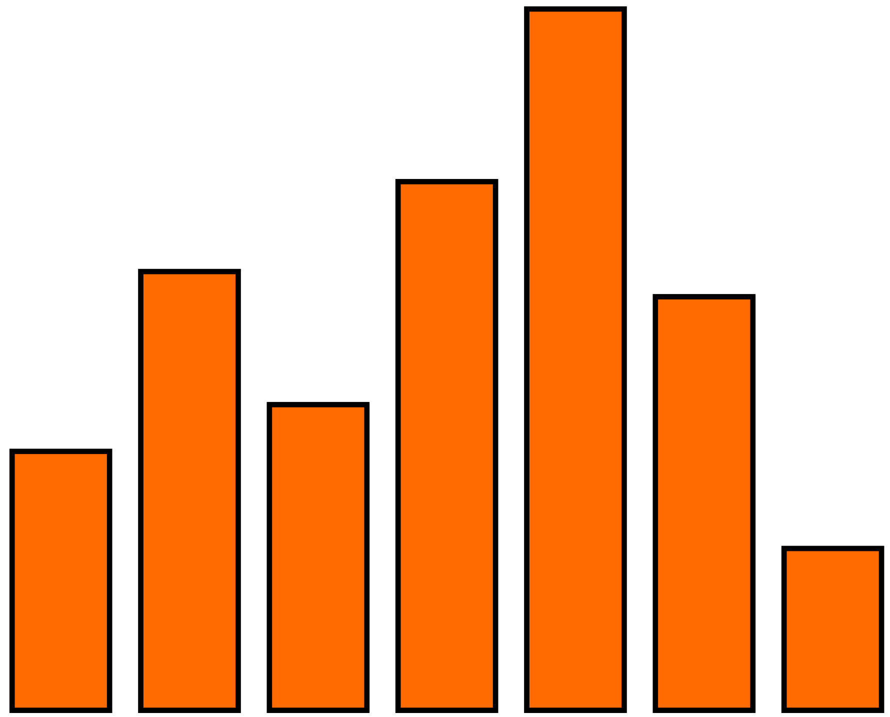

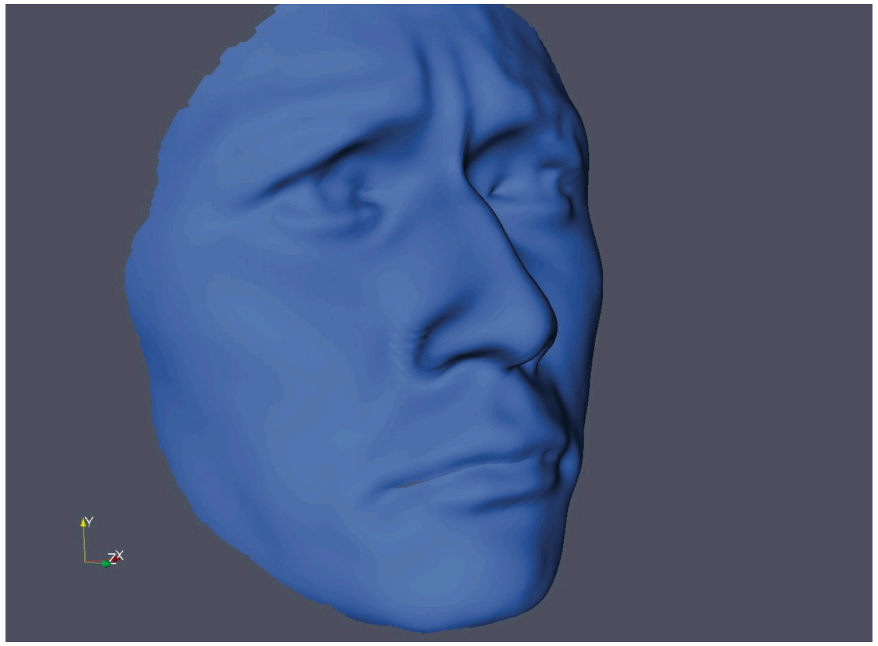

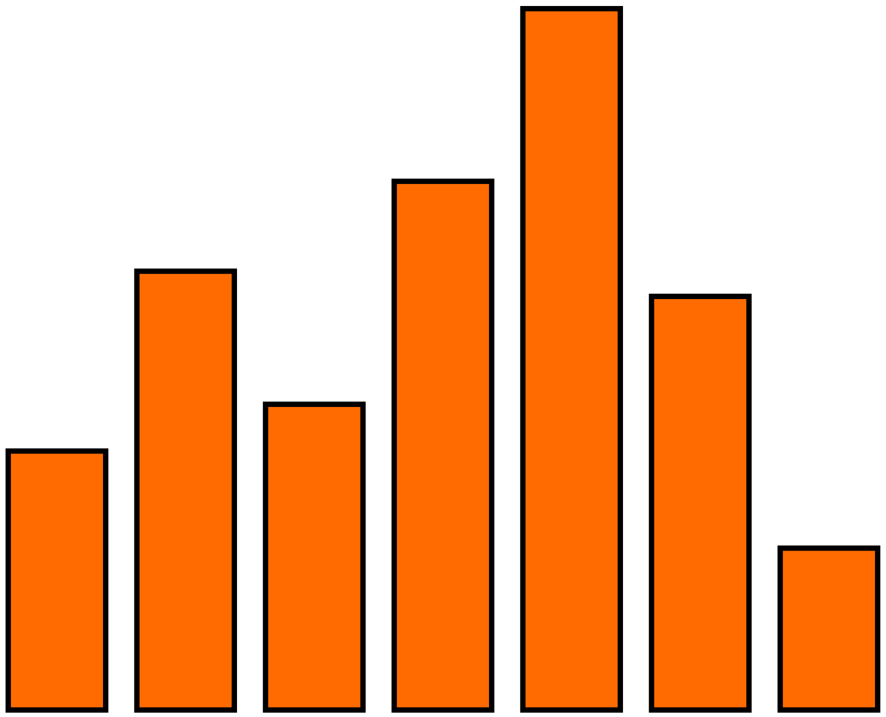

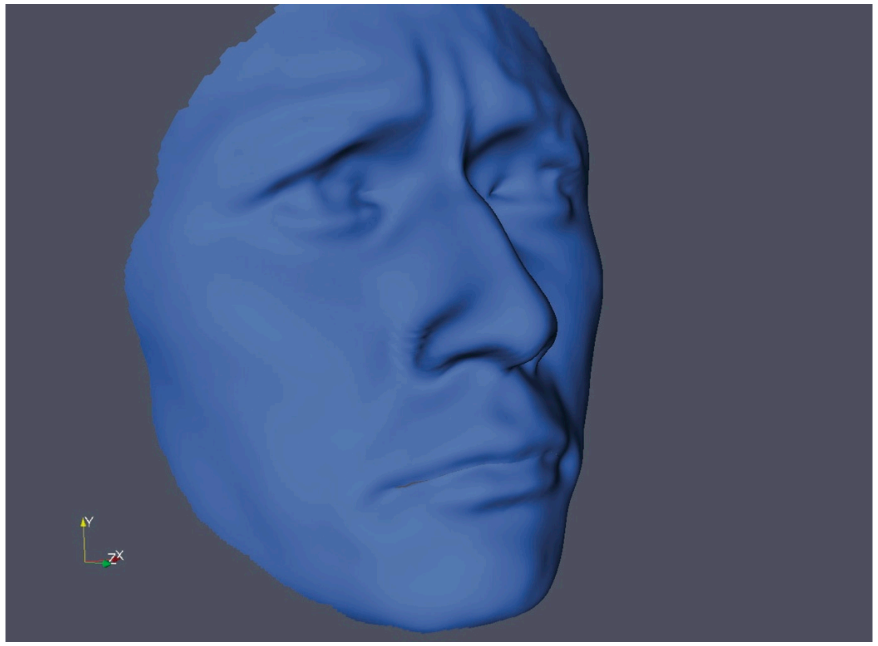

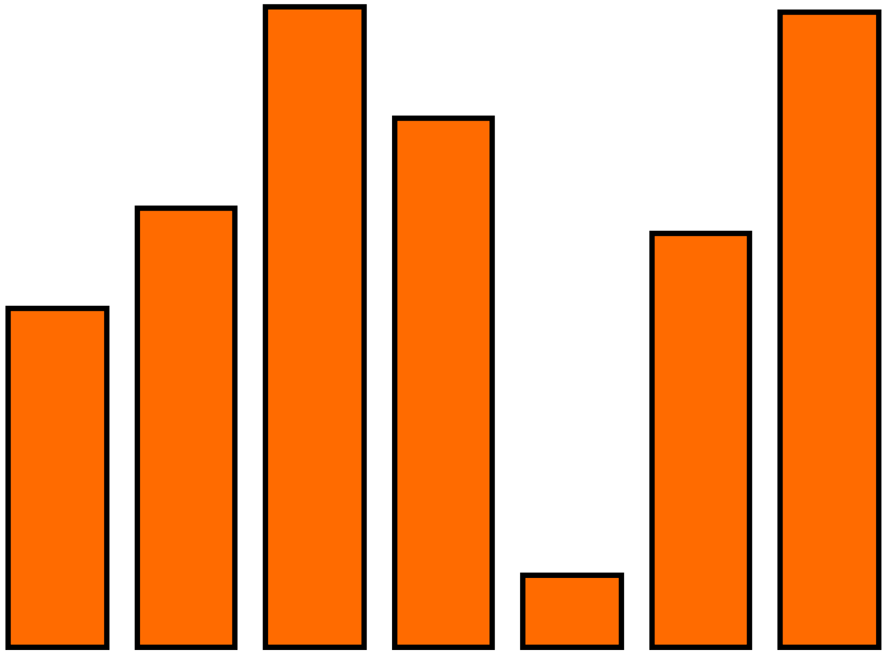

