Point Sets up to Rigid Transformations are Determined by the Distribution of their Pairwise Distances

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Survey of results of M. Boutin and G. Kemper

Outline

- Motivation
- Preliminaries
- Folklore Lemma
- Main Result
- Computation
- Extensions and Open Problems

Motivation

- Shape Distributions of [osada]
- Distribution of different functions on randomly selected points considered.
- Different dissimilarity measures between distributions also considered.

Motivation

- Comparing the distribution of pairwise distances w.r.t L_1 norm appears to be a good classifier.
- This paper provides some reasoning on why this might be the case: When the distribution of pairwise distances are identical, then the points in general position are the same up to a rigid transformation.

Preliminaries

- General Position
- Rigid Transformation
- Distribution of Pairwise Distances
- Reconstructibility from Pairwise didstances

General Position

- As defined by Matousek:

  Let a set of n points in $\mathbb{R}^d$ be specified by a vector $t = (t_1, t_2, \ldots, t_m)$ for $m = dn$. Then, a general position condition is a condition that can be expressed as $\bigwedge_i p_i(t) \neq 0$ for a countable number of polynomials p_i.

- Set of points not in general position has measure zero.

Rigid Transformation

- Definition:

Let p be a point in $\mathbb{R}^m$. Then a rigid transformation is a function $R : \mathbb{R}^m \to \mathbb{R}^m$ that can be written as $R(p) = Mp + T$ where M is an orthogonal m-by-m matrix and T is a column vector in $\mathbb{R}^m$.

Pairwise Distances

- Distribution:

Given a set of points P, let the distribution of pairwise distances $d(P)$ be the multiset $\{||p_i - p_j||\}_{i<j}$.

- Reconstructibility:

We say that $p_1, \ldots, p_n \in \mathbb{R}^m$ is reconstructible from pairwise distances if for every $q_1, \ldots, q_n \in \mathbb{R}^m$ with the same distribution of pairwise distances, there exists a rigid transformation R and a permutation π of $\{1, \ldots, n\}$, such that $P(p_i) = q_{\pi(i)}$ for every $i \in 1, \ldots, n$.

Folklore Lemma

- Tricky to work with rigid transformations directly.
- Compare distance matrices instead.
- Folklore Lemma:

Let $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ be points in $\mathbb{R}^m$. If $||p_i - p_j|| = ||q_i - q_j||$ for every i, j in $1, \ldots, n$, then there exists a rigid transformation R such that $R(p_i) = q_i$ for all i.

Folklore Lemma

- Proof:

Let $x_i = p_i - p_n$ and $y_i = q_i - q_n$ for $i = 1, \ldots, n$. We claim that $\langle x_i, x_j \rangle = \langle y_i, y_j \rangle$ for all i, j. To see this, we have the following calculation:

$$\begin{aligned}
\langle x_i, x_j \rangle &= \langle p_i - p_n, p_j - p_n \rangle \\
&= \frac{\langle p_i - p_n, p_i - p_n \rangle + \langle p_j - p_n, p_j - p_n \rangle - \langle p_i - p_j, p_i - p_j \rangle}{2} \\
&= \frac{||p_i - p_n|| + ||p_j - p_n|| - ||p_i - p_j||}{2} \\
&= \frac{||q_i - q_n|| + ||q_j - q_n|| - ||q_i - q_j||}{2} \\
&= \frac{\langle q_i - q_n, q_i - q_n \rangle + \langle q_j - q_n, q_j - q_n \rangle - \langle q_i - q_j, q_i - q_j \rangle}{2} \\
&= \langle q_i - q_n, q_j - q_n \rangle = \langle y_i, y_j \rangle
\end{aligned}$$

Now, let $X = [x_1, \cdots, x_n]$ and $Y = [y_1, \cdots, y_n]$. We have shown above that $X^T X = Y^T Y$. Now, $X^T X$ is symmetric and positive semidefinite, and hence can be written as $Q \Lambda Q^T$ for an orthogonal Q and a non-negative diagonal matrix Λ. Since Λ, is non-negative, we can write $X^T X = Y^T Y = Q \Lambda^{1/2} \Lambda^{1/2} Q^T$. Therefore, using the singular value decomposition, we can write X as $U_X \Lambda^{1/2} Q^T$ and Y as $U_Y \Lambda^{1/2} Q^T$ for orthogonal U_X and U_Y. Then, we can write $Y = MX$ for orthogonal $M = U_Y U_X^T$. Moreover, it is easy to verify that $M x_i = y_i$ for $i = 1, \ldots, n$. To finish the proof, we note that $M(p_i - p_n) = q_i - q_n$ or $q_i = M p_i + q_n - M p_n$ so there is a rigid transformation from p_i to q_i with the orthogonal matrix M and translation vector $q_n - M p_n$.