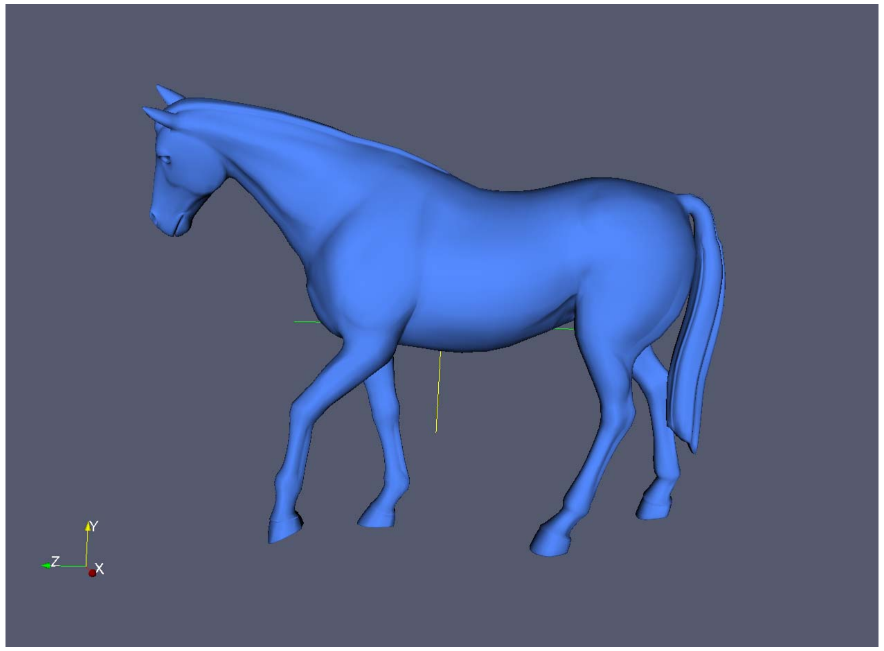

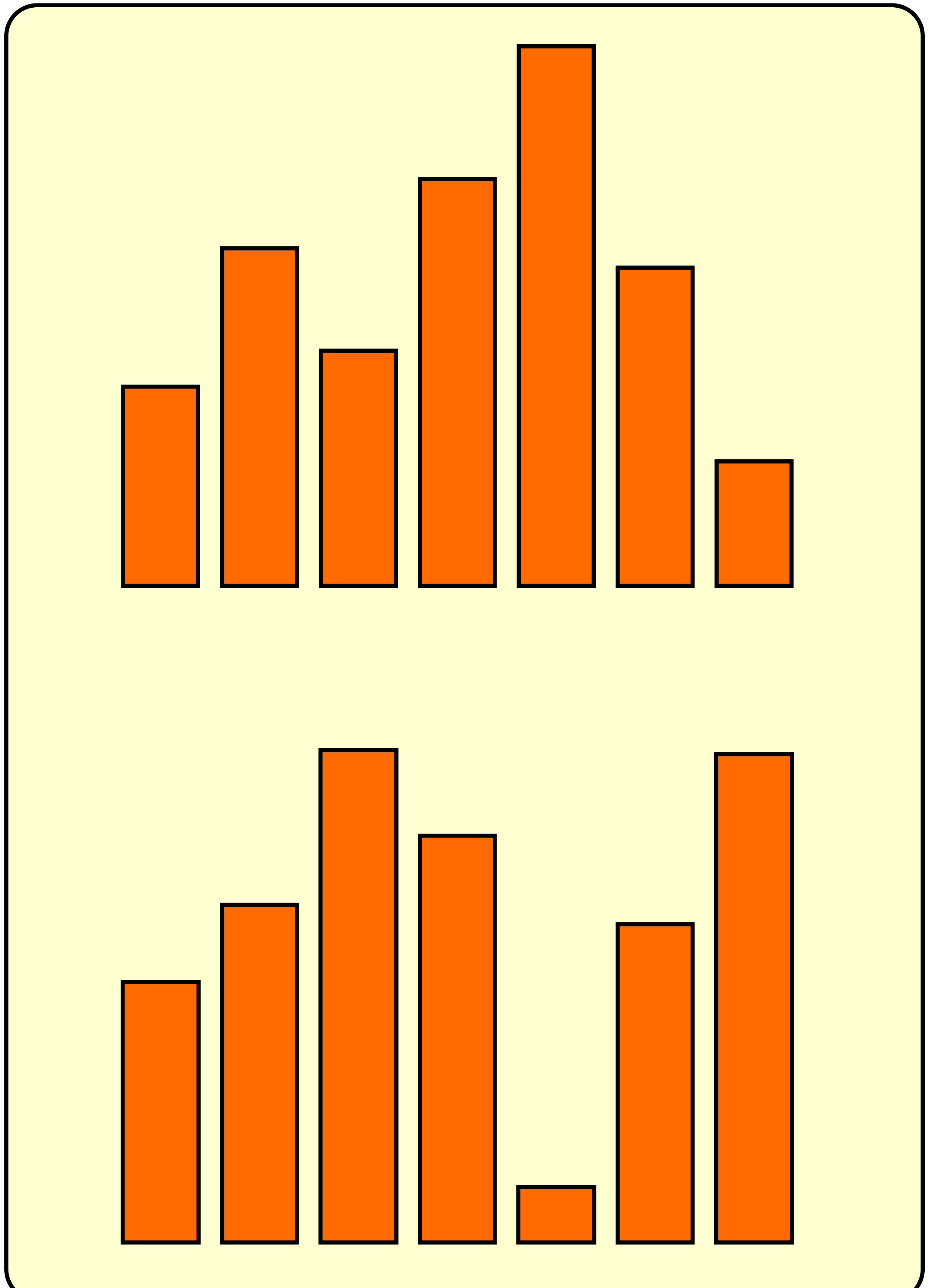

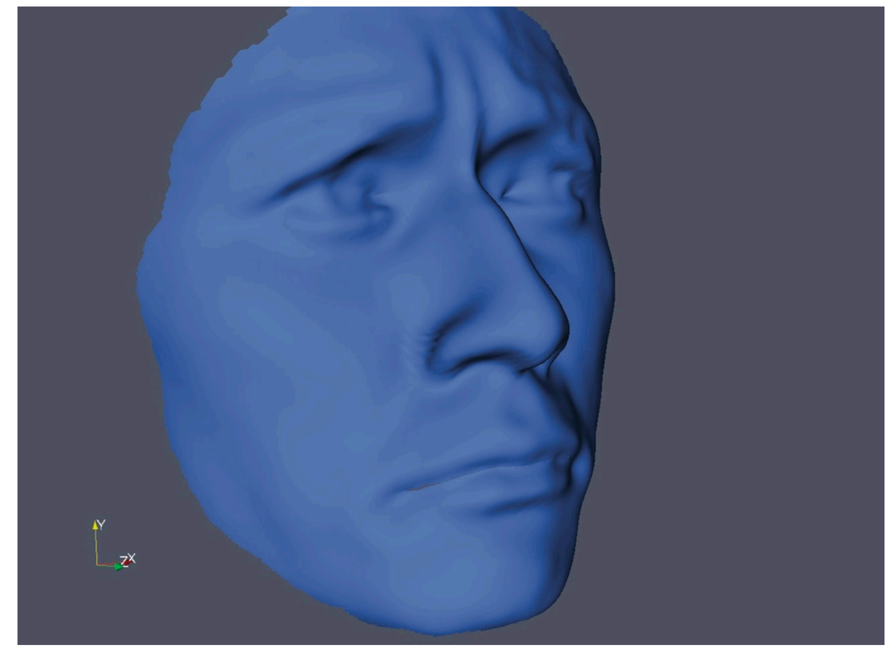

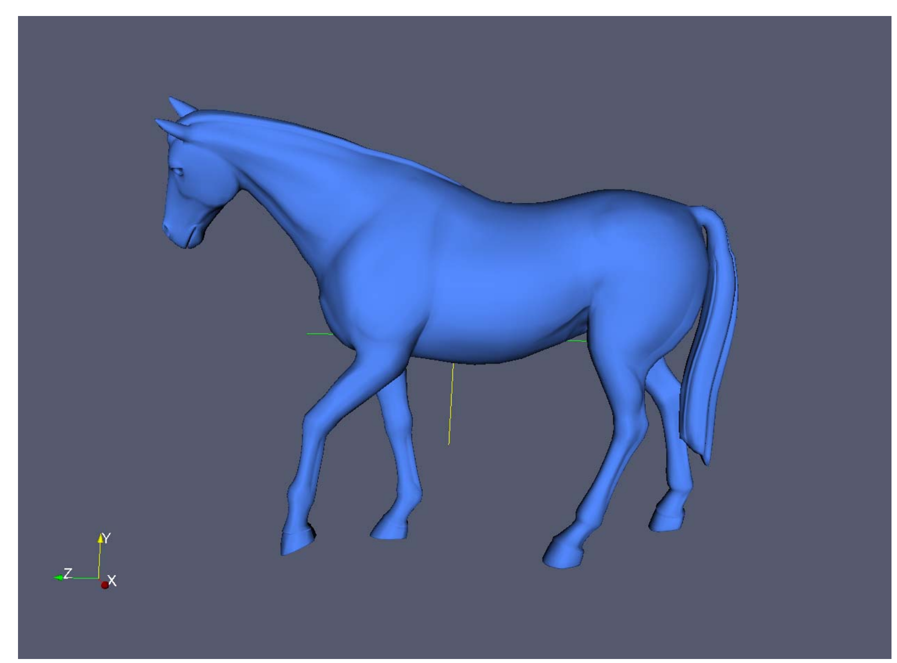

Shape Contexts

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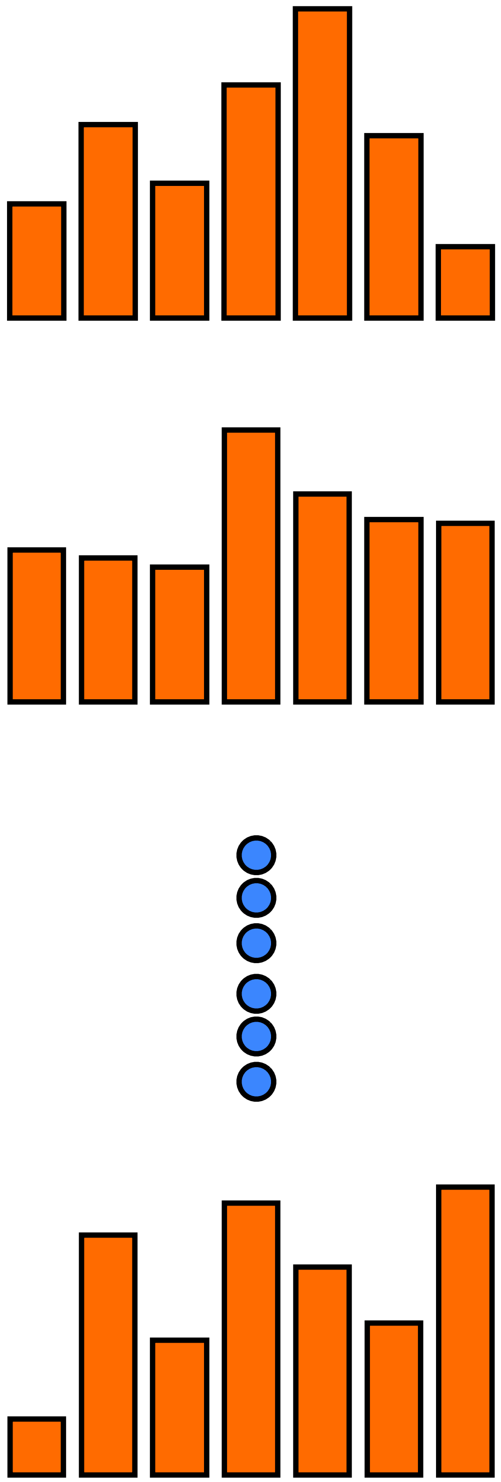

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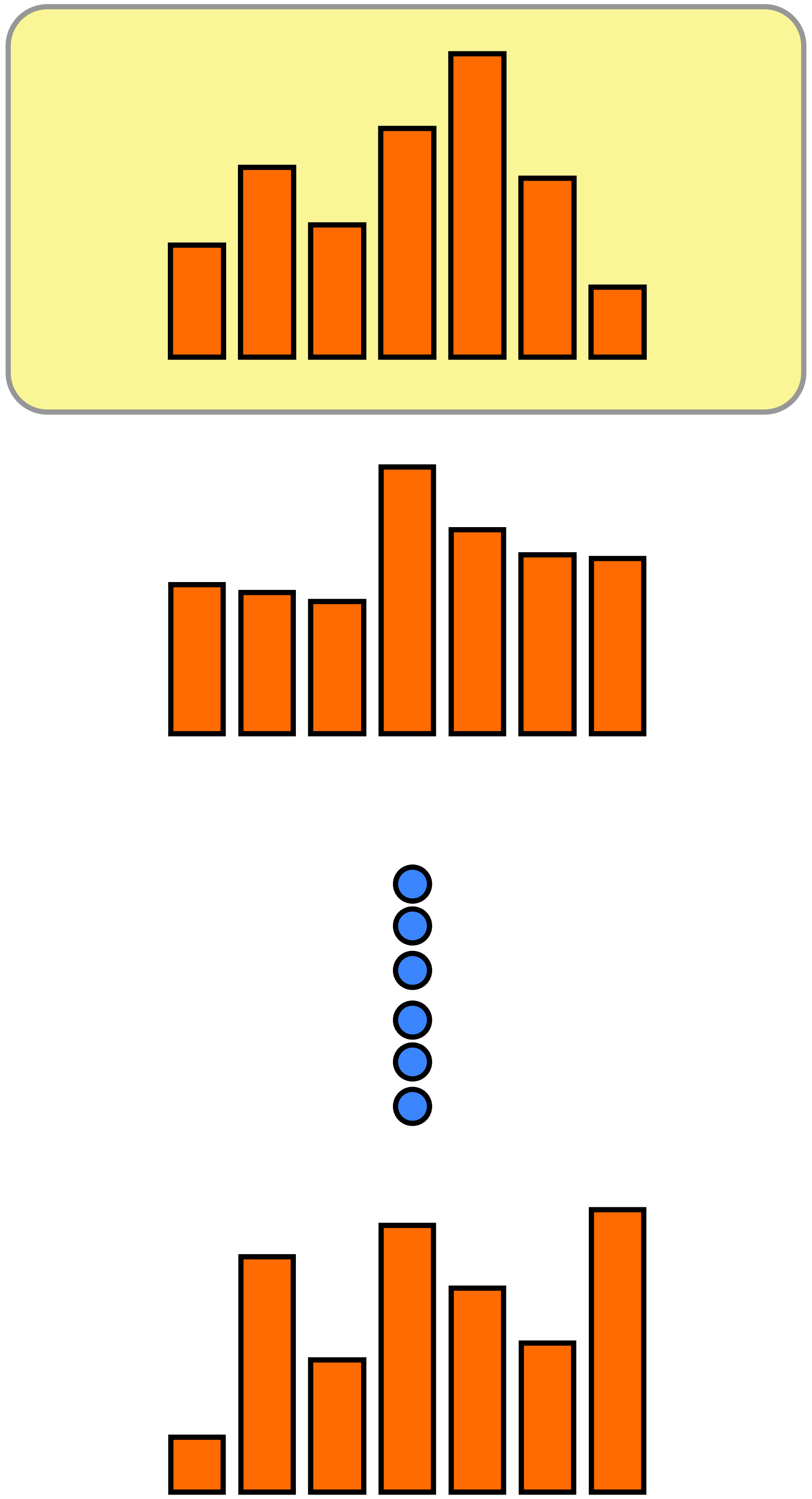

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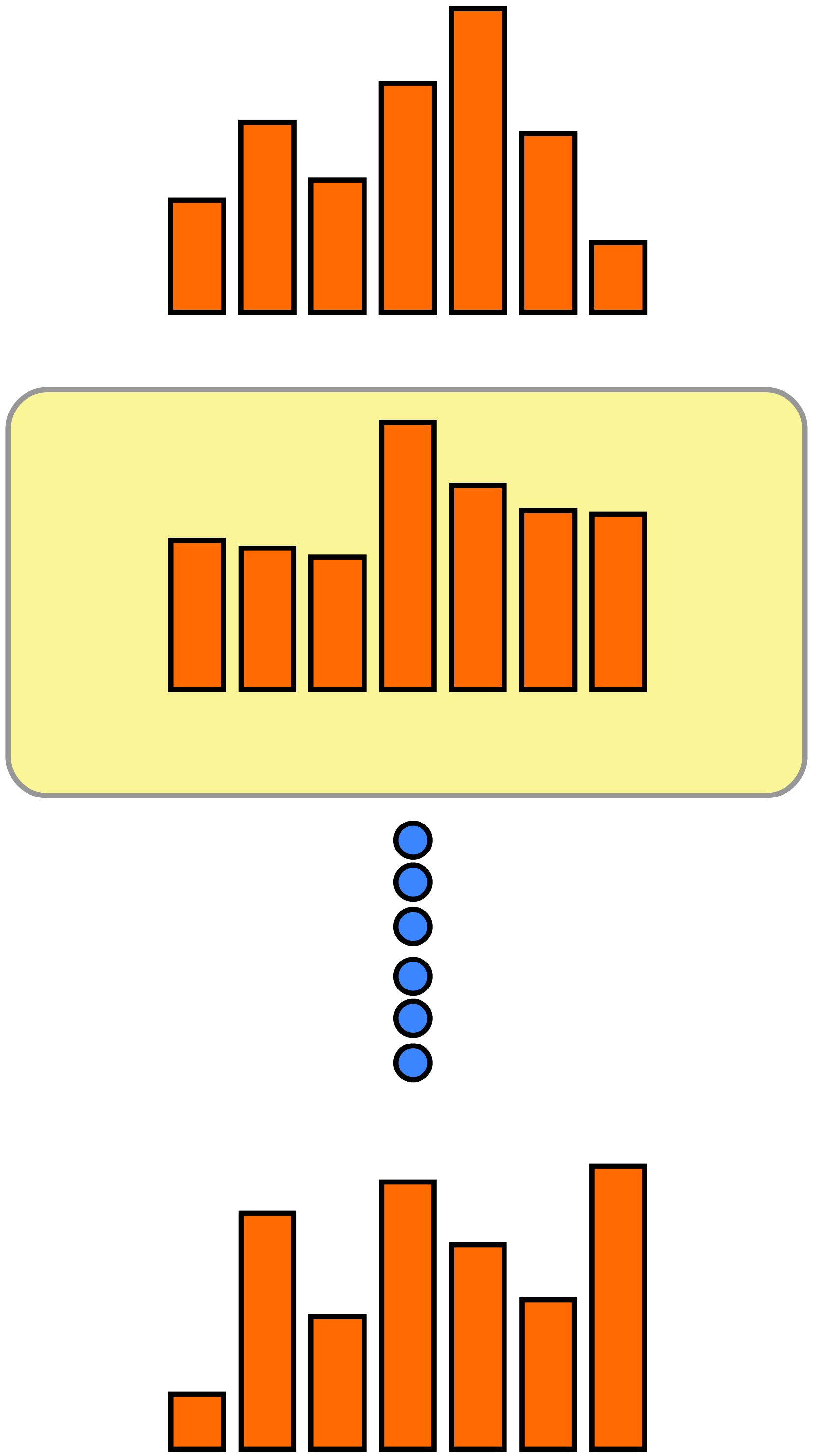