Digression: Counterexample

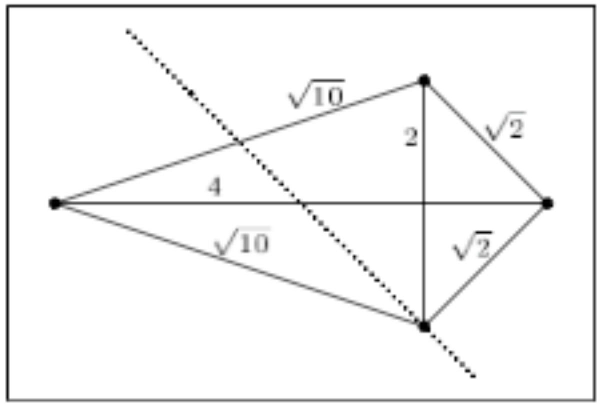
- 1D Counterexample:

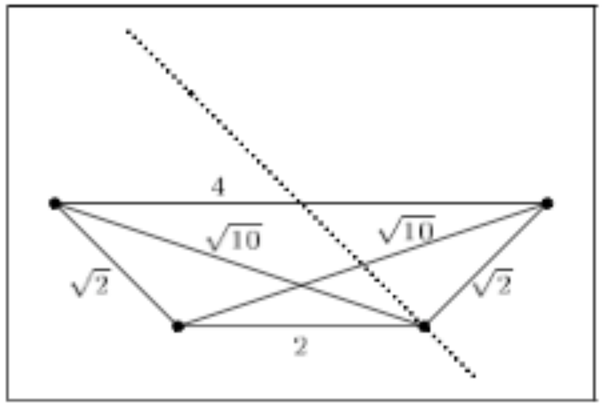
$$P = \{0, 1, 4, 10, 12, 17\} \quad Q = \{0, 1, 8, 11, 13, 17\}$$

It can be verified that the distribution of distances is the multiset:

$$d(P) = d(Q) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17\}$$

- 2D Counterexample:





Main Idea

- With Folklore Lemma in mind, show that if two point sets have the same distribution of pairwise distances, they have the same distance matrices up to a relabeling of the points.

Some Notation

- Convention:

Let $P = \{p_1, \ldots, p_n\}$ be a labeled set of n points. Then we let $d_P : \mathcal{P} \to \mathbb{R}$ be defined as $d_P(\{i,j\}) = ||p_i - p_j||^2$.

- Corollary:

If n-point configurations P and Q have the same distribution of distances, then there exists a permutation ϕ of $\mathcal{P}$ such that $d_P(\phi(\{i,j\})) = d_Q(\{i,j\})$ for all $\{i,j\} \in \mathcal{P}$.

A Weaker Result

Let P be a configuration of n points.Then, there exists neighborhoods of p_i such that if Q if a configuration of n points with q_i in the neighborhood of p_i and having the same distribution of distances as P, then P and Q are the same up to a rigid transformation and a relabeling of the points.

- **Proof:**

Suppose for the purpose of contradiction that there exists a sequence of configurations $\{Q_k\}_{k=1}^{\infty}$ converging to P such that none of Q_k can be mapped to P via rigid transformation and relabeling, but there exists a sequence of permutations $\{\phi_k\}_{k=1}^{\infty}$ such that $d_P(\phi_k(\{i,j\})) = d_{Q_k}(\{i,j\})$. Since there are finitely many permutations ϕ_k, we can pick ϕ_1, for instance, and let $\{R_l\}_{l=1}^{\infty}$ be the subsequence of $\{Q_k\}_{k=1}^{\infty}$ where $\phi_k = \phi_1$. Then, taking the limit $| \rightarrow \infty$, we have $d_P(\phi_1(\{i,j\})) = \lim_{l\to\infty} d_{R_l}(\{i,j\})$. Since $\{R_l\}_{l=1}^{\infty}$ converges to P, we then have $d_P(\phi_1(\{i,j\})) = d_P(\{i,j\})$. But then, we have $d_P(\{i,j\}) = d_{Q_k}(\{i,j\})$ and by the Folklore Lemma, there is a rigid transformation from P to Q_k, so we have a contradiction.