Hamza-Krim

$$\begin{pmatrix} 0 & d_{12} & d_{13} & d_{14} & \dots \\ d_{12} & 0 & d_{23} & d_{24} & \dots \\ d_{13} & d_{23} & 0 & d_{34} & \dots \\ d_{14} & d_{24} & d_{34} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\frac{\sum_j d_{1,j}}{N}$$

$$\frac{\sum_j d_{2,j}}{N}$$

$$\vdots$$

$$\underline{\sum_j \quad {}_{N,j}}$$

Hamza-Krim

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Hamza-Krim

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$$\frac{\sum_j d_{2,j}}{N}$$

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Shapes as mm-spaces, [M07]

$$(X, d_X, \mu_X)$$

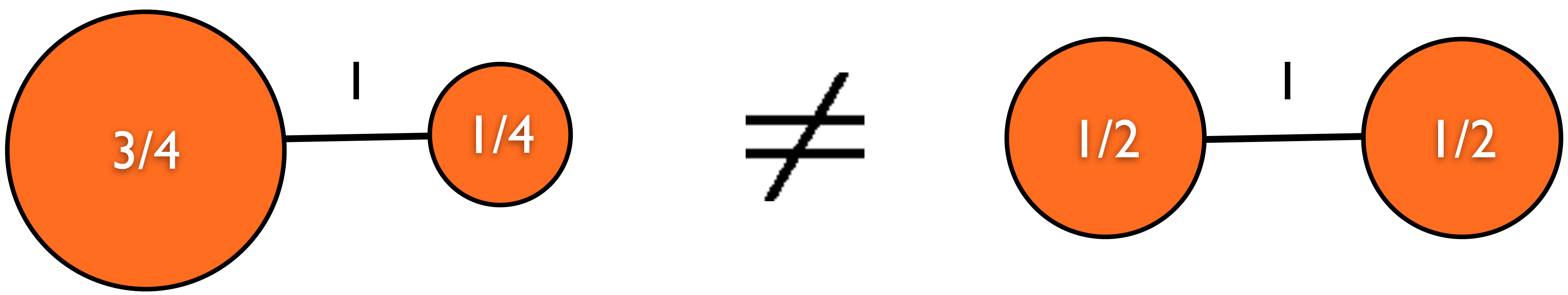
Remember:

1. Specify representation of shapes.
2. Identify invariances that you want to mod out.
3. Describe notion of isomorphism between shapes (this is going to be the zero of your metric)
4. Come up with a *metric* between shapes (in the representation of 1.)

- Now we are talking of triples (X, d_X, μ_X) where X is a set, d_X a metric on X and μ_X a probability measure on X.
- These objects are called *measure metric spaces*, or mm-spaces for short.
- two mm-spaces X and Y are deemed *equal* or *isomorphic* whenever there exists an isometry $\Phi : X \to Y$ s.t. $\mu_Y(B) = \mu_X(\Phi^{-1}(B)$ for all (measurable) sets $B \subset Y$.

Remember

Now, one works with **mm-spaces**: triples (X, d, ν) where (X, d) is a compact metric space and ν is a Borel probability measure. Two mm-spaces are *isomorphic* iff there exists isometry $\Phi : X \to Y$ s.t. $\mu_X(\Phi^{-1}(B)) = \mu_Y(B)$ for all measurable $B \subset Y$.



Shape signatures for mm-spaces Let (X, d_X, μ_X) be an mm-space.

- **Shape Distributions [osada]**: construct histogram of interpoint distances,
$$F_X : \mathbb{R} \to [0,1]$$
given by
$$t \mapsto \mu_X \otimes \mu_X \left(\{(x,x')|\, d_X(x,x') \le t\}\right)$$

- **Shape Contexts [BK,BK-1]**: at each $x \in X$, construct histogram of $d_X(x,\cdot)$,
$$C_X : X \times \mathbb{R} \to [0,1]$$
given by
$$(x,t) \mapsto \mu_X \left(\{x'|\, d(x,x') \le t\}\right)$$

- **Hamza-Krim** [HK-01]: Let $p \in [1,\infty]$. Then, at each $x \in X$ compute mean distance to rest of points,
$$s_{X,p} : X \to \mathbb{R}$$
$$x \mapsto \left(\int_X d_X^p(x,x')\mu_X(dx')\right)^{1/p}$$

Shape Distributions [osada]: construct histogram of interpoint distances,

$$F_X : \mathbb{R} \to [0,1]$$

given by

$$t \mapsto \mu_X \otimes \mu_X \left(\{(x,x') | \, d_X(x,x') \leqslant t\} \right).$$

For each $t \in [0,1]$ let $A_t \subset X \times X$ be given by

$$A_t = \{(x,x') | \, d_X(x,x') \leqslant t\}.$$

Then,

$$F_X(t) = \mu_X \otimes \mu_X(A_t) = \sum_{(x,x') \in A_t} \mu_X(x) \, \mu_X(x').$$

Note that

- $A_0 = \mathrm{diag}(A \times X) = \{(x,x) | \, x \in X\}$ and thus, $F_X(0) = \sum_{x \in X} (\mu_X(x))^2$. For uniform distribution, $F_X(0) = \frac{1}{\#X}$.
- For $T \geqslant \mathbf{diam}\,(X)$, $A_t = X \times X$. Hence, $F_X(T) = \sum_{(x,x') \in X \times X} \mu_X(x) \, \mu_X(x') = \sum_x \mu_X(x) \sum_{x'} \mu_X(x') = 1 \cdot 1 = 1$.

Shape Contexts [**BK,BK-1**]: at each $x \in X$, construct histogram of $d_X(x, \cdot)$,

$$C_X : X \times \mathbb{R} \to [0, 1]$$

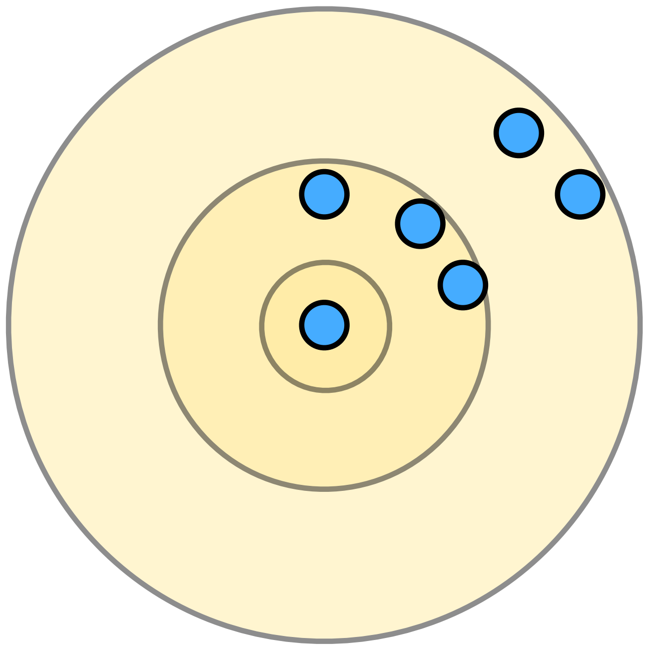
given by

$$(x, t) \mapsto \mu_X\left(\{x' \,|\, d(x, x') \leqslant t\}\right).$$

Clearly, $C_X(x, t) = \mu_X\left(\overline{B}(x, t)\right)$, i.e. $C_X(x, t)$ is combined weights of all the points in X whose distance to x is less than or equal t. In the finite case:

$$C_X(x, t) = \sum_{x' \in \overline{B}(x,t)} \mu_X(x').$$

- $C_X(x, 0) = \mu_X(x)$
- $C_X(x, t) = 1$ for $t \geqslant \mathbf{diam}\,(X)$.



Hamza-Krim (a.k.a. eccentricities) [Hamza-Krim] Let $p \in [1, \infty]$. Then, at each $x \in X$ compute mean distance to rest of points,

$$s_{X,p} : X \to \mathbb{R}$$

$$x \mapsto \left(\int_X d_X^p(x, x') \mu_X(dx') \right)^{1/p}$$

and for $p = \infty$ (if $\operatorname{supp}[\mu_X] = X$),

$$x \mapsto \max_{x' \in X} d_X(x, x').$$

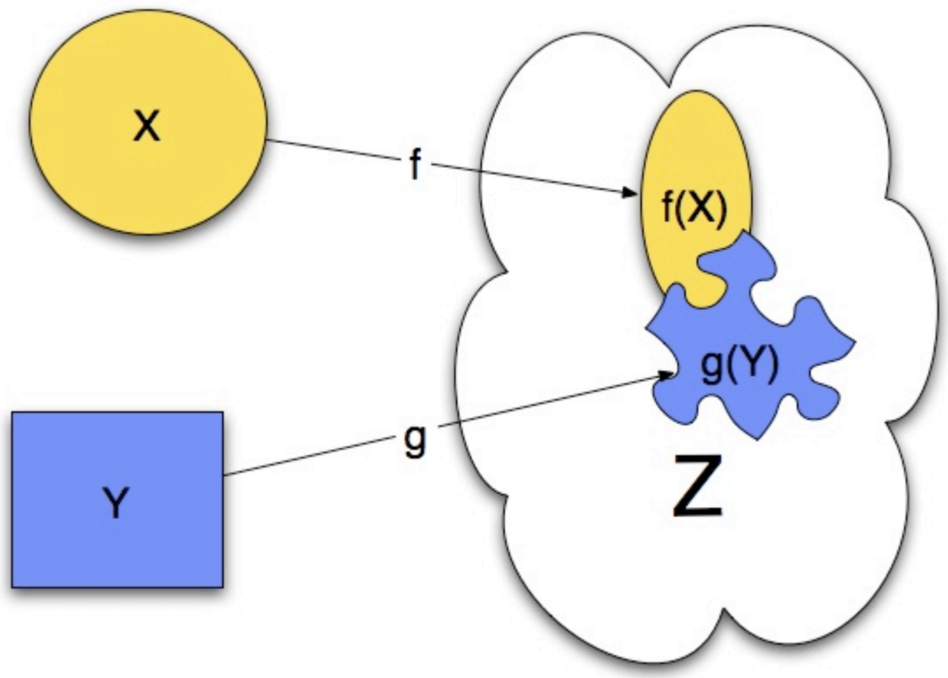
In the finite case, for each $x \in X$ and $p \in [0, \infty]$,

$$s_{X,p}(x) = \left(\sum_{x' \in X} d_X^p(x, x') \right)^{1/p}$$

GH distance

GH: definition

$$d_{\mathcal{GH}}(X,Y) = \inf_{Z,f,g} d_{\mathcal{H}}^{Z}(f(X), g(Y))$$



It is enough to consider $Z = X \sqcup Y$ and then we obtain

$$d_{\mathcal{GH}}(X,Y) = \inf_{d} d_{\mathcal{H}}^{(Z,d)}(X,Y)$$

Recall:
Proposition
Let (X,d) be a compact metric space and $A, B \subset X$ be compact. Then

$$d_{\mathcal{H}}(A,B) = \inf_{R \in \mathcal{R}(A,B)} \|d\|_{L^\infty(R)}$$

Main Properties

1. Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces then

$$d_{\mathcal{GH}}(X, Y) \leq d_{\mathcal{GH}}(X, Z) + d_{\mathcal{GH}}(Y, Z).$$

2. If $d_{\mathcal{GH}}(X, Y) = 0$ and (X, d_X), (Y, d_Y) are compact metric spaces, then (X, d_X) and (Y, d_Y) are isometric.

3. Let $\mathbb{X}_n = \{x_1, \ldots, x_n\} \subset X$ be a finite subset of the compact metric space (X, d_X). Then,

$$d_{\mathcal{GH}}(X, \mathbb{X}_n) \leq d_{\mathcal{H}}(X, \mathbb{X}_n).$$

4. For compact metric spaces (X, d_X) and (Y, d_Y):

$$\begin{aligned} \frac{1}{2}\left|\mathrm{diam}(X) - \mathrm{diam}(Y)\right| &\leq d_{\mathcal{GH}}(X, Y) \\ &\leq \frac{1}{2}\max\left(\mathrm{diam}(X), \mathrm{diam}(Y)\right) \end{aligned}$$

Stability

$$|d_{\mathcal{GH}}(X,Y) - d_{\mathcal{GH}}(X_n, Y_m)| \leq r(X_n) + r(Y_m)$$

for finite samplings $X_n \subset X$ and $Y_m \subset Y$, where $r(X_n)$ and $r(Y_m)$ are the covering radii.

Critique

- Was not able to show connections with (sufficiently many) pre-existing appraches
- Computationally hard: currently only two attempts have been made:
  - [**MS04,MS05**] and [**BBK06**] only for surfaces.
  - [**MS05**] gives probabilistic guarantees for estimator based on sampling parameters.
  - Full generality leads to a hard **combinatorial optimization problem**: QAP.

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Desiderata

- Obtain an L^p version of the GH distance that:
  - retains theoretical underpinnings
  - its implementation leads to easier (continuous, quadratic, with linear constraints) optimization problems
  - can be related to pre-existing approaches (shape contexts, shape distributions, Hamza-Krim,..) via lower/upper bounds.

goal

Gromov-Hausdorff

$$\Downarrow$$

Gromov-<u>Wasserstein</u>

(Kantorovich, Rubinstein, Earth Mover's Distance, Mass Transportation)

correspondences and the Hausdorff distance

Definition [Correspondences]

For sets A and B, a subset $R \subset A \times B$ is a *correspondence* (between A and B) if and and only if

- $\forall\ a \in A$, there exists $b \in B$ s.t. $(a, b) \in R$
- $\forall\ b \in B$, there exists $a \in A$ s.t. $(a, b) \in R$

Let $\mathcal{R}(A, B)$ denote the set of all possible correspondences between sets A and B. Note that in the case $n_A = n_B$, correspondences are larger than bijections.

correspondences

Note that when A and B are finite, $R \in \mathcal{R}(A, B)$ can be represented by a matrix $((r_{a,b})) \in \{0,1\}^{n_A \times n_B}$ s.t.

$$\sum_{a \in A} r_{ab} \geq 1 \ \ \forall b \in B$$

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Proposition

Let (X, d) be a compact metric space and $A, B \subset X$ be compact. Then

$$d_{\mathcal{H}}(A, B) = \inf_{R \in \mathcal{R}(A,B)} \|d\|_{L^\infty(R)}$$

correspondences and measure couplings

Let (A, μ_A) and (B, μ_B) be compact subsets of the compact metric space (X, d) and μ_A and μ_B be **probability measures** supported in A and B respectively.

Definition [Measure coupling] Is a probability measure μ on $A \times B$ s.t. (in the finite case this means $((\mu_{a,b})) \in [0,1]^{n_A \times n_B}$)

- $\sum_{a \in A} \mu_{ab} = \mu_B(b) \;\; \forall b \in B$
- $\sum_{b \in B} \mu_{ab} = \mu_A(a) \;\; \forall a \in A$

Let $\mathcal{M}(\mu_A, \mu_B)$ be the set of all couplings of μ_A and μ_B.
Notice that in the finite case, $((\mu_{a,b}))$ must satisfy $n_A + n_B$ *linear* constraints.

correspondences and measure couplings

Proposition $[(\mu \leftrightarrow R)]$

- Given (A, μ_A) and (B, μ_B), and $\mu \in \mathcal{M}(\mu_A, \mu_B)$, then

$$R(\mu) := \mathrm{supp}(\mu) \in \mathcal{R}(A, B).$$

- König's Lemma. [gives conditions for $R \to \mu$]

Wasserstein distance

$$d_{\mathcal{H}}(A, B) = \inf_{R \in \mathcal{R}(A,B)} \|d\|_{L^\infty(R)}$$

$$\Downarrow \ (R \leftrightarrow \mu)$$

$$d_{\mathcal{W},\infty}(A, B) = \inf_{\mu \in \mathcal{M}(\mu_A,\mu_B)} \|d\|_{L^\infty(R(\mu))}$$

$$\Downarrow \ (L^\infty \leftrightarrow L^p)$$

$$d_{\mathcal{W},p}(A, B) = \inf_{\mu \in \mathcal{M}(\mu_A,\mu_B)} \|d\|_{L^p(A\times B,\mu)}$$

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$$\boxed{d_{\mathcal{W},p}(A,B) = \inf_{\mu \in \mathcal{M}(\mu_A,\mu_B)} \|d\|_{L^p(A\times B,\mu)}}$$

The plan

$$
\begin{tikzcd}
d^{(1)}_{\mathcal{G}\mathcal{H}} \arrow[rr, equal] \arrow[dd] & & d^{(2)}_{\mathcal{G}\mathcal{H}} \arrow[dd] \\
& d_{\mathcal{H}} \arrow[ul] \arrow[ur] \arrow[dd, Rightarrow] & \\
d^{(1)}_{\mathcal{G}\mathcal{W},p} \arrow[rr, equal, "?" near start, "?" near end] & & d^{(2)}_{\mathcal{G}\mathcal{W},p} \\
& d_{\mathcal{W},p} \arrow[uul] \arrow[uur] &
\end{tikzcd}
$$

The plan

$$
\begin{tikzcd}
d^{(1)}_{\mathcal{GH}} \arrow[rr, equal] \arrow[dd] & & d^{(2)}_{\mathcal{GH}} \arrow[dd] \\
& d_{\mathcal{H}} \arrow[ul] \arrow[ur] \arrow[dd, Rightarrow] & \\
d^{(1)}_{\mathcal{GW},p} \arrow[rr, equal, "?" near start, "?" near end] & & d^{(2)}_{\mathcal{GW},p} \\
& d_{\mathcal{W},p} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

The plan

$$
\begin{tikzcd}
d^{(1)}_{\mathcal{GH}} \arrow[rr, equal] \arrow[dd] & & \boxed{d^{(2)}_{\mathcal{GH}}} \arrow[dd] \\
& d_{\mathcal{H}} \arrow[ul] \arrow[ur] \arrow[dd, Rightarrow] & \\
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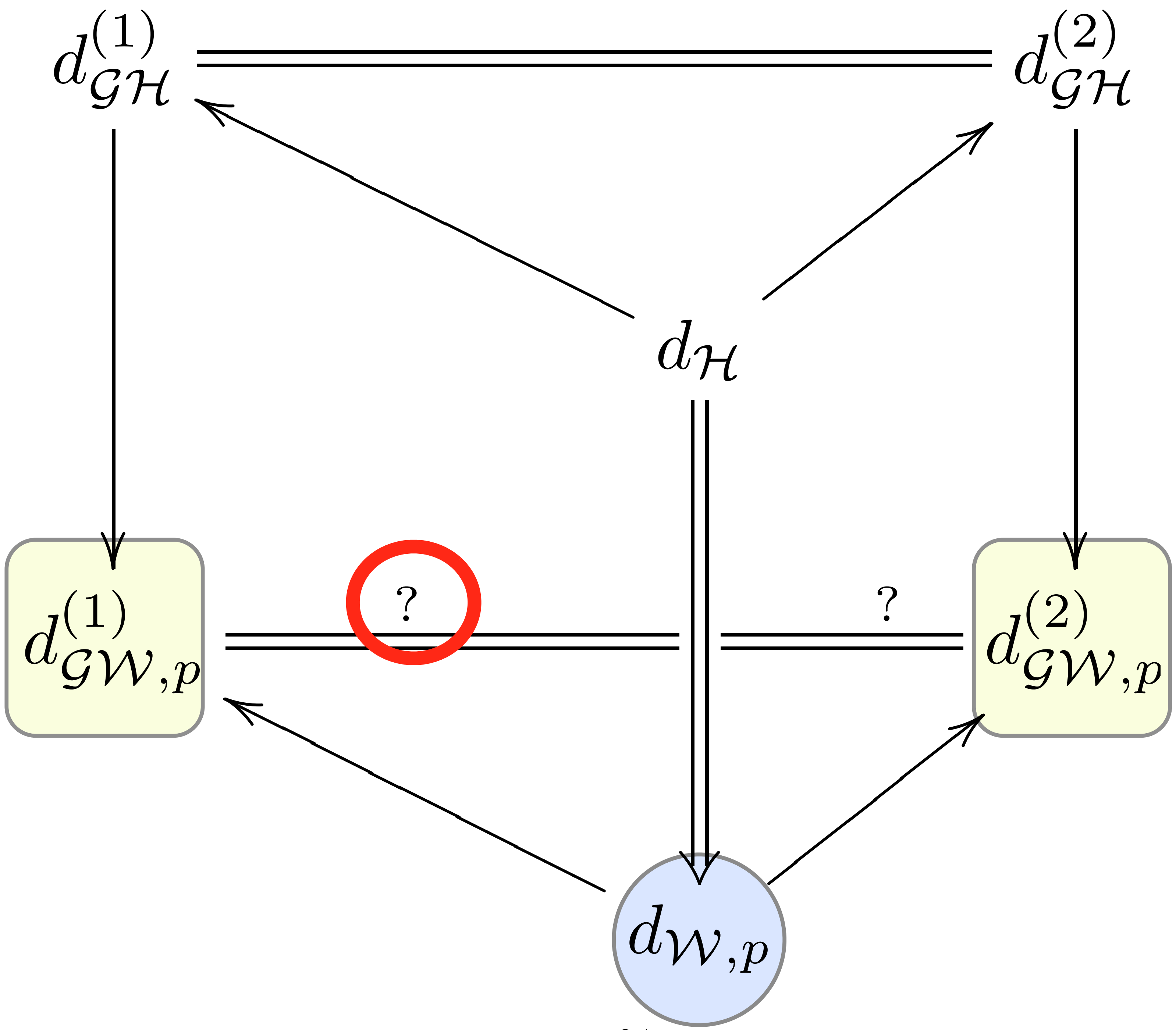
\end{tikzcd}
$$

The plan



$d^{(1)}_{\mathcal{GH}}$
$d^{(2)}_{\mathcal{GH}}$
$d_{\mathcal{H}}$
?
?
$d^{(1)}_{\mathcal{GW},p}$
$d^{(2)}_{\mathcal{GW},p}$
$d_{\mathcal{W},p}$

$$\frac{\mathbf{GH}}{\mathbf{H}} = \frac{\mathbf{GW}}{\mathbf{W}}$$

correspondences and GH distance

The GH distance between (X, d_X) and (Y, d_Y) admits the following expression:

$$\boxed{d_{\mathcal{GH}}^{(1)}(X,Y) = \inf_{d \in \mathcal{D}(d_X, d_Y)} \inf_{R \in \mathcal{R}(X,Y)} \|d\|_{L^\infty(R)}}$$

where $\mathcal{D}(d_X, d_Y)$ is a metric on $X \sqcup Y$ that reduces to d_X and d_Y on $X \times X$ and $Y \times Y$, respectively.

$$\begin{array}{cc} & \begin{array}{cc} X & Y \end{array} \\ \begin{array}{c} X \\ Y \end{array} & \begin{pmatrix} d_X & \mathbf{D} \\ \mathbf{D}^T & d_Y \end{pmatrix} \end{array} = d$$

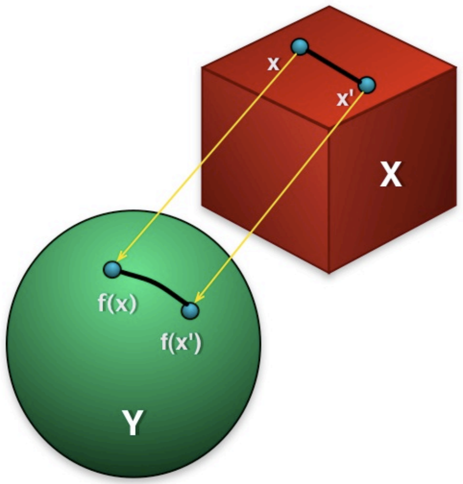
In other words: you need to **glue** X and Y in an optimal way. Note that $\mathbf{D}$ consists of $n_X \times n_Y$ positive reals that must satisfy $\sim n_X \cdot C_2^{n_Y} + n_Y \cdot C_2^{n_X}$ linear constraints.

Another expression for the GH distance

For compact spaces (X, d_X) and (Y, d_Y) let

$$d^{(2)}_{\mathcal{GH}}(X,Y) = \frac{1}{2}\inf_R \max_{(x,y),(x',y')\in R} |d_X(x,x') - d_Y(y,y')|$$

We write, compactly,



$$\boxed{d^{(2)}_{\mathcal{GH}}(X,Y) = \frac{1}{2}\inf_R \|d_X - d_Y\|_{L^\infty(R\times R)}}$$

Equivalence thm:

Theorem [Kalton-Ostrovskii, see [**BBI**]]
For all X, Y compact,

$$\begin{tikzcd} d_{\mathcal{GH}}^{(1)} \arrow[rr, equal] \arrow[d, equal] & & d_{\mathcal{GH}}^{(2)} \arrow[d, equal] \\ \inf_{d,R} \|d\|_{L^\infty(R)} & & \frac{1}{2} \inf_R \|d_X - d_Y\|_{L^\infty(R \times R)} \end{tikzcd}$$

Relaxing the notion of correspondence

$$\begin{tikzcd}
d_{\mathcal{GH}}^{(1)} \arrow[rr, equal] \arrow[dd] & & d_{\mathcal{GH}}^{(2)} \arrow[dd] \\
& d_{\mathcal{H}} \arrow[ul] \arrow[ur] \arrow[dd, Rightarrow] & \\
d_{\mathcal{GW},p}^{(1)} \arrow[rr, equal, "?"] & & d_{\mathcal{GW},p}^{(2)} \\
& d_{\mathcal{W},p} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
d_{\mathcal{GH}}^{(1)} \arrow[rr, equal] \arrow[d, equal] & & d_{\mathcal{GH}}^{(2)} \arrow[d, equal] \\
\inf_{d,R} \|d\|_{L^\infty(R)} & & \frac{1}{2}\inf_R \|d_X - d_Y\|_{L^\infty(R\times R)}
\end{tikzcd}$$

$$\begin{tikzcd}
d^{(1)}_{\mathcal{GH}} \arrow[r, equal] \arrow[d, equal] & d^{(2)}_{\mathcal{GH}} \arrow[d, equal] \\
\inf_{d,R} \|d\|_{L^\infty(R)} \arrow[d] & \frac{1}{2} \inf_R \|d_X - d_Y\|_{L^\infty(R\times R)} \arrow[d] \\
\inf_{d,\mu} \|d\|_{L^p(\mu)} \arrow[d, equal] & \frac{1}{2} \inf_\mu \|d_X - d_Y\|_{L^p(\mu\otimes\mu)} \arrow[d, equal] \\
d^{(1)}_{\mathcal{GW},p} & d^{(2)}_{\mathcal{GW},p}
\end{tikzcd}$$

Now, one works with *mm-spaces*: triples (X, d, ν) where (X, d) is a compact metric space and ν is a Borel probability measure. Two mm-spaces are *isomorphic* iff there exists isometry $\Phi : X \to Y$ s.t. $\mu_X(\Phi^{-1}(B)) = \mu_Y(B)$ for all measurable $B \subset Y$.

The first option, proposed and analyzed by K.L Sturm [**St06**], reads

$$d^{(1)}_{\mathcal{GW},p}(X,Y) = \inf_{d\in\mathcal{D}(d_X,d_Y)} \inf_{\mu\in\mathcal{M}(\mu_X,\mu_Y)} \left(\sum_{x,y} d^p(x,y)\mu_{x,y} \right)^{1/p}$$

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The second option reads **[M07]**

$$d^{(2)}_{\mathcal{GW},p}(X,Y) = \inf_{\mu \in \mathcal{M}(\mu_X, \mu_Y)} \left(\sum_{x,y} \sum_{x',y'} |d_X(x,x') - d_Y(y,y')|^p \mu_{x,y} \mu_{x',y'} \right)^{1/p}$$

The **first** option,

$$d^{(1)}_{\mathcal{GW},p}(X,Y) = \inf_{d\in\mathcal{D}(d_X,d_Y)} \inf_{\mu\in\mathcal{M}(\mu_X,\mu_Y)} \left(\sum_{x,y} d^p(x,y)\mu_{x,y}\right)^{1/p}$$

requires $\mathbf{2}(\mathbf{n_X} \times \mathbf{n_Y})$ variables and $\mathbf{n_X} + \mathbf{n_Y}$ plus $\sim \mathbf{n_Y} \cdot \mathbf{C_2^{n_X}} + \mathbf{n_X} \cdot \mathbf{C_2^{n_Y}}$ linear constraints. When $p = 1$ it yields a *bilinear* optimization problem.

Our **second** option,

$$d^{(2)}_{\mathcal{GW},p}(X,Y) = \inf_{\mu\in\mathcal{M}(\mu_X,\mu_Y)} \left(\sum_{x,y}\sum_{x',y'} |d_X(x,x') - d_Y(y,y')|^p \mu_{x,y}\mu_{x',y'}\right)^{1/p}$$

requires $\mathbf{n_X} \times \mathbf{n_Y}$ variables and $\mathbf{n_X} + \mathbf{n_Y}$ linear constraints. It is a *quadratic* (generally non-convex :-( ) optimization problem (with linear and bound constraints) for all p.

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Then one would argue for using $d^{(2)}_{\mathcal{GW},p}$.

$$d^{(2)}_{\mathcal{GW},p} = \mathbf{D}_p$$

Numerical Implementation

The numerical implementation of the second option leads to solving a **QOP** with linear constraints:

$$\min_U \tfrac{1}{2} U^T \mathbf{\Gamma} U$$
$$\text{s.t. } U_{ij} \in [0,1],\; U\mathbf{A} = \mathbf{b}$$

where $U \in \mathbb{R}^{n_X \times n_Y}$ is the *unrolled* version of μ, $\mathbf{\Gamma} \in \mathbb{R}^{n_X \times n_Y \,\times\, n_X \times n_Y}$ is the unrolled version of $\Gamma_{X,Y}$ and $\mathbf{A}$ and $\mathbf{b}$ encode the <u>linear</u> constrains $\mu \in \mathcal{M}(\mu_X, \mu_Y)$.

This can be approached for example via gradient descent. The QOP is non-convex in general!

Initialization is done via solving one of the several *lower bounds* (discussed ahead). All these lower bounds lead to solving **LOP**s.

For details see [**M07**].

Can GW (1) be equal to GW (2)?

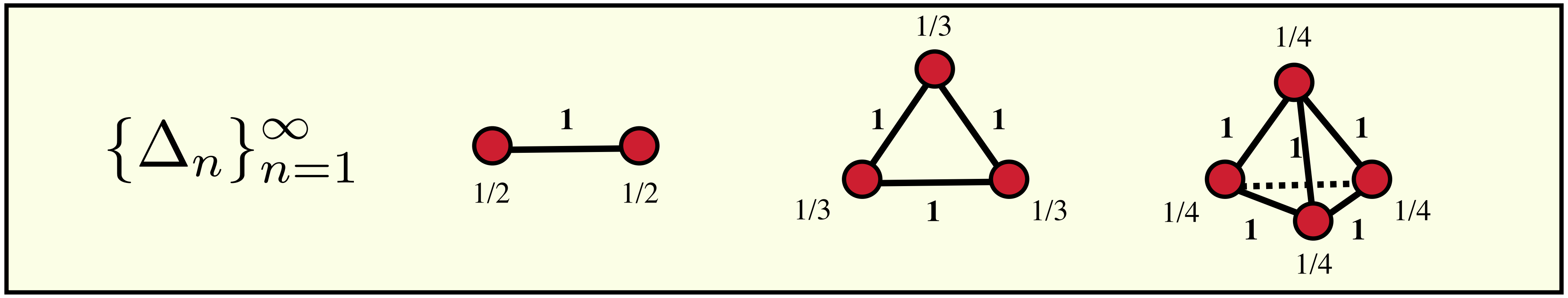
- Using the same proof as in the Kalton-Ostrovskii Thm., one can prove that
$$d^{(1)}_{\mathcal{GW},\infty} = d^{(2)}_{\mathcal{GW},\infty}.$$

- Also, it is obvious that for all $p \geq 1$
$$d^{(1)}_{\mathcal{GW},p} \geq d^{(2)}_{\mathcal{GW},p}.$$

- But the equality does not hold in general. One counterexample is as follows: take $X = (\Delta_{n-1}, ((d_{ij} = 1)), (\nu_i = 1/n))$ and $Y = (\{q\}, ((0)), (1))$. Then, for $p \in [1, \infty)$
$$d^{(1)}_{\mathcal{GW},p}(X,Y) = \frac{1}{2} > \frac{1}{2}\left(\frac{n-1}{n}\right)^{1/p} = d^{(2)}_{\mathcal{GW},p}(X,Y)$$

- Furthermore, these two (tentative) distances are **not equivalent**!! This forces us to analyze them separately. The delicate step is proving that $\mathrm{dist}(X,Y) = 0$ implies $X \simeq Y$.

- K. T. Sturm has analyzed GW (1).



Properties of $d^{(2)}_{\mathcal{GW},p}$

1. Let X, Y and Z mm-spaces then

$$d_{\mathcal{GW},p}(X,Y) \leq d_{\mathcal{GW},p}(X,Z) + d_{\mathcal{GW},p}(Y,Z).$$

2. If $d_{\mathcal{GW},p}(X,Y) = 0$ then X and Y are isomorphic.

3. Let $\mathbb{X}_n = \{x_1, \ldots, x_n\} \subset X$ be a subset of the mm-space $(X, d, \ )$. Endow $\mathbb{X}_n$ with the metric d and a prob. measure $\ _n$, then

$$d_{\mathcal{GW},p}(X, \mathbb{X}_n) \leq d_{\mathcal{W},p}(\ ,\ _n).$$

4. $p \geq q \geq 1$, then $\mathbf{D}_p \geq \mathbf{D}_q$.

5. $\mathbf{D}_\infty \geq d_{\mathcal{GH}}$.

The parameter p is not superfluous

The simplest lower bound one has is based on the triangle inequality plus

$$2 \cdot d^{(2)}_{\mathcal{GW},p}(X,\{q\}) = \left(\int_{X\times X} d_X(x,x')\,\nu(dx)\nu(dx')\right)^{1/p} := \mathbf{diam}_p(X)$$

That is

$$d^{(2)}_{\mathcal{GW},p}(X,Y) \geq \frac{1}{2}|\mathbf{diam}_p(X) - \mathbf{diam}_p(Y)|$$

For example, when $X = S^n$ (spheres with uniform measure and usual intrinsic metric):

- $p = \infty$ gives $\mathbf{diam}_\infty(S^n) = \pi$ for all $n \in \mathbb{N}$
- $p = 1$ gives $\mathbf{diam}_1(S^n) = \pi/2$ for all $n \in \mathbb{N}$
- $p = 2$ gives $\mathbf{diam}_2(S^1) = \pi/\sqrt{3}$ and $\mathbf{diam}_2(S^2) = \sqrt{\pi^2/2 - 2}$

Claim: When $X = S^d$, $s_{X,1}(x) = \pi/2$ for all $x \in S^d$ and $d = 1, 2, 3, \ldots$. Let $A; S^d \to S^d$ be the antipodal map. Then, for all $x_0 \in X$ and $x' \in X$,

$$d_X(x_0, x') + d_X(A(x_0), x') = \pi$$

.

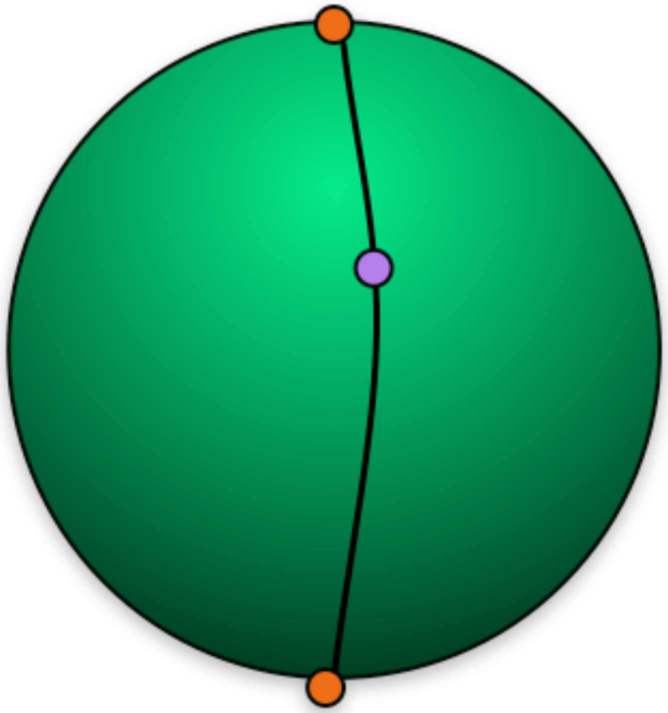
Then, integrating out the variable x' w.r.t. to μ_X,

$$\pi = \int_X \left(d_X(x_0, x') + d_X(A(x_0), x')\right) \mu_X(dx') = \int_X d_X(x_0, x')\, \mu_X(dx') + \int_X d_X(A(x_0), x')\, \mu_X(dx')$$

and then by definition,

$$\pi = s_{X,1}(x_0) + s_{X,1}(A(x_0)) = \pi.$$

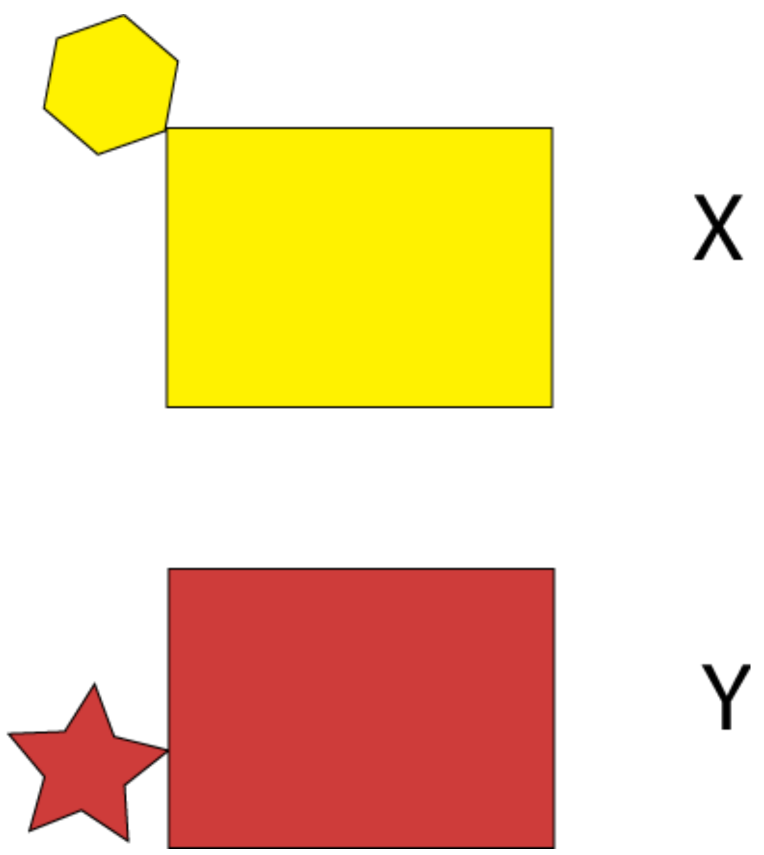
But by symmetry/homogeinity, $s_{X,1}(x_0)$ is independent of x_0, hence the claim.



$$\frac{\mathbf{GH}}{\mathbf{H}} = \frac{\mathbf{GW}}{\mathbf{W}}$$

Gromov's Box distance

$$\begin{array}{rcl} \underline{\square}_\lambda(X,Y) & \simeq & \inf\{\varepsilon > 0 | \,\exists\, X' \subset X,\, Y' \subset Y \\ & \text{s.t.} & d_{\mathcal{GH}}(X',Y') \leq \varepsilon \\ & \text{and} & \max(\mu_X(X\backslash X'), \mu_Y(Y\backslash Y')) \leq \lambda \cdot \varepsilon\} \end{array}$$

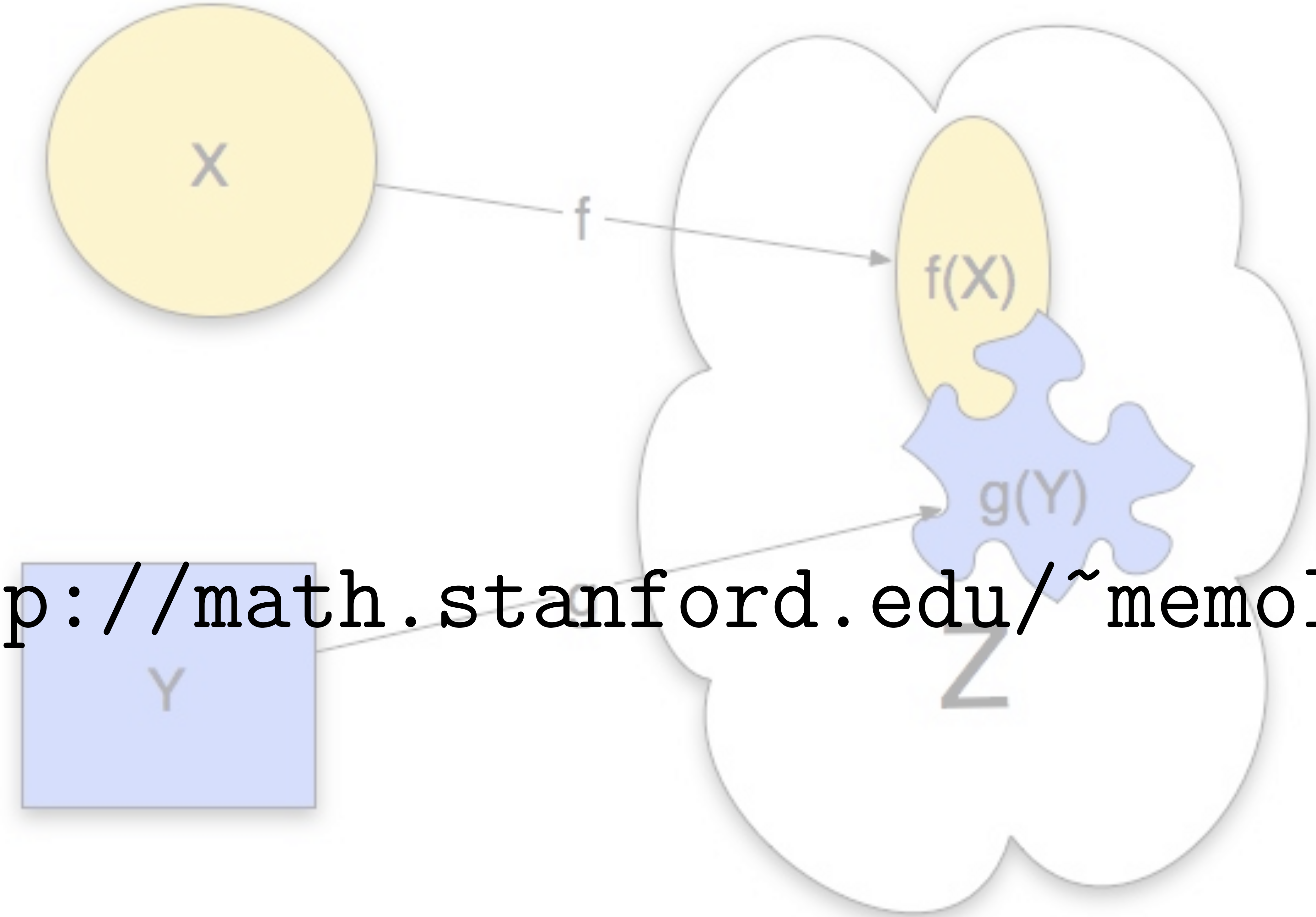


Discussion

- Implementation is easy: Gradient descent or alternate opt.
- Solving lower bounds yields a seed for the gradient descent. These lower bounds are compatible with the metric in the sense that a layered recognition system is possible: given two shapes, (1) solve for a LB (this gives you a μ), if value small enough, then (2) solve for GW using the μ as seed for your favorite iterative algorithm.
- Easy extension to partial matching.
- Interest in relating GH/GW ideas to other methods in the literature. Interrelating methods is important also for applications: when confronted with N methods, how do they compare to each other? which one is better for the situation at hand?
- Latest developments:
  - Partial matching [**M08-partial**].
  - Euclidean case [**M08-euclidean**].
  - Persistent Topology based methods (Frosini et al., Carlsson et al.)
- No difference between continuous and discrete. Probability measures take care of the 'transition'.

Next Class:

- Other properties of $\mathbf{D}_p$
- Lower bounds for $\mathbf{D}_p$ using shape distributions, eccentricities and shape contexts.



http://math.stanford.edu/~memoli