Relabelings

A permutation ϕ of $\mathcal{P}$ is a relabeling if there exists a permutation π of $\{1,\ldots,n\}$ such that $\phi(\{i,j\}) = \{\pi(i),\pi(j)\}$.

- Corollary:

  If there exists a relabeling ϕ of $\mathcal{P}$ such that $d_P(\phi(\{i,j\})) = d_Q(\{i,j\})$, then there is a permutation π of $\{1,\ldots,n\}$ such that $p_{\pi(1)},\ldots,p_{\pi(n)}$ has the same distance matrix as $q_1,\ldots q_n$, and therefore, by the Folklore Lemma, there is a rigid transformation from $p_{\pi(1)},\ldots,p_{\pi(n)}$ to $q_1,\ldots q_n$.

- We want to show most point sets with the same distribution of distances have a relabeling.

Key Lemma

Suppose $n \neq 4$. Then a permutation ϕ of $\mathcal{P}$ is a relabeling if and only if for all pairwise distinct indices i, j, k we have $\phi(\{i,j\}) \cap \phi(\{i,k\}) \neq \emptyset$.

- **Proof:**

For every $n \leq 3$, every ϕ is a relabeling and $\phi(\{i,j\}) \cap \phi(\{i,k\}) \neq \emptyset$. Therefore, we may assume that $n \geq 5$. The only if part of the statement is clear by the definition of a relabeling, so we will show the if direction.

Suppose we have ϕ a permutation of $\mathcal{P}$ such that for all pairwise distinct indices i, j, k we have $\phi(\{i,j\}) \cap \phi(\{i,k\}) \neq \emptyset$. We note that since ϕ is a permutation, the intersection must then contain only one element. Then, we argue that for i, j, k, l pairwise distinct, $\phi(\{i,j\}) \cap \phi(\{i,k\}) \cap \phi(\{i,l\}) \neq \emptyset$. Suppose otherwise. Then, we can write $\phi(\{i,j\}) = \{a,b\}$, $\phi(\{i,k\}) = \{a,c\}$, and $\phi(\{i,l\}) = \{b,c\}$. Now, since we have more than 4 points, we can choose m distinct from i, j, k, l. $\phi(\{i,m\})$ must intersect $\{a,b\}$, $\{a,c\}$, and $\{b,c\}$. Since $\phi(\{i,m\})$ only has two elements, it must be one of the sets $\{a,b\}$, $\{a,c\}$, and $\{b,c\}$, but then that would violate ϕ being a permutation. Therefore, $\phi(\{i,j\}) \cap \phi(\{i,k\}) \cap \phi(\{i,l\}) \neq \emptyset$.

Then if we fix an i and choose any distinct j, k, $\phi(\{i,j\}) \cap \phi(\{i,k\})$ must contain an distinct element a and the above shows that a belongs to any $\phi(\{i,l\})$ where l is distinct from i, j, k. Therefore $\bigcap_{l \neq i} \phi(\{i,l\}) = a$ and we can define the map σ from $\{1, \ldots, n\}$ to itself where $\sigma(i) = a$.

To show that σ is a permutation, we simply need to show that it is injective. To do this, we let M_i be the set of all pairs with i in them. Then, $\phi(M_i) \subseteq M_{\sigma(i)}$. But ϕ is a permutation and $|M_i| = |M_{\sigma(i)}|$ so therefore $\phi(M_i) = M_{\sigma(i)}$. Now consider i, j with $\sigma(i) = \sigma(j)$. Then, $M_{\sigma(i)} = M_{\sigma(j)}$ so $\phi(M_i) = \phi(M_j)$. But ϕ is a permutation so $M_i = M_j$ and therefore $i = j$ so σ is injective.

Now, consider $\phi(\{i,j\})$. By the above discussion, it contains both $\sigma(i)$ and $\sigma(j)$, so $\phi(\{i,j\})$ must be $\{\sigma(i), \sigma(j)\}$. Since σ is a permutation, ϕ is therefore a relabeling.

4 Points on a Plane

The proof that most point configurations are reconstructible from pairwise distances relies on a certain determinant that is zero when four points p_i, p_j, p_k, p_l lie on a plane:

$$\det \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} = 0$$

where $a = -2d_P(\{i,l\})$, $b = d_P(\{i,j\}) - d_P(\{i,l\}) - d_P(\{j,l\})$, $c = d_P(\{i,k\}) - d_P(\{i,l\}) - d_P(\{k,l\})$, $d = -2d_P(\{j,l\})$, $e = d_P(\{j,k\}) - d_P(\{j,l\}) - d_P(\{k,l\})$ and $f = -2d_P(\{k,l\})$.

This determinant can be expanded as a polynomial:

$$\begin{aligned} g(U,V,W,X,Y,Z) = {} & 2U^2Z + 2UVX - 2UVY - 2UVZ - 2UXW - 2UXZ + \\ & 2UYW - 2UYZ - 2UWZ + 2UZ^2 + 2V^2Y - 2VXY - \\ & 2VXW + 2VY^2 - 2VYW - 2VYZ + 2VWZ + 2X^2W - \\ & 2XYW + 2XYZ + 2XW^2 - 2XWZ \end{aligned}$$

where $U = d_P(\{i,j\})$, $V = d_P(\{i,k\})$, $W = d_P(\{i,l\})$, $X = d_P(\{j,k\})$, $Y = d_P(\{j,l\})$ and $Z = d_P(\{k,l\})$.

Main Result

Let $n \geq 5$ and P be a configuration of n points in $\mathbb{R}^2$. Suppose for choices of indices $a, b, c, d, e, f, g, h, i, j, k$ such that the pairs $U = \{a, b\}$, $V = \{d, e\}, W = \{f, g\}, X = \{h, i\}, Y = \{j, k\}, Z = \{a, c\}$ are distinct, we have:

$$g(d_P(U), d_P(V), d_P(W), d_P(X), d_P(Y), d_P(Z)) \neq 0$$

then P is reconstructible from pairwise distances.

- **Proof:** Suppose that Q is an n-point configuration in $\mathbb{R}^2$ with the same distribution of distances as P. Then, there exists a permutation ϕ such that $d_Q(\{i, j\}) = d_P(\phi(\{i, j\}))$ for all $i \neq j$. We then try to show that ϕ^{-1} is a relabeling, and thus ϕ is also a relabeling.

Now pick any pairwise distinct indices r, s, t, u. Now since q_r, q_s, q_t, q_u lie on a plane,

$$g(d_Q(\{r, s\}), d_Q(\{r, t\}), d_Q(\{r, u\}), d_Q(\{s, t\}), d_Q(\{s, u\}), d_Q(\{t, u\})) = 0$$

But then, it follows that

$$g(d_P(\phi(\{r, s\})), d_P(\phi(\{r, t\})), d_P(\phi(\{r, u\})), d_P(\phi(\{s, t\})), d_P(\phi(\{s, u\})), d_P(\phi(\{t, u\}))) = 0$$

Therefore, it follows that $\phi(\{r, s\})$ and $\phi(\{t, u\})$ are disjoint, otherwise the pairs would satisfy the conditions of the U, V, W, X, Y, Z stated above and

$$g(d_P(U), d_P(V), d_P(W), d_P(X), d_P(Y), d_P(Z)) \neq 0$$

so we have show that ϕ maps disjoint sets $\{r, s\}$ and $\{t, u\}$ to disjoint sets. We take the contrapositive and note that if $\phi(\{r, s\})$ and $\phi(\{t, u\})$ intersect, then $\{r, s\}$ and $\{t, u\}$ also necessarily intersect. Thus, for all i, j, k we have

$$\phi^{-1}(\{i, j\}) \cap \phi^{-1}(\{i, k\}) \neq \emptyset$$

and hence ϕ^{-1} is a relabeling. But then there exists a permutation π of $\{1, \ldots, n\}$ such that $\phi^{-1}(\{i, j\}) = \{\pi(i), \pi(j)\}$. Then, clearly, $\phi(\{i, j\}) = \{\pi^{-1}(i), \pi^{-1}(j)\}$ and thus ϕ is also a relabeling. Thus, by Corollary 4.5, there is a rigid transformation and a relabeling that maps P to Q.

Generalization

- Can be generalized for points in R^m for n greater than or equal to m+2.
- Similar method.

Computation

- Experiments testing this general position condition are not very fast.
- $O(n^{11})$ time.

More Extensions

- Oriented rigid transformations.
- Scalings.
- Graphs with edge weights using distribution of sub-triangles.

Open Problems

- Complete test for reconstructibility.
- Relate to Gromov-Hausdorff: Point sets are close w.r.t GH distance when their pairwise distance distributions are close under some measure